Panel Sessions at COMPUMAG 'Rio

In the previous issue of the Newsletter (Volume 5, No 2, 1 July 1998) we presented a report from one of the panel sessions held at Compumag Rio on "Mesh Generation". We now publish a report from the other panel session on "Edge Elements", which was organised and chaired by Professor Oszkar Biro from IGTE, Technical University of Graz, Austria, who also collected and edited the contributions of the panelists. As on the previous occasion the report does not summarize the session but is a collection of individual opinions expressed by the panelists on specific issues.

Panel Session "Edge Elements"

COMPUMAG'97, Rio de Janeiro, Brazil

Tangential Vector Finite Elements

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Tangential vector finite elements are a consequence of geometry: Consider the "roof top" surface formed by approximating a two dimensional potential function *f* by first-order finite elements. Since the finite element approximation is continuous along the element edges, the derivative tangent to each element edge will be continuous. However, the normal derivative along each edge will be discontinuous since the finite element approximation has a crease there. Consequently, approximating *f* by first-order finite elements, implies that the gradient of *f* must be approximated by functions having continuous tangential components but discontinuous normal components. Further, the resulting two component vectors are constants inside each finite element. This simple observation lies at the heart of tangential vector finite elements.

Now consider the curl operator. The curl operator has a domain, a range, and a nullspace. Restricting the discussion to 2D for simplicity, we notice that the range of the curl operator is a one component vector (a scalar times a unit vector). In a differential equation, the range of the curl must be set equal to something. Thus, in the lowest order case, we will approximate the scalar in this range by zeroth-order finite elements (constants). Note that the dimension of this approximation per element is one. The question is: What functions must be in the domain of the curl operator so that its range is zeroth-order finite elements?

The domain space of the curl consists of two parts: (a) a gradient space as described above, and (b) a non-gradient space which generates constants. Thus, in addition to having a function in the domain to generate the constants in the range, we need to add functions to complete the nullspace. The nullspace of the curl operator is the gradient space. It is shown in [1] that the dimension of the gradient space of a finite element approximation is one less than that of the original space. First-order elements have a dimension of three, so the dimension of its gradient space is two, as observed above. Since the dimension of the gradient space is two and the dimension of the non-gradient space is one, it follows that the dimension of the domain space is three. Further, these functions must possess tangential continuity as described above. The above observation leads to edge elements: define a vector finite element by its three constant tangential components along each side of the element. Two of these functions generate the nullspace described above, the third contributes the nontrivial range.

This analysis is extended to higher order and to three dimensions in [1,2]. It is also shown in these references that a proper treatment of the domain, range and nullspace of the curl operator is required to ensure correct results.

If edge elements are the answer, what is the question?

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There is now consensus on the relevance of edge elements, but why their analytical form should be the well-known one ($\lambda^i\nabla\lambda^j - \lambda^j\nabla\lambda^i$, if one denotes by λ^i the barycentric function of node *i*), *that* is puzzling. Why not $\lambda^i\nabla\lambda^j$, for instance, with two shape functions per edge (and hence, two DOF's)? Such an edge element has tangential continuity, too, and has the apparent advantage of "first-order completeness" (the property that any globally linear vector field lies in the generated finite space). So why the first form (with 6 DoFs per tetrahedron) rather than the second one (with 12)? Why are edge elements what they are?

Two kinds of answers may be given. First: take seriously the idea that edge elements are to (say) the magnetic field, what scalar nodal ones are to the magnetic potential. A point *x* can be represented as a weighted sum of mesh nodes, $x = \sum_i \lambda^i x_i$, so we are justified in interpolating the potential from its values at nodes: we just set $\varphi(x) = \sum_i \lambda^i \varphi_i$, where φ_i is the DoF at node *i*, and this works. So it would work the same if we just could represent a line segment *xy* as a linear sum of edges, like this (imagine *xy* and the edges *e* as oriented vectors): $xy = \sum_e \mu_e(x,y)e$. Then we would say that the magnetomotive force along *xy* is $\sum_e \mu_e(x,y)H_e$, where the H_e s are the mmf's along the edges. Since for a small segment *xy* the mmf is approximately $H(x)\cdot xy$, this provides an approximation of the field *H*. Note how natural the idea is: edge elements — assuming we don't know them yet — should yield all possible mmf's from edge-mmf's, the same way as nodal elements yield potentials at all points from node potentials. A calculation shows that, if edge *e* goes from *i* to *j*, then $\mu_e(x,y) = (\lambda^i\nabla\lambda^j - \lambda^j\nabla\lambda^i)(x)\cdot(y-x)$.(Details are in **ICS Newsletter, 1**, 3 (1995), pp. 3-6.) So when this working programme is fulfilled, what pops up is the Whitney edge element.

The second line of argument would take much more space to develop. In a nutshell: We are not looking for the edge element in isolation. We want finite elements for *all* the potentials and fields that appear in Maxwell's equations, some of which are edge-based indeed, but some of which are node-based, face-based, etc., as well. So it's a *consistent* representional *system* for all of these at once that should be looked for, i.e., a consistent family of "cell" elements, for cells of all dimensions. "Consistent" here is understood as implying two things: first, homogeneity in polynomial degree; second, preservation at the discrete level of such properties as ker(rot) = range(grad), which are so essential to avoid spurious modes (besides other reasons). A technique, based on the so-called "Poincaré gauge", was sketched during the panel session, that allows a systematic construction of consistent families of cell elements. And not surprisingly, when polynomial degree is set to one, and for a tetrahedral mesh, it's the family of Whitney elements that is generated by this process

Higher-Order Edge-Based Vector Finite Elements

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Another interesting problem is the extension of edge elements to other types of geometric finite elements or higher orders of approximation. For first order elements, either tetrahedral or hexahedral, the choice of degrees of freedom is relatively direct and simple. For example, the tetrahedral first order edge elements has six degrees of freedom, the line integrals of the unknown field along the tetrahedron's edges. This choice is justified by the fact that fields having tangential continuity across inteface boundaries, like the electric or magnetic field intensity, get a more tangible physical sense through their line integrals. These are, in fact, the measurable and physically perceptible quantities, like potential drops and electromotive or magnetomotive force.

In higher order elements, however, different kinds of degrees of freedom appear. Although they are still related to tangential projections of the unknown field, the kind of projection can be essentially different from a simple tangential component and can refer to an edge, face or the whole element's volume. In general, an edge - based higher order element can have edge-, face- or volume- related degrees of freedom. This diversity serves the purpose of linear independence of degrees of freedom, whereas their number is strictly determined by the same property, which is formally called unisolvence. On the other hand, the shape functions are strictly associated with the particular choice of degrees of freedom, in the sense that different choices of degrees of freedom may lead to different shape function expressions. Edge-based vector shape functions fulfil the property of tangential continuity across the interface boundaries between adjacent elements and a property of decoupling between degrees of freedom and shape functions. The latter simply states that the "value" of a shape function, as computed via the definition of degrees of freedom should be unity for the degree of freedom that is associated with it and zero for the others. It must be mentioned here that different degrees of freedom result in different classes of elements.

Of particular importance is also the property of correct modelling of irrotational fields, which ensