Technical Article

Computational Electromagnetism and Geometry

***Abstract* — Electromagnetic theory can be formally expressed in many ways, and an appropriate formalism is context dependent. In this article we discuss some essential aspects of the theory for the needs of software development for electromagnetic design. The aim is to explain in plain words the key geometric properties behind the constitutive laws and differentiation and their relation to numerical methods.**

I. Introduction

Years ago, remember reading somewhere: "Everything we know about the world is from the boundary". Unfortunately, I'm no longer able to say where this was written and by whom. Nevertheless, this phrase points towards an essential attribute of how we make observations: Whatever we know of the physical world is in a way or another interpolated from the boundary. For example, if we see a house, we don't know, what is there inside. Once we step in, we see the furniture, but we are not able to say from which material they are made of and so on.

Electromagnetic theory makes no exception to this. Say, in electric engineering one makes observations on the currents through boundary surfaces and voltages between boundary points. Then one may employ electromagnetic theory to interpolate what happens within domains. Fig. 1. shows an elementary example of this.

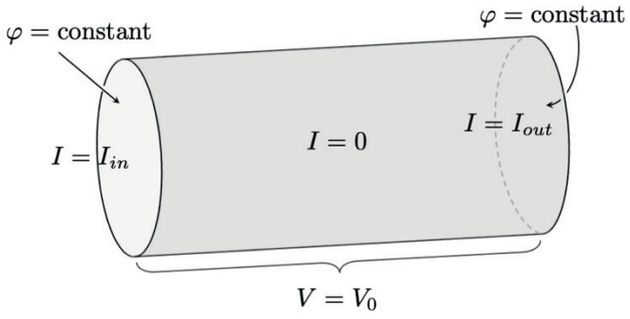


Fig. 1. Once we know the currents and voltages on the boundary, we become able to interpolate the corresponding fields within the domain.

Mathematically such an idea of interpolating inwards boils down to the Hodge map in the constitutive laws and to the Stokes theorem, which establishes a duality between the differential and the boundary operator.

During the formal education field theories are typically presented with vector calculus. Historically this goes back to O. Heaviside. In the end of 19th century, he condensed Maxwell's original version of electromagnetic theory expressed with quaternion equations to equations given in vector notation. This is also now the commonly employed notation in textbooks and university basic courses.

While vector calculus has proven its power as the basic language of electromagnetism, the side-effect is, it buries the geometric character of electromagnetism into the myriad details. Or in other words, while classical vector calculus can be employed to powerfully explain electromagnetic theory, eventually it will also make it unnecessary difficult to recognize what seemingly unrelated thigs do have in common.

The priorities in contemporary engineering design have shifted towards multi-physical problems. To develop software for such needs, a compact view on field theories is an advance. This improves the chances to come up with concise software that is easier to debug and maintain.

In this article we explore the geometric characteristics of the constitutive laws and differentiation. The priority is to explain and highlight the essential aspects and their usage in electromagnetism. Then we exploit the geometric view to explain some key properties of commonly employed numerical methods in computational electromagnetism.

We do not aim for a rigorous exposition. The corresponding technical and mathematical details can be found in the vast literature of differential geometry to which we give some recommendations in the last section.

II. From Vectors to Multivectors

In electromagnetism the term vector points usually to a so-called 1-vector. This is a certain p-vector, that is, a multivector. In short, up to dimension n=3 a p-vector is the triplet {p-plane, p-area, orientation}. In case of an ordinary vector p=1, and accordingly a 1-vector can be identified with the triplet {line, length, orientation}; Just think of the arrows which are typically employed to graphically express vectors. Such an arrow is a segment of line, and the length of the segment indicates the magnitude of the vector. Orientation is indicated with the head on top of the line segment.

A 2-vector is the triplet {plane, area, orientation}, and a 3-vector can be identified with {space, volume, orientation}, see fig. 2.

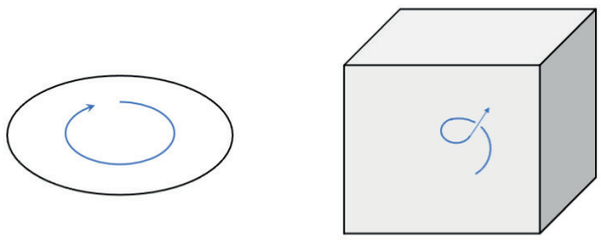


Fig. 2. Multivectors expressed graphically. On the left: a 2-vector. The disk is part of a plane, and it has a certain area. Orientation corresponds with the rotation direction. On the right: a 3-vector. The box indicates the magnitude, and the spiral shows the orientation. Notice, the shape of the disk or cube is not meaningful.

As the name suggests, multivectors are elements of vector spaces. Consequently, like ordinary vectors, p-vectors can also be added and scaled as they fulfil the required axioms.

There exists also an algebra with a product for multivectors. The algebra and the product are known by the name *exterior algebra* and *wedge product*, respectively. The idea is that the wedge product is related to the construction of a so-called signed area. In dimension n=3 this corresponds to the construction of oriented parallelograms and parallelepipeds.

For the purposes of this article, it is sufficient to restrict the dimension to $n \leq 3$.

The wedge product of an *r* and *s*-vector yields an (*r*+*s*)-vector. If $r + s > n$ then the product vanishes. Let *u*, *v*, and *w* be multivectors and α a scalar. The wedge product has following basic properties that correspond with the idea of oriented parallelograms and parallelepipeds:

$$u \wedge v = -v \wedge u,$$
$$u \wedge \alpha v = \alpha(u \wedge v),$$
$$(u \wedge v) \wedge w = u \wedge (v \wedge w),$$
$$u \wedge u = 0.$$

For an example, the wedge product $\wedge$ of two 1-vectors *u* and *v* yields a 2-vector $u \wedge v$. This can be interpreted as an oriented parallelogram, see fig. 3.

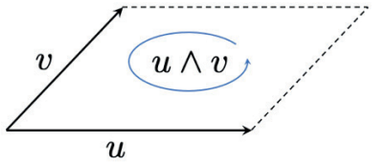


Fig. 3. The wedge product of 1-vectors *u* and *v* is a 2-vector $u \wedge v$, which can be interpreted to correspond to an oriented parallelogram.

To create some intuition on 2-vectors, let us consider the notion of *torque*. In terms of classical vector analysis torque is the cross product of displacement vector *r* and force *f*,

$$\tau = r \times f.$$

This formalization, however, includes some challenges. For, it is already rather cumbersome to interpret torque as a vector perpendicular both to *r* and *f*. No explanation seems to match satisfactorily with our intuition of torque. Furthermore, perpendicularity is another issue. It is a metric notion, and consequently, to talk of perpendicularity one needs an inner product for vectors. But this, again, does not seem to coincide with our observations on torque.

Say, if we have a tight nut, our intuition tells us immediately in which direction to pull the wrench. This direction is not $r \times f$. Or, if two people pull the wrench instead of one, it is plain the torque will also increase. Nothing suggests we needed metric to understand what torque is about.

If instead of $\tau = r \times f$ we define torque as a 2-vector

$$\tau = r \wedge f,$$

the magnitude of $r \wedge f$ will equal to that of $r \times f$, but this time the interpretation of τ supports our intuition: Torque affects in the plane spanned by vectors *r* and *f* and it has orientation, the rotation direction, that matches with intuition. Furthermore, if the torque is modeled as a 2-vector, there is no need for metric. This simple example reveals vector calculus is not void of problems.

Next, let us briefly recognize the wedge product of a 2-vector *w* and a 1-vector *v* is a 3-vector $w \wedge v$ that can be viewed as an oriented parallelepiped, see fig. 4.

