*Technical article*

# Multiscale finite element methods for eddy current problems in laminated iron MSFEM4ECP

*Abstract* — **This article is about the ongoing research project "Multiscale Finite Element Methods for Eddy Current Problems" MSFEM4ECP in laminated iron.**

**The simulation of eddy currents in laminated iron cores by the finite element method (FEM) is of great interest in the design of electrical machines and transformers. In extrem cases the iron core is made of grain oriented ferromagnetic laminates, the material properties are anisotropic and exhibit a magnetic hysteresis. The scales vary from the meter range for the iron core to the thickness of single laminates (typically in the range of $0.2 - 0.3mm$). Clearly, modeling each laminate individually is not a feasible solution. Many finite elements (FEs) have to be used in such a model leading to extremely large nonlinear systems of equations. An accurate simulation of eddy currents and the iron losses in laminated ferromagnetic cores with reasonable computer resources is not solved satisfactorily. It is still one of the major challenges in computational electromagnetics. Laminated cores represent simple speaking a periodic microstructure and therefore are well suited for multiscale finite element methods (MSFEMs).**

**Simulations with MSFEM show a boundary layer quite similar to that which occurs in corresponding brute force models of such cores with anisotropic material properties. An accurate approximation of the boundary layer is essential for an exact evaluation of the iron losses. However, this requires many FE layers, which considerably increases the total number of FEs in the model. The periodic nature of the lamination is interrupted by step lap joints, ventilation ducts or disturbed by skewing leading to complex geometries which are costly in FE modeling on its own.**

**An accurate approximation by the FEM with standard polynomials also in case of equations with rough coefficients, for instance materials with a microstructure, and problems with a boundary layer, requires extremely fine meshes. Therefore, new multiscale finite element methods (MSFEM) are developed to cope with the microstructure, where the standard polynomial basis is augmented by special functions incorporating a priori information into the ansatz space to avoid fine FE meshes. Then, the MSFEM is combined with the harmonic balance method to reduce the computational costs furthermore. To provide a comprehensive solution for the present topic, approaches for the boundary layer and for the above geometrical difficulties are designed and integrated into MSFEM. Hysteresis is considered by an appropriate hysteresis model. Fast adapted numerical integration methods, a very important issue for an efficient MSFEM in this context, are developed which do not affect the accuracy of the approximation. All approaches are developed for the time and frequency domain and for both potential formulations, the magnetic and the current vector potential. In order to give a rigorous justification of the used techniques, sharp error estimators are developed which admit a multiscale structure as well as allow a cheap computation. A benchmark to provide measurement data and the supercomputer VSC to compute very expensive reference solutions will ensure an optimal development of the new MSFEM approaches.**

**The aim is to create highly accurate numerical solutions consuming minimal computer resources to run on personal computers without any difficulty. All new MSFEM approaches will be incorporated into the open source hp-FEM code Netgen/NGSolve.**

## I Introduction

Computational electromagnetics in general and thus, computer aided design of electric devices has undergone a great development over the last decades and has become without any doubt an indispensable standard tool for engineers. The remarkable scientific progress in the simulation of eddy current problems by the finite element method is, well documented in many outstanding publications in corresponding journals like IEEE Transaction on Magnetics, Compel, etc., and has ever been one of the main foci of scientists participating relevant conferences like Compumag, CEFC, CEM, IGTE Symposium, etc.

It is fair to say that the simulation of eddy currents in three-dimensional arrangements with complex geometries can be solved routinely even in ferromagnetic materials with pronounced saturation effects. In this context, the finite element method (FEM) has proved its versatility [1, 2]. The computation of eddy currents in laminates is clearly of enormous practical importance. Unfortunately, this is still far from being a routine task in computational electromagnetics. This can be explained by the fact that modeling each laminate leads to extremely high computational costs, i.e., computation times and memory requirements. In order to overcome this unpleasant limitation, much effort has been undertaken in the development of various different methods in the last two decades. Nevertheless, the ability to accurately simulate eddy currents and thus the associated losses in laminated ferromagnetic cores with reasonable computer resources is still one of the persistent striking problems which is by far not solved satisfactorily yet, but is of great importance and poses an exceptional challenge.

Accurate knowledge about the flux distribution in laminated cores is important to reduce both iron losses (eddy current, hysteresis and anomalous losses [3, 4]) and sources for forces (magnetostriction, inter-laminar forces [5, 6]). Extremely large difficulties arise from the ratio of the thickness of the steel sheets, typically in the range of 0.2-0.3$\mathrm{mm}$ (or even essentially smaller: the air gap between laminates), to the overall dimensions of cores, which can be a few meters and the large number of the laminates, i.e., up to more than one thousand. The flux varies strongly in the vicinity of corners of air gaps. The permeability of grain-oriented electrical steel sheets, which are anisotropic and exhibit a magnetic hysteresis, has to be taken into account and modeled correctly. An adequate modeling of each laminate by the finite element method requires a very high number of finite elements and leads to an extremely large nonlinear system of equations absolutely inappropriate for a fast routine analysis [7, 8].

MSFEM for eddy currents in laminated ferromagnetic media seems to be a very promising method to overcome all difficulties [8, 9]. The aim is a radical reduction of the extremely high computational costs, i.e., computation times and memory requirements, to run such eddy current simulations on personal computers without any difficulty.

## II State of the Art

The attempt is made to give an overview about methods dealing with eddy currents in conducting thin laminates so far. The authors are aware of the fact that the overview is by no means complete.

Several methods are based on the idea to decompose the magnetic flux into the main magnetic flux parallel to the lamination and a magnetic stray flux, which impinges the core in the direction normal to the lamination. Due to the stray flux large laminar eddy current loops are caused, whereas the main flux leads to very narrow eddy current loops [10, 7].

*Brute Force Methods*:
So-called *Brute Force Methods* apply either an anisotropic electric conductivity [11, 12] and eventually an anisotropic permeability [13, 14] or prescribe a current vector potential having a single component normal to the lamination [15] in finite element models. The discretization is similar to one that would be used if the core is not laminated. These methods consider only the laminar eddy currents caused by the stray flux. Therefore, the associated eddy current losses are too small [12], because the losses caused by the eddy currents due to the main flux are neglected.

*Two Step Methods*:
To improve the results, i.e., taking into account the small eddy current loops caused by the main flux parallel to the laminations, two step methods have been developed. The first step is the analysis described in the previous paragraph with an anisotropic material or with a single component current vector potential. This solution is corrected in the second step exploiting different approaches. The simplest methods carry out a one dimensional correction, see for example [12]. One dimensional methods are only true when edge effects (or fringing effects [16]) can be neglected. Therefore, several three-dimensional corrections were developed based on different assumptions to take into account the main magnetic flux and small eddy current loops in the second step.

Some of them assume that the average values of field components parallel to the lamination across the thickness of the laminates in the first step equal to the averages of the corresponding field components in the reference solution of finite element models considering the laminates. Several methods have been developed for problems with linear material properties. Some of them are summarized subsequently. A comparison of the coefficients after averaging the fundamental solution of the electric and magnetic field of the diffusion equation with the average values of the field components in the first step expanded in Fourier series were carried out in [17]. Three different correction methods have been proposed in [18]. The first one is quite similar to the previous one but without Fourier expansion, the second method exploits the finite integration technique (FIT, [19]) or "cell method" [20] which offers an easy possibility to prescribe edge-voltages and edge-excitations, facet-fluxes and facet-currents, respectively. The third method utilizes the current vector potential formulation, which facilitates the setting of a normal component of the current density to zero on the surface of the laminates.

Another kind of assumptions are that either the tangential component of the magnetic field intensity or the normal component of the magnetic flux density together with that of the electric current density [21] coincides in the anisotropic model with those in the reference model. To facilitate the prescription of the boundary values the first option uses a current vector potential and the second one uses a mixed formulation, i.e., a magnetic and a current vector potential. In the second step boundary value problems of single laminates with boundary conditions according to one of these assumptions are solved. A model with anisotropic conductivity and nonlinear magnetic properties is simulated in the first step for nonlinear problems. The agreement of the tangential components of the magnetic field intensity is employed to reconstruct the true solution in [22]. In the second step, the cell method and piece-wise constant permeability according to the saturation of the solution in the first step are used. A relatively modest accuracy in the total losses (eddy current and hysteresis losses) compared to measurements was obtained. Both methods described in [21] where extended to nonlinear materials in [7]. The reconstructed eddy current losses are in good agreement with that of the reference solution.

All *two step methods* have in common that the field is reconstructed in single laminates one after another. Thus, only a few arithmetic operations are required in the second step. This is fast and the additional memory requirement is negligible small. The computation of the losses is also computationally cheap.

*Methods which provide the solution in one step*:
Homogenization methods have been proposed where the total magnetic flux is considered in one step in the finite element formulation to solve static magnetic field problems. The magnetic scalar potential has been employed in [23] and the cell problem was solved to get the periodic micro-shape function. The single component magnetic vector potential was used in [24] and an asymptotic expansion of the solution was carried out to determine the periodic micro-shape function.

Several homogenization methods for eddy current problems have also been developed up to now. To model the eddy currents due to the main magnetic flux the well known $1D$ approach was selected for instance in [25] and [26], respectively, where the main magnetic flux density is assumed to be constant across the thickness of a laminate and thus the current density varies linearly. A comprehensive analysis of eddy current losses occurring in one laminate can be found in [25]. [26] shows how to incorporate all kinds of iron losses [3, 4] into a $2D$ FE model and its importance by means of the wave form of the total current feeding an Epstein frame. The previous $1D$ eddy current modeling was generalized to $3D$ in [27]. The complex representation of the current density and the magnetic field strength of the $1D$ analytic solution is used to consider the main flux in a laminate for arbitrary frequencies and linear materials. The effect of the stray field is considered by an anisotropic conductivity. Currents perpendicular to the plane of the laminates along the edges, so-called edge effects, are neglected. No air gaps are considered. A symmetric solution within one laminate is enforced. The last restriction has been eliminated by the approach in [28] and an anisotropic permeability has been introduced too. A time domain homogenization technique for laminated media using $3D$-FEM is presented in [29]. The magnetic flux density and the magnetic field strength of the main flux are expanded in different even polynomial basis functions. This means an extension of the methods assuming a constant main magnetic flux density in the $1D$ model. The solution is enforced to be symmetric within the thickness of one laminate. The constitutive law (material relation) is imposed in a weak sense. The homogenization technique takes account of an air gap between the laminates and of the stray field in terms of anisotropic material properies. The method disregards net currents and neglects edge effects. The magnetic flux density parallel to the lamination is expanded into orthogonal even polynomials, so-called skin effect sub-basis functions, in [30]. Coefficients appearing in the integrals of the weak form due to the skin effect sub-basis functions valid for a single laminate are homogenized by averaging them across the lamination. Homogenized coefficients are approximated by finite elements of a coarse mesh. The dual $\mathbf{h}$-conforming formulation to [30], where the so-called $\mathbf{b}$-conforming formulation was presented, is shown in [16]. The homogenization technique presented in [29] has been extended to nonlinear materials in [31]. The arising nonlinear algebraic equation system is solved by the method of Newton-Raphson.

The nonlinear problem is solved in two nested loops called main- and sub-analysis in [32]. The $1D$ model with nonlinear material properties and Dirichlet boundary conditions for the magnetic vector potential prescribing the total main magnetic flux from the main-analysis within one laminate is solved in the sub-analysis. An anisotropic electric conductivity and a reluctivity according to the solution of the sub-analysis are considered in the main-analysis to solve the $3D$ problem. A $3D$ sub-analysis with linear material properties is carried out in [33] for a problem with a rotational magnetic flux.

Recently, a coupled formulation using the magnetic vector potential and a single current vector potential has been proposed by [34]. The effect of the main magnetic flux is modeled by the $1D$ model assuming a constant magnetic flux density approximated by the magnetic vector potential. The large current loops due to the magnetic stray flux are taken into account by the single component current vector potential. Different FE meshes are used for the potentials and the associated different boundary layers. Control volume, i.e., additional FEs, are defined to impose a matching magnetic flux density in a weak sense.

An other approach is shown in [35]. It is assumed that the variation of the highly oscillating electromagnetic field due to the lamination is piecewise linear. The $3D$ finite element model includes also edge effects.

A $2D$ FE method considering the main magnetic flux with a $1D$ diffusion equation across the lamination and using a multiharmonic ansatz of the magnetic vector potential including hysteresis is shown in [36] for the frequency domain. The arising nonlinear algebraic equation system is solved by the fixed-point iteration. The coupling between the $1D$-model and $2D$- FE model has been solved by a nested scheme first, then the $1D$ diffusion model has been introduced directly into the $2D$ FE model in [37].

A $2D$ FE model using the $1D$-model for eddy currents and a complex reluctivity to consider hysteresis in the frequency domain for a rotating machine is presented in [38]. Various frequencies occur because the machine is fed by a pulse width modulation (PWM) based frequency converter. The complex reluctivity represents the hysteresis and includes also the eddy current effects. Losses are interdependent, see [39]. Penetration depth varies with the frequency and the shape of the hysteresis loops vary with the penetration depth. All kind of losses in a FE analysis of electrical machines are considered in [40]. The paper [41] studies amongst others the interdependency of static magnetization properties and the static magnetostriction, dynamic hysteretic dependency of the magnetostriction on the supply flux density and the increase of iron losses due to applied mechanical stress.

A pragmatic two-step homogenisation technique for ferromagnetic laminated cores is presented in [42]. First, a 1D hysteresis model is determined for a specific material and then homogenized by averaging across the laminate. An algebraic approximation is carried out to use the model efficiently in 2D FE-formulations in the second step. To confront very specific literature with the present project it is cited in the subsequent sections.

## III Multiscale Finite Element Method MSFEM

The problem we are confronted with exhibits two different scales. A large scale characterized by large scale dimensions, for instance the length $L$, the height $H$, the width $W$ and the dimensions of the windows, $E$ and $F$, of a transformer core shown in Fig. 1. At the small scale or microscale the dimensions are the thickness $d$ of the laminates, the width $d_0$ of the air gap in between the laminates (see Fig. 2) and the penetration depth $\delta$. The ratio between the scales is extremely large, about $10^5$ up to $10^6$.

Mapped polynomial shape functions are used by the FEM to approximate the unknown solution. The standard FEM performs well as long as the solution or the coefficients in the equations are smooth. However, to obtain an accurate approximation also in case of equations with rough coefficients, for instance materials with a micro-structure (laminated iron core), problems with singularities or with boundary layers, extremely fine meshes are required. This is the reason for the prohibitively large equation systems which require exorbitant amounts of computer resources to obtain an accurate solution.

To avoid large equation systems the generalized finite element method (GFEM) as a general framework for equations with rough coefficients or for problems with singularities seems to be a very promising option [43, 44]. The standard polynomial basis is augmented by special functions including a priori information into the ansatz space

$$u_h(x) = \sum_{i=1}^{n}\sum_{j=1}^{m} u_{ij}\varphi_i(x)\phi_j(x) = \sum_{i=1}^{n}\sum_{j=1}^{m} u_{ij}\psi_{ij}(x), \qquad (1)$$

where $n$ is the number of standard polynomials $\varphi_i$, $m$ is the number of special functions $\phi_j$, see Fig. 2, and $u_{ij}$ are the coefficients of the approximated solution $u_h$. The special functions, which are custom tailored ansatz functions, may stem from an analytic solution or, for example, from a FE solution of a basic problem, i. e. these functions are known. The local basis of the special functions approximates well the solution locally. Multiplication of standard polynomials $\varphi_i$ with special functions $\phi_j$ yields the new basis functions $\psi_{ij}$.

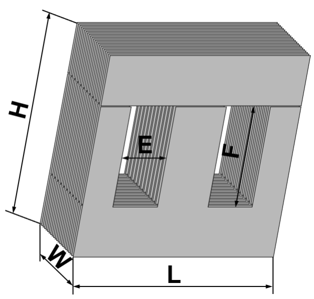


Figure 1: Large scale dimensions.

## IV Scientific activities

The performance of GFEM depends strongly on the subspaces. Special subspaces are designed which incorporate the microstructure of the solution. The number of the degrees of freedom should be independent of the size of the microstructure to keep the required computer resources as small as possible. The aim is to create highly accurate numerical solutions consuming minimal computer resources.

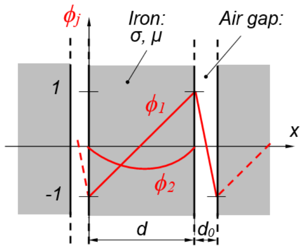


Figure 2: Small scale dimensions and micro-shape functions $\phi_i$.

- Optimal design of special functions for the local basis taking into account the microstructure or of singularities.

- The numerical integration is a very important issue in the implementation of the homogenization method as observed, for instance, in [9]. Fast adapted numerical quadrature methods are developed which ensure that the errors in the integration do not affect the accuracy of the approximation.

- Implementations are made which facilitate essential boundary conditions (periodic or symmetric boundary conditions).

Backward Euler method is used for the time discretization of the ordinary differential equations and Newton−Raphson method to solve the arising nonlinear algebraic system of equations. The magnetic hysteresis is considered by an appropriate hysteresis model [45, 46, 47, 48].
The new MSFEM approaches are and will be incorporated into the open source hp-FEM code Netgen/NGSolve [49]. Since the solutions of potential functions $\boldsymbol{A}_0$, $A_1$, $A_2$, $w_1$ etc. (the meaning of these quantities can be seen in section V $Some\ Results$ (see (5)) are very smooth, p-refinement fits particularly well to this kind of problems.
The activities of the project are presented in the following subsections.

## A Harmonic Balance Method

Most of the sources of eddy current problems alternate harmonically in time, and only the solution of the steady state has to be calculated. However, in case of nonlinear materials the solution is not harmonic any more, but still periodic. Thus, the solution can be represented as a Fourier series. This can be exploited advantageously by the so-called multi-harmonic ansatz or harmonic balance method, i.e., a truncated Fourier series expansion at a finite number [50, 51, 52, 53]. Only a few harmonics are required for a sufficiently accurate approximation. That's why the harmonic balance method is superior to the time stepping method particularly in case of a transient that takes a long time.
Applying Fourier block diagonalization to the equation system obtained by time stepping of one period yields $N$ decoupled equations of the system, where $N$ is the number of time steps of one period and the number of unknowns in each linear system is just the number of degrees of freedom at a time instant. Thus, the steady-state of the solution can be calculated at the cost of time stepping through one period only with the additional minor expense of complex arithmetics. Exploiting the fixed-point method for nonlinear materials a nonlinear term appears on the right-hand side only and Fourier block diagonalization can be applied again [54]. Thus, the periodic nonlinear problem can be solved very efficiently. Harmonic balance and the fixed-point method yield also in the nonlinear case decoupled equation systems for the individual harmonics [55]. An optimal choice of the fixed-point permeability $\mu_{FB}$ has been presented in [56] to ensure a minimal contraction number for the fixed-point method.
The aim in this project part is to combine the multiscale approach described in section V $Some\ Results$ (see (5)) with the harmonic balance method to reduce the computational costs drastically compared with the common time stepping method.

## B Boundary Layer

Numerical investigations with FEM and homogenization (V $Some\ Results$) revealed also a pronounced boundary layer quite similar to that which occurs in corresponding brute force models of laminated iron cores with anisotropic material properties. The boundary layer covers several thicknesses of laminates at the periphery of the core in the model. An accurate approximation of the boundary layer is essential for the exact computation of the iron losses in this layer. The classical FEM performs poorly unless $h$ is sufficiently small, i.e., a very strong refined FE mesh has to be used, which is prohibitively expensive.
To capture the special behavior of the boundary layer tailored functions will be included in the FE ansatz space. The goal is to achieve the same accuracy as with standard FEM and usual polynomials but with significantly fewer degrees of freedom. The new approach for the boundary layer will be integrated into MSFEM. An optimal MSFEM approach taking into account the boundary layer will be realized in the following steps. Appropriate special functions will be selected to augment the standard FE basis. Approaches will be successively extended up to 3D and nonlinear material.

## C Geometric problems

Corner and T-joint regions of transformer cores are major sources for losses and noise. A step-lap-technique is applied to reduce losses and noise. Several parameters (overall size, number of laminates, overlap length, lamination factor, sheet width, number of overlap steps, air-gap length, etc. [5]) have an impact on the losses and noise. Therefore, the optimal design of the joint regions is very important and requires a detailed simulation.
An early study of step lap joints with FEM were carried out by [57]. To obtain an optimal configuration of the core joints a homogenization technique for the static magnetic field has been developed in [24]. An equivalent reluctivity has been derived in [58] and [59] assuming that the energy stored in the static magnetic field problem, which tends to a minimum, is the same in the homogenized and in the original problem. The air gap length of joints for different magnetic flux densities have been considered in [60] to calculate an equivalent magnetization curve (and Preisach model) and conductivity for efficient two-dimensional simulations of overlap joints in transformer cores. An accurate and efficient simulation of the static magnetic field of step lap joints using anisotropic higher order FEM, where the laminates are modeled by FEs, can be found in [6]. Recently, a method which starts with a 2D static magnetic calculation and considers successively lamination, eddy currents, joint configuration and nonlinearity in some steps was proposed by [61].
Step lap joints interrupt the periodic structure of lamination. The scale of joints varies extremely, from almost the overall size of the transformer core over the overlap length of laminates to the thickness of the laminates. An efficient and accurate FE approach is very challenging. This explains why there is no sat-

isfactory FE approach available which comprises all relevant aspects, a three-dimensional FE model with a homogenization approach of the laminates considering eddy currents and nonlinear material, and which provides a solution in one step, up to now.

In the frame of the present project point we turn our attention to the calculation of the quasi-static magnetic field and neglect acoustics, mechanics etc. The aim is to design an accurate and efficient FE approach which enables a fully 3D simulation, considers nonlinear and anisotropic material properties, incorporates corner and T-joint regions and works for a possible wide range of design parameters. The idea for an appropriate FE approach of the step lap joints is to observe the joint regions also as a periodic structure at a intermediate scale between the small scale (laminates) and the large scale (macroscopic dimensions of the core) to benefit from a homogenization approach. In case no homogenization approach works satisfactorily, because the periodicity of the structure can not be exploited advantageously, substructuring will be the second choice.

## D Movement and Maxwell

The stator and even the rotor of large electric machines are laminated. Three-dimensional simulations, where the laminates are resolved in FE models are rather seldom and clearly far away from a routine task although symmetries are exploited. Simulations are typically carried out for single laminates of electric machines which are assumed to be infinitely long or of a couple of slices of electric machines which vary significantly along the rotation axis due to skewing.

Early contributions to problems with movement and the coupling of independently generated finite element meshes by Lagrange multiplier or by overlapping elements can be found in [62] and [63], respectively. We presented a domain decomposition technique based on Nitsche's method to discretize transmission conditions on non-matching meshes [64]. This technique is a very interesting alternative to approaches as moving band, locked-step or interpolated sliding surface and a semi-analytical air-gap macro technique as a special application of finite and boundary element coupling [65, 66, 67]. This project part will greatly benefit from the domain decomposition technique based on Nitsche's method among other things.

Skewing is applied to induction machines to reduce undesirable effects (torque ripple, acoustic noise and harmonic currents). Multislice technique is common practice to consider skewing. To avoid problems in the interpolation in the axial direction and to take into account end effects, a full FE model is clearly preferred. The periodic nature of the lamination is interrupted by ventilation ducts and disturbed by skewing. Ventilation ducts and skewing are different kind of geometric difficulties from how to consider them in the multiscale homogenization approach point of view. These difficulties bring additional challenges into the topic MSFEM.

The goals in this project part are the integration of the moving term into the MSFEM, the implementation of essential periodic boundary conditions for MSFEM taking into account the domain decomposition technique based on Nitsche's method to be able to exploit possible symmetries [64].

## E Necessity of Boundary Layer Correctors

To visualize the influence of interface effects in MSFEM, a simplified scalar elliptic problem is solved using a first order MSA (for details see [68]). Figure 3 shows the total error of the multiscale solution compared to a reference solution, measured in the energy norm for different mesh sizes $h$ and laminate widths $d$. For every $h$ a decrease in the error with an order of $\mathcal{O}(\sqrt{d})$

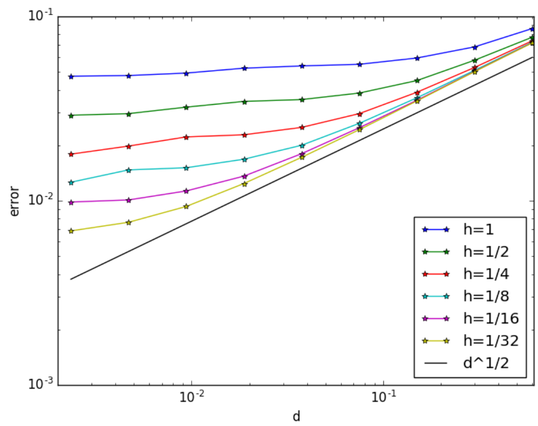


Figure 3: Energy error of the multiscale solution.

can be observed, which eventually flattens out when the modeling error of the multiscale ansatz is dominated by the FEM discretization error. Considering the local errors shown in Fig. 4 it becomes clear that a significant proportion of the error arises in a close proximity of the interface between $\Omega_0$ and $\Omega_m$ (compare with Fig. 11). Figure 5 shows that these interface errors are

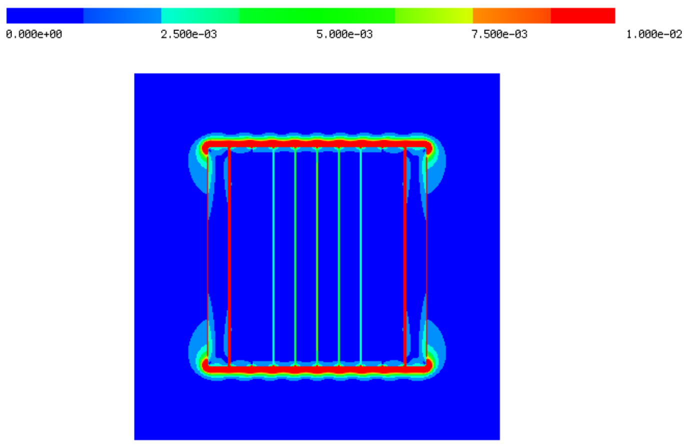


Figure 4: Energy error distribution.

the reason for the lower order of convergence. Calculating the total error only for areas with a fixed minimum distance from the interface yields an energy error of $\mathcal{O}(d)$. In order to preserve the rate of convergence for the whole domain, specific boundary layer correctors will have to be developed to neutralize the error peaks along the interface of the layered material.

## F Error Estimation

While the various used ansatz techniques show good experimental results with a stable rate of convergence, finding reliable and efficient error estimates is a complex task and still the subject of ongoing studies. A promising method which has worked well in several settings is a modification of error estimation by reconstruction of the discrete flux [69, 70].

An energy equality similar to the Prager-Synge equation is the starting point of the equilibration method. When considering the Poisson problem, the unmodified equation of Prager and Synge

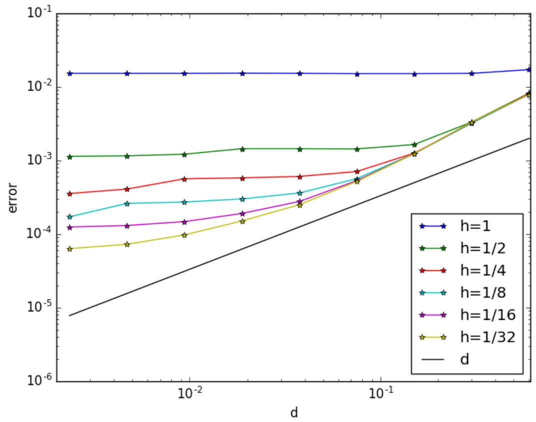


Figure 5: Energy error away from the interface.

gives the following relation for the exact solution $u$ and the discrete solution $v$:

$$\|\lambda^{1/2}(\nabla u - \nabla v)\| \leq \|\lambda^{1/2}(\nabla v - \lambda^{-1}\sigma)\| \tag{2}$$

This inequality holds for every flux $\sigma \in H(\mathrm{div})$ satisfying $\mathrm{div}\sigma + f = 0$ with $\lambda$ being a coefficient in the elliptic problem. Similar inequalities can be proven for problems with a mass term or equations of Maxwell type using $H(\mathrm{curl})$ FEs.

The equilibration technique shifts the problem of finding a good upper bound for the energy error to the construction of a $\sigma \in H(\mathrm{div})$ which needs to be close to the discrete flux $\lambda\nabla v$ while at the same time being cheap to calculate.

Applying these ideas to the multiscale setting, the aim of cheap construction enforces $\sigma$ to admit a structure similar to the multiscale solution, i.e. being composed of functions defined on the coarse grid multiplied by predefined oscillating functions. Good results have been achieved by first multiplying $\nabla u_0$ with appropriate means of $\lambda$, found by asymptotic calculations as done in [71], followed by classical reconstruction techniques in order to make it a $H(\mathrm{div})$ function satisfying the conditions of the theorem of Prager and Synge. This smooth function is then enriched by additional correctors which incorporate the oscillating nature of the discrete flux while being of curl type in order to preserve the reconstructed divergence.

The next problem is the actual calculation of the error estimation according to (2). This requires the integration of products of highly oscillating functions with functions defined on the coarse grid. While this problem has been studied extensively for example in [72], classical techniques require the oscillating functions to be a smooth, while in the multiscale setting the oscillating ansatz functions are in general only continuous and might even lose continuity by multiplication with the equation-specific coefficient functions, which calls for modified methods.

A possible solution for the 1D case is to choose the asymptotic expansion (3) where $\varphi$ is an arbitrary, not necessarily continuous periodic function and $f$ is assumed to be smooth.

$$\int_a^b \varphi(x) f(x)\, dx \approx \sum_{n=0}^{N} \int_a^b \overline{\varphi_n} f^{(n)}(x)\, dx. \tag{3}$$

The constants $\overline{\varphi_n}$ in (3) have to be calculated a priori in order to ensure an equality for $f$ chosen as a monomial of degree up to $N$. Finding such $\overline{\varphi_n}$ is straightforward in the MSFEM setting with $\varphi$ taken as a piecewise polynomial in each period.

This technique can be easily applied to the two dimensional case, since the oscillating function only depends on one coordinate:

$$\int_\Omega \varphi(x) f(x,y)\, d\Omega = \int_a^b \varphi(x) \underbrace{\int_{c(x)}^{d(x)} f(x,y)\, dy}_{=:\tilde{f}(x)}\, dx \tag{4}$$

In (4) $c(x)$ and $d(x)$ are a parametrization of the boundaries of the integration domain. Figure 6 shows the result of a numerical example compared to known exact integral values with $\varphi$ being a discontinuous, piecewise quadratic polynomial over one period. For each additional term in the expansion (3) an additional order of convergence can be observed. The problem of an increased number of function evaluations in order to calculate the needed derivatives can be mitigated to an extend by making use of the properties of the function $\tilde{f}$ and the shape of the integration domain, which is a triangle in the classical FEM setting. Still there are further improvements possible and the search for efficient integration remains another important aspect of efficient error estimation.

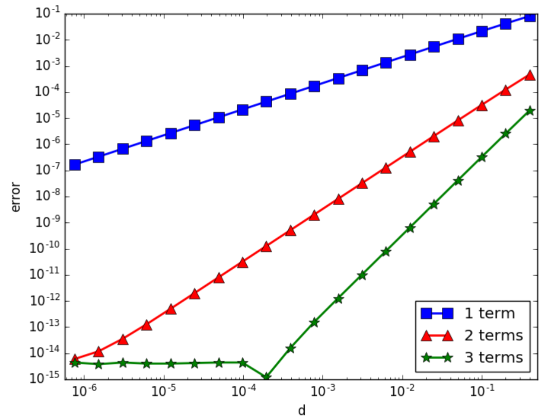


Figure 6: Absolute error for decreasing laminate width $d$.

## G Benchmark

Simulation results of the new methods will be evaluated by a relevant benchmark (BM). To this end a transformer is designed, manufactured and measurements will be carried out at the input terminals and of the magnetic field distribution (see, for instance [7]). It is intended to publish the setup and the measurement data. The preliminary design, numerical data and planned measurements are presented below.

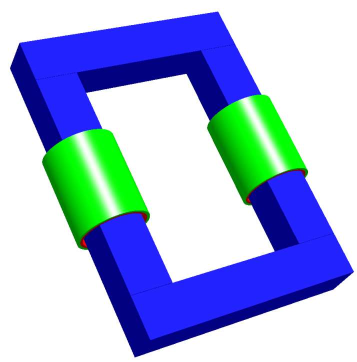

Figure 7: Benchmark with core (blue) and coils (green).

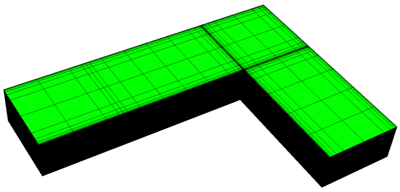

Figure 8: Hexahedral mesh of the laminated core, one eighth of the core, laminates resolved.

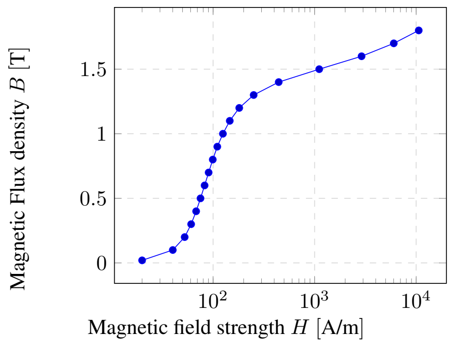


Figure 9: Magnetization curve, non-oriented steel M400-50A.

*Preliminary design*:
The BM with a simple structure is shown in Fig. 7. It consists of a laminated iron core and two cylindrical coils. The following features are or will be considered in the design:

- A wide range of saturation of the iron core.
- A design with and without a large magnetic stray flux.
- Grain and non-grain oriented laminates will be used for the setup.
- The BM shall also be furnished with step-lap-joints.

Initially, the BM is made of non-grain oriented iron laminates (see Fig. 9) and without step-lap joints to keep the BM as simple as possible. A small size of the BM is selected that can easily be operated in a laboratory on one hand and is large enough to make sense for homogenization or multiscale methods. The BM exhibts three planes of symmetry, which enables counterchecks of measurements and a simulation of only one eighth of the problem, see Fig. 8. Four small air gaps of $d = 0.5\text{mm}$ between limbs and yokes are provided to ensure a well defined distance between them and to allow for measurements of the magnetic fluxes by means of thin wire coils (loops). The yokes can be rotated so that the orientation of the lamination of yokes and limbs coincides or includes an angle of 90 degrees to study the entering of the magnetic flux from the limbs via the air gaps to the yokes. Positive and negative mutual magnetic coupling of the coils can be realized simply by changing the interconnection of the input terminals. Later, the BM will be extended by grain oriented iron laminates and step-lap joints.

*Numerical data*:
Hexahedral finite elements are well suited to model single laminates. To avoid the necessity to model the cylindrical coils also with hexahedral elements the Biot-Savart field of the coils has been exploited. The hand made mesh with hexahedral edge finite elements of $2^{nd}$ order leads to about $15$ millions of unknowns for the full model and to about $1.1$ millions of unknowns for a model with an anisotropic conductivity [21]. The saturation of the iron core is very sensitive with respect to the input voltage of the coils. Series resistors are used to control the saturation. Therefore, the finite element model is coupled with a network. Backward Euler method is used for the time stepping scheme and Newton's method to solve the nonlinear problem. Simulation results of the input values are shown in Fig. 10

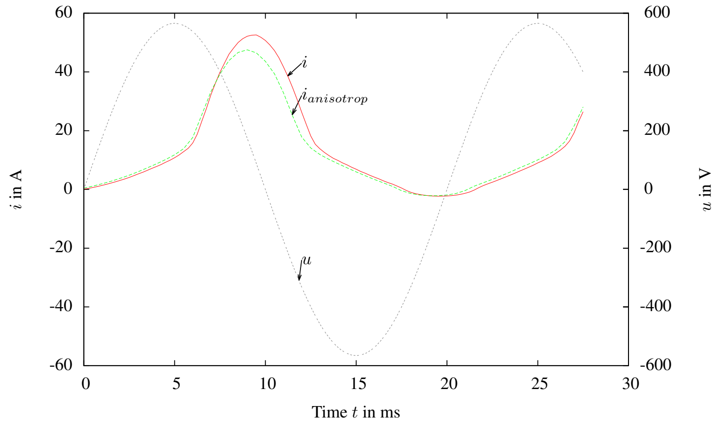


Figure 10: Input values.

*Measurements*:
Measurements to be considered are electrical quantities at the input terminals of the coils as well as magnetic fluxes. A Hall-sensor and wire loops will be used to measure the magnetic stray field close to the cylindrical coils and the core and the main magnetic flux in single laminates, respectively. Electrical quantities are the instantaneous values of the voltage u(t), current i(t) and of the power loss p(t).

## V Some Results

An eddy current problem as shown in Fig. 11 consisting of a laminated medium $\Omega_m$ surrounded by air $\Omega_0$, i.e., $\Omega = \Omega_m \cup \Omega_0$ with the magnetic vector potential $\boldsymbol{A}$ is considered.

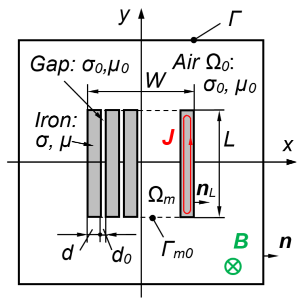


Figure 11: Sketch of the eddy current problem.

### A Higher Order Multiscale Approach with A in 2D

To get also an accurate solution for problems, where the penedration depth $\delta$ is essentially smaller than $d$, the approach in [9] has been extended by higher order terms for the laminar part as well as for the edge effect of eddy currents leading to the multiscale approach (MSA), see also [73]:

$$\tilde{\mathbf{A}} = \mathbf{A_0} + \phi_1 \begin{pmatrix} 0 \\ A_1 \end{pmatrix} + \phi_3 \begin{pmatrix} 0 \\ A_3 \end{pmatrix} + \phi_5 \begin{pmatrix} 0 \\ A_5 \end{pmatrix} + \nabla(\phi_1 w_1) + \nabla(\phi_3 w_3) + \nabla(\phi_5 w_5) \quad (5)$$

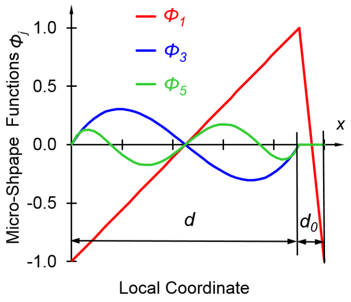


Figure 12: Odd micro-shape functions.

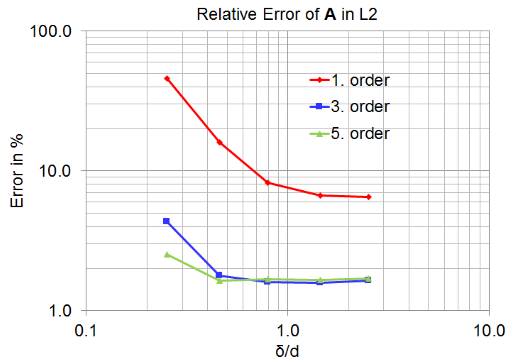


Figure 13: Relative error for different order of MSAs in the $L_2$-Norm.

Since the magnetic flux density **B** is an even function across the laminates, only odd higher order terms for the MSA with **A** are considered. The higher order multiscale functions (MSFs) $\phi_3$ and $\phi_5$, see Fig. 12, are living in $\Omega_c$, i.e. on iron subintervalls. The MSFs $\phi_3$ and $\phi_5$ are bubble functions and, thus, they do not disturb the required continuity of **A**. The choice of the FE subspaces $\mathbf{A}_{0h} \in \mathcal{U}_h \subset H(\text{curl}, \Omega)$, $A_{1h}$, $A_{3h}$ and $A_{5h} \in \mathcal{V}_h \subset L_2(\Omega_m)$, $w_{1h}$, $w_{3h}, w_{5h} \in \mathcal{W}_h \subset H^1(\Omega_m)$ and $\phi_1$, $\phi_3$ and $\phi_5 \in H_{per}(\Omega_m)$ is quite natural. The polynomial order of the basis has been chosen according to the deRham complex [49]. Natural boundary conditions hold for $A_i$ and $w_i$ at the interface between $\Omega_m$ and $\Omega_o$. The solutions of $\mathbf{A_{0h}}$, $A_{1h}$, $A_{3h}$, $A_{5h}$, $w_{1h}$, $w_{3h}$ and $w_{5h}$ are very smooth, a rather coarse FE-mesh suffices to approximate them accurately. The reduction of the required computer resources can be seen in Tab. I below and Tab. II in the following section.
A small 2D problem with 10 laminates has been studied. The reference solution (RS) is obtained by a FE-model where the single laminates are resolved by FEs. Fig. 13 shows results for different dimensions of the local space. Adding of $3^{rd}$ order terms improves the accuracy essentially. The $5^{th}$ order approach performs clearly better than the $3^{rd}$ order one for small $\delta$. The computational costs are compared in Table I. A fairly good improvment has been achieved for this small problem.

Table I: No. of degrees of freedom.

| | Total No. | $H(\text{curl}, \Omega)$ | $L_2(\Omega_m)$ | $H^1(\Omega_m)$ |
|---|---|---|---|---|
| RS | 24,745 $^{a}$) | 24,745 | - | - |
| MSFEM | 1,675 $^{b}$) | 676 | 152 | 181 |

$^{a}$) For $6^{th}$ order $H(\text{curl})$ - elements for the smallest $\delta$.
$^{b}$) For the $5^{th}$ order MSA.

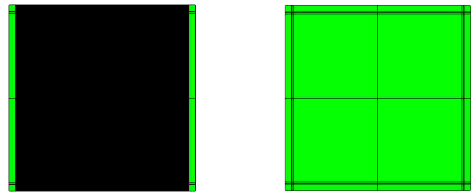

Figure 14: FE model for RS (left) and for MSFEM (right).

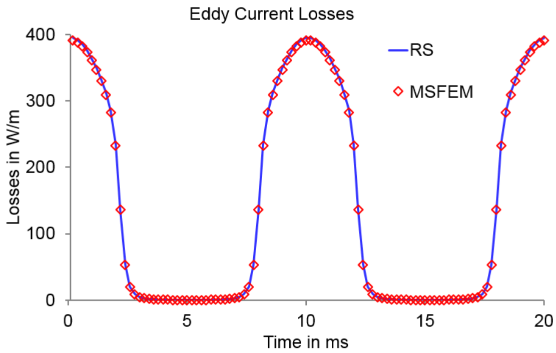


Figure 15: Eddy current losses.

## B Large Nonlinear Problem in 2D

To show the capacity of MSFEM to cope with large and nonlinear problems compared with standard FEM the problem with FE models in Fig. 14 has been studied. The nonlinear problem with a magnetization curve like in Fig. 9 consists of 1000 laminates. Here, the $1^{st}$ order MSA of (5) has been used. Backward Euler method was used for the time discretization and Newton's method to solve the nonlinear problem. The agreement of the losses is very satisfactory as can be seen in Fig. 15. The reduction of computational costs of MSFEM compared with RS is impressive as shown in Table II. The computational requirements of MSFEM for this large problem are almost the same as those for the small problem in the previous subsection A with only 10 laminates.

Table II: No. of degrees of freedom.

| | Total No. | $H(\text{curl}, \Omega)$ | $L_2(\Omega_m)$ | $H^1(\Omega_m)$ |
|---|---|---|---|---|
| RS | 164, 430 | 164, 430 | - | - |
| MSFEM | 1, 289 | 840 | 208 | 241 |

## C Higher Order MSFEM Approach with **A** for 3D

The feasible three-dimensional higher order multiscale approach

$$\tilde{\mathbf{A}} = \mathbf{A}_0 \;+\; \phi_1 \begin{pmatrix} 0 \\ A_{12} \\ A_{13} \end{pmatrix} + \phi_3 \begin{pmatrix} 0 \\ A_{32} \\ A_{33} \end{pmatrix} \;+\; w_1 \begin{pmatrix} \phi_{1x} \\ 0 \\ 0 \end{pmatrix} + w_3 \begin{pmatrix} \phi_{3x} \\ 0 \\ 0 \end{pmatrix} \quad (6)$$

with respect to Cartesian coordinates, where the normal vector of the lamination points in x-direction has been assumed.
A stack of 100 iron laminates and with the dimensions $(25 \times 25 \times 75)$ in mm is immersed in a homogeneous time harmonic magnetic field and serves as a numerical example. A thickness of both, iron layer and air gap, of $d+d_0 = 0.25\text{mm}$, an unfavorable fill factor of 0.9, a conductivity of $\sigma = 2 \cdot 10^6 \text{S/m}$ and a relative

permeability of $\mu_r = 50,000$ were selected. The error in the eddy current losses compared with RS are shown in Fig. 16, AM stands for the model with an ansiotropic conductivity.

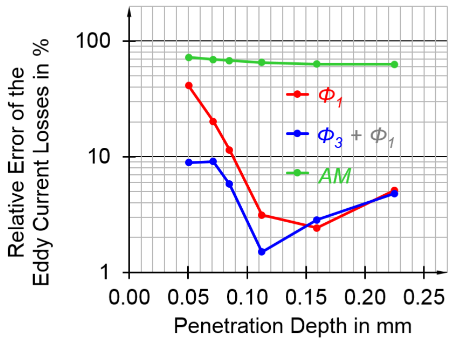


Figure 16: Comparison of the eddy current losses in the frequency domain.

A comparison of the required computer resources is summarized in Table III. The number of unknowns required by the higher order MSFEM is only about 10% that of the reference solution.

Table III: No. of degrees of freedom in thousands of unknowns.

| | Total No. | $H(\text{curl}, \Omega)$ | $L_2(\Omega_m)$ | $H^1(\Omega_m)$ |
|---|---|---|---|---|
| RS | 4 443 $^{a}$) | 4 443 | - | - |
| MSFEM1$^{b}$) | 389 | 323 | 27 $^{d}$) | 12 $^{d}$) |
| MSFEM3$^{c}$) | 455 | 323 | 27 $^{d}$) | 12 $^{d}$) |

$^{a}$) $2^{nd}$ order finite elements, $^{b}$) $1^{st}$ order MSA, $^{c}$) $3^{rd}$ order MSA, $^{d}$) holds for one quantity in approach (6).

### D Numerical Integration for MSFEM

Fast and accurate numerical integration to assemble the FE-matrices is an essential component for a powerful MSFEM. To this end the periodic and possible nonlinear nature of the laminated medium has to be exploited (see also section F).
*Averaging*: Averaged coefficients across a laminate are used instead of the exact ones in the assembly of FEs in case of linear material parameters of laminated cores [10].
*Interpolation*: First, appropriate polynomial interpolations of the coefficients (for example the conductivity) are calculated with a very high integration order. Then the usual assembly is carried out with these interpolations. This technique avoids the computationally expensive evalution of all basis functions and partly of their derivatives for the high integration order [9].
*Nonlinear material*: Above techniques are not possible for nonlinear materials. Although the solution varies within a laminate sometimes strongly the variation of the solution from one laminate to the neighboring one is moderate. A nested integration is carried out where the Gaussian points of the standard integration rule represent the outer loop. For each of these points a 1D integration is made across the corresponding iron and air layer in the inner loop [9].

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