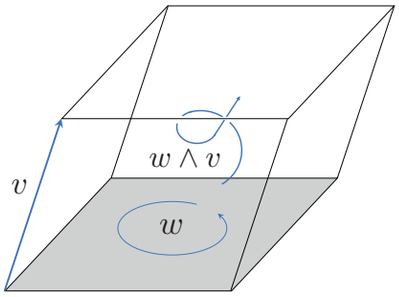


Fig. 4. The wedge product of 2-vector *w* and 1-vector *v* corresponds to 3-vector $w \wedge v$. For convenience, we may illustrate it geometrically as a parallelepiped. Still, one should not understand multivectors had a shape.

III. Identification of Vectors

Next, let us now pinpoint what is the source of confusion created when vector calculus is employed in formalizing field theories. To concretize the reasoning, we exploit charge density as a working example.

The very idea of charge density is to express pointwise the ratio between charge and volume. Charge density can be aptly given as a scalar function. The potential of the electrostatic field is another notion that is expressed as a scalar function.

While charge density and the electrostatic potential are both formalized as scalar fields, it is still clear that the two are structurally slightly different. For, even if we try hard, it becomes very difficult to find an interpretation for the electrostatic potential as a density. This indicates, the usage of a scalar function for both purposes does not only pack but also loose information. As a result, the modeler should keep record of different scalar fields of different type. In the ideal case, the formalism provided the modeler with this information.

A. Metric structure and Hodge map

The reason for two kinds of scalar fields is hidden in the identification vector calculus makes between p-vectors and $(n - p)$-vectors.

The electrostatic potential is a field of real numbers. In this case *p*=0 as real numbers are also 0-vectors. (Real numbers form a vector space whose elements can be added and scaled.) Charge density is a field of 3-vectors, and hence, the two fields are not of the same type.

To work out, how vector calculus creates an identification between the 0 and 3-vectors, we denote a real number, that is, a 0-vector by α. The unit 3-vector is denoted by ω_0, and the inner product of $a, b \in \mathbb{R}$ by $(a, b) = |ab|$.

Now, for all real numbers β one can find a 3-vector *w* such that equation

$$\beta \wedge w = \omega_0(\beta, \alpha) \quad (1)$$

holds. To interpret this, notice first that $\beta \wedge w$ is the same as *w* scaled by β, that is, $\beta \wedge w = \beta w$. Accordingly, instead of (1) we may write

$$\beta w = \omega_0 |\beta|\, |\alpha| = \omega_0 |\beta\alpha|\,. \quad (2)$$

This in turn is equivalent to

$$\pm w = \frac{\beta}{|\beta|} w = \omega_0 |\alpha|\,. \quad (3)$$

And now we see, for any real number α we always find a 3-vector w whose magnitude matches with $|\alpha|$. (The sign in $\pm w$ influences only on the orientation of w.) This 3-vector w is special as it can be recognized as the image of α under the so-called Hodge map.

The *Hodge operator* $\star$ is a linear map between p and (n–p)-vectors. The map is defined as follows: Let (u, v) be the inner product between p-vectors u and v. Then, (n–p)-vector $\star v$ is the image of Hodge map on p-vector v, if

$$u \wedge \star v = \omega_0(u, v) \tag{4}$$

holds for all p-vectors u.

The Hodge map yields the identification between p and (n–p)-vectors: Given a p-vector v, it can be identified with an (n–p)-vector $\star v$. Notice, eq. (1) is a special case of this when $p = 0$ and $n = 3$.

B. Geometric interpretation

To create a geometric interpretation, we pick first a non-zero 1-vector v and notice that eq. (4) should hold for all 1-vectors u. Then we exploit the fact that on the right-hand side of (4) the inner product (u, v) vanishes when u is perpendicular to v. Consequently, in such a case on the left-hand side the wedge product between 1-vector u and 2-vector $\star v$ should also vanish. In other words, the volume of the parallelepiped corresponding with $u \wedge \star v$ should go to zero when u is perpendicular to v. The basic properties of the wedge product say, this is precisely the case when u is on the plane of $\star v$.

To find the magnitude of $\star v$, we choose $u = v$. In this case the right-hand side of (4) corresponds with a parallelepiped whose volume is $|v|^2$. This implies, on the left-hand side of (4) the magnitude of $\star v$ should also be $|v|$ as then the volume of the parallelepiped corresponding with the wedge product of v and $\star v$ is also $|v|^2$, see fig. 5.

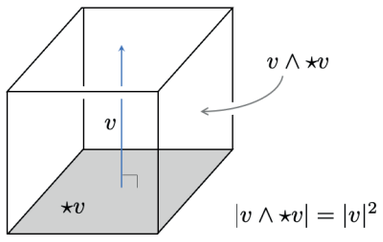


Fig. 5. 2-vector $\star v$ is perpendicular to 1-vector v, and its magnitude is $|v|$.

If the Hodge map is taken twice, one has $\star\star = \pm 1$, where the sign depends on n and p. However, in dimension three, the sign is always positive and $\star\star$ is an identity map. Accordingly, in fig. 5, the Hodge map takes also $\star v$ back to v. The Hodge map is central in potential and field theory.

When field theories are expressed with classical vector calculus, the hidden identification between (n–p) and p-vectors is easily a source of confusion that blocks one from seeing how the different things are related to each other. Above we discussed already about modelling torque as a vector. Now we have enough tools to exemplify, how the issue gets back to the metric identification between 2 and 1-vectors: We have

$$\star(r \wedge f) = r \times f \tag{5}$$

That is, 1-vector $r \times f$ is a proxy for the 2-vector $r \wedge f$, and if this is not recognized, the attempts to give a physical meaning to the axial direction of vector $r \times f$ get easily rather confusing.

IV. Electromagnetic fields

Next, we move to modeling electromagnetic fields. In a discussion with Alain Bossavit, he once said: "The electrostatic field is equivalent to knowing all voltages". This thought goes back to the question, how do we know about fields at the first place?

To answer this question, notice that in electrostatics we can observe the work needed in displacing a charge from one point to another. The work observations are transformed to voltages by peeling out the effect of charge.

Voltages themselves are about functions. They map oriented paths to real numbers. Accordingly, knowing all voltages in some domain corresponds with knowing the numbers voltage yields on all possible oriented paths.

Operating with all oriented paths is, however, bit clumsy. For this reason, the notion of a field sounds preferable. To work out the electric field from all voltages, notice first that oriented paths can be constructed from virtual segments, i.e., from the tangent vectors of the paths. Then, consider a point and some oriented path. If we i) take a small segment of the path around the point, ii) record the voltage over this segment, and iii) divide the voltage with the length of the segment while the segment shrinks towards the chosen point, such limit process provides us with a voltage related to the tangent vector.

Next, if we know the voltages for all oriented paths through the point, then we also know the map from all tangent vectors to voltages. Such a function mapping 1-vectors to real numbers is a *1-covector*. It is an element of the space dual to vectors. A field of such convectors is a *differential form*, and more precisely a 1-form. In this case the 1-form is called the electric field strength e. It is the local pointwise counterpart to voltage.

The other way around, if we need to know the voltage U over a given oriented path l, this is easily obtained from e by integrating it over l:

$$U = \int_l e \ .$$

(Notice, the integral is the sum of the voltages e yields on the tangent vectors. As the path l involves all the information of its tangent vectors, there is no need to separately denote the tangent vectors in the notation.)

As voltages can be measured only over paths with finite length, the notion of electric field strength e is already a part of the electromagnetic model rather than an object we could explicitly observe or measure.

Extending the idea of "all voltages" to the other basic notions of electromagnetic theory is now straightforward. Magnetic field strength h has to do with the magnetomotive force which is a map from oriented paths to real numbers. Accordingly, also h is modeled naturally as a 1-form.

A. p-forms

Electric currents, electric flux, and magnetic flux are related to surfaces. The limit process to work out differential forms j, d,

and b (the current density, electric flux density, and magnetic flux density, respectively) is the same as in case of e.

Let us employ b as a working example: If we know the magnetic fluxes on all oriented surfaces, then at any point we can pick a surface and generate a sequence of patches on it converging towards a tangent 2-vector and record simultaneously the magnetic fluxes over the patches. At each point such a limit process provides us with a 2-covector that maps 2-vectors to real numbers representing magnetic fluxes. A field of such covectors is the 2-form magnetic flux density denoted by b. 2-forms j and d are constructed in the same way.

Conversely, electric currents, electric fluxes, and magnetic fluxes can be integrated from j, d, and b, respectively:

$$I = \int_s j, \quad \Psi = \int_s d, \quad \Phi = \int_s b \ .$$

Summing up, 2-covectors j, d, and b are the local counterparts of electric current, electric flux, and magnetic flux.

The model of charge density should now be clear. Charge Q is a map from oriented volumes to real numbers. If charge density ρ is modeled as a 3-form, then locally at a point, maps 3-vectors to real numbers representing net charges. This way 3-covector ρ becomes a local counterpart to charge Q.

Remark I: There exists a subtle detail related to orientation: Electric current density, electric flux density, charge density, and magnetic field strength are so-called *twisted forms* that act on outer oriented multivectors. Their orientation depends on the orientation of space. For example, electric current is about the net charge through, not on, a surface. To take sides on the penetration direction, one must specify which side of the surface is positive and which negative. This is not a property of the surface alone, but instead the penetration direction depends also on the orientation of the space in which the surface lies.

V. Constitutive laws

Covectors map vectors to real numbers, but they are also vectors. They do fill the axioms of addition and scaling. Consequently, the Hodge operator also maps p-covectors to $(n-p)$-covectors. Furthermore, the Hodge map will enable us to formally express the geometric observation that field lines are always perpendicular to the equipotential layers.

We employ the pair $\{e, j\}$ as a working example. As we just recognized, covectors e and j are the local counterparts to U and I, respectively. Given a conductive domain with a pair of connectors, the product UI is the power lost in heat, see fig. 6.

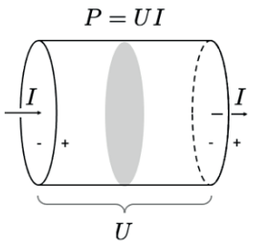


Fig. 6. A typical textbook graphical illustration of voltage between end connectors and current through a conductive domain. The power lost in heat is the product of voltage and current.

In this kind of setting the observations on voltage and current are made from the boundary of the domain. The very idea of field theory is to provide us with tools that make it possible to interpolate inwards from the boundary to model what happens within the domain. Accordingly, as e and j are the local counterparts of U and I, a constitutive relation between e and j is a natural proxy to Ohm's law. Indeed, if we denote conductivity by σ and write

$$j = \star\sigma e$$

then by the definition of the Hodge map we also have

$$e \wedge j \ = \ e \wedge \star\sigma e \ = \ \omega_0(e, \sigma e) \qquad (6)$$

Before working out the meaning of eq. (6), let us hasten to draw it graphically, see fig. 7.

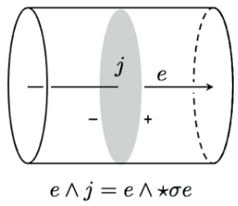


Fig.7. Graphical illustration of the Hodge relation $e \wedge j$, where 1-covector e yields a real number representing voltages on the vector drawn with black, and 2-covector j returns the current through the gray patch. The product $e \wedge j$ is a 3-covector that maps the volume to a real number.

The visible similarity between fig. 6 and 7 is not a coincidence. The wedge product 1-covector e and 2-vector j is a 3-vector. Its magnitude can be deduced from right hand side of (6). In terms of vector calculus, the inner product $(e, \sigma e)$ of covectors corresponds to $\boldsymbol{e} \cdot \sigma \boldsymbol{e} = \sigma |\boldsymbol{e}|^2$. This is the density of power converting to heat. Consequently, $e \wedge j$ is a 3-vector whose magnitude represents the power transferring to heat in the virtual volume of a point.

In this sense $j = \star\, \sigma e$ can be interpreted as a local counterpart to Ohm's law $I = \mathrm{G}U$, where G is the inverse of resistance. The other way around, the integral of $e \wedge j$ over the whole domain is the power,

$$P = \int_\Omega e \wedge j = \int_\Omega e \wedge \star\sigma e \, .$$

The other two constitutive laws of electromagnetism, the dielectric and the magnetic one, can be given in the same way by writing

$$d = \star\varepsilon e \, ,$$
$$b = \star\mu h \, .$$

The products $e \wedge d$ and $b \wedge h$ are related to the energy stored in the electric and magnetic field, respectively.

VI. Differentiation

We move now to differentiation and differential equations. The differential and boundary operators are dual to each other. The very idea is, the integral of a field over a boundary should equal to the integral of its derivative over the domain bounded by the boundary. This is the Stokes theorem

$$\int_\Omega \mathrm{d}f \ = \ \int_{\partial\Omega} f \ .$$

Here, d is the differential operator of differential forms known by the name *exterior derivative*.

The Stokes theorem provides us now with a geometric definition for the exterior derivative:

$$\mathrm{d}f \;=\; \lim_{|\Omega|\to 0}\left(\frac{1}{|\Omega|}\int_{\partial\Omega} f\right) \;.$$

The definition is also the same for all degrees of differential forms $p = 0, \ldots, n$, and it has a very geometric character, see fig. 8.

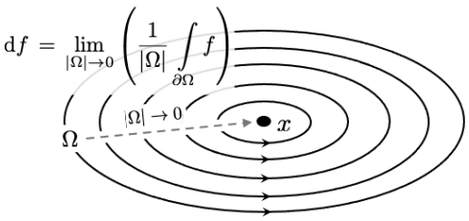


Fig. 8. Exterior derivative d of field f at point x is defined by a sequence of integrals over the boundary of domain Ω while Ω shrinks to point x.

The definition can be put to words by stating, the derivative of field f is equivalent to knowing all the integrals of f over bounding cycles. (Bounding cycles are boundaries of oriented objects, such as pairs of nodes that are the end nodes of a path, loops that are boundaries of surfaces, or surfaces that are boundaries of volumes.)

The definition is also the first part of the answer to the question, when does a field have a potential? As the derivative is defined with bounding cycles, conversely, the existence of a potential has to do with the condition whether all the cycles –that is, the objects whose boundary vanish– are also bounding cycles. If this is not the case, the domain should be cut to make the condition to hold. (This is, however, not the same issue as simply connectedness. The difference boils down to the distinction between homology and homotopy.)

Remark II: The definition of d calls for no coordinate; Coordinate systems make it possible to represent derivatives with numbers with respect to a chosen system, but we should not need coordinates to understand or to talk meaningfully of derivatives. For example, the notion of speed or velocity is already plain without any coordinate system.

Remark III: The definition requires no metric; Neither the concept of a boundary nor the integration over a boundary depends on the metric structure. This property can be effectively exploited to simplify the treatment of problems with moving or deforming bodies.

Remark IV: The (metric) counterparts of the exterior derivative in vector calculus are the grad, curl, and div operators. The fact that the same operator has three proxies is another example of vector calculus making it bit difficult to recognize what seemingly different things have in common.

VII. Numerical methods

To apply what was explained above to numerical techniques employed in solving electromagnetic boundary value problems, would like first to stress out that the difference between the numerical techniques and the actual electromagnetic theory lies in the dimension of the underlying functional spaces: Numerical methods in computational electromagnetism, such as finite elements or finite differences, are about solving boundary value problems in *finite dimensional spaces*.

On one hand the finiteness implies, not all properties of the "continuous model" can be retrieved. On the other hand, whatever numerical technique is employed, the approach should converge towards the continuous model when the mesh grain tends to zero.

In more precise terms the continuous model is a second order boundary value problem consisting of a pair of first order differential equations and of the constitutive laws. Accordingly, the solutions of finite dimensional problems should converge towards these conditions.

The *finite element method* starts from imposing one of the differential equations in the strong sense. This implies, the numerical solutions fulfil exactly the corresponding differential equation. In addition, the constitutive law is, or constitutive laws are satisfied at all (ordinary) points of space. However, the remaining differential equation is imposed only in the weak sense with a finite amount of test functions. Consequently, the finiteness of the method implies one gives up from fulfilling exactly the other differential equations.

In the *finite difference method* (and in generalized finite differences, the finite integration technique etc.) one generates a dual pair of interlocked meshes and imposes the pair of differential equations exactly on the meshes, whereas the constitutive law is imposed only on the finite amount of intersection points between the primal and dual mesh. In this case one gives up from fulfilling the constitutive laws at all points of space.

Historically, these finite elements and finite difference have been discovered and developed from very different starting points. For this reason, they also appear only weakly related to each other. However, both methods must still converge towards the same differential equations when the mesh grain tends to zero.

Differentiation itself can be interpreted as sequences of integrals over bounding cycles while these cycles shrink. In case of finite elements one may consequently ask, shouldn't it be possible to find in the same way as in finite differences a dual mesh that makes also the other differential equation exact in the sense the integrals over bounding cycles hold exactly? And this is, indeed, the case as A. Bossavit has shown in ref. [1], see fig. 9.

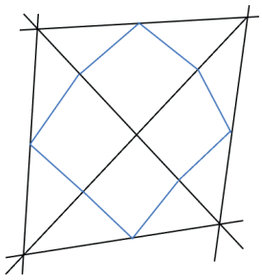


Fig. 9. The black lines are part of a finite element mesh, where one of the differential equations is imposed to hold exactly. In finite elements the other differential equation is imposed in the weak sense. But if Whitney elements are employed, the other differential equation still holds exactly in the sense of integrals over the bounding cycle shown with blue color. For example, in case of magnetostatics, if one imposes $\mathrm{d}b = 0$ on the finite element mesh shown with black lines and sets $b = \star\, \mu h$, then the Ampère law $\int_{\partial s} h = \int_s j$ will hold exactly for the bounding cycle ∂s shown with blue.

This means, while numerically finite elements and finite differences have different properties, they are still variants of the same approach. The two have different numerical properties, but these are about the superstructures which,

though, are important for the end users. Say, the end user may prefer the efficiency of finite difference kind of approaches in solving wave problems while find finite elements more appropriate in modeling electromagnetic problems in the quasistatic regime.

However, the infrastructure lies in the differential equations and in the constitutive laws, and the way we understand their information content has a significant effect on the software we produce. The aim of software development is to come up with concise systems that enable users to tackle large range of problems with the methods they find preferable. A recipe to meet such a goal lies in what mathematician M.F. Atiyah wrote [2]: "*The aim of theory really is, to a great extent, that of systematically organizing past experience in such a way that the next generation, our students and their students and so on, will be able to absorb the essential aspects in as painless a way as possible, and this is the only way in which you can go on cumulatively building up any kind of scientific activity without eventually coming to a dead end.*"

Electromagnetic theory is perhaps the strongest and best understood theory of physics. The underlying mathematics is also well understood, and the literature is vast. The background of numerical methods is also extensively studied. However, the knowledge is scattered, and the books of mathematics written for completely different purposes tend to make a difficult reading for engineers. Accordingly, there is much space for a systematic reorganization of the knowledge to support and guide software development to meet the new challenges.

VIII. Literature

A very good entry level reading on the geometry of electromagnetism and differential geometry can be found in a series of papers Bossavit wrote in 1998, see refs. [3], [4], [5], [6].

Another good starting point to differential geometry is [7]. Mathematically a more involved but also an excellent book in field theories and differential geometry can be found in [8].

A profound wiki for mathematics, physics, and philosophy is available online called by the name nLab [9].

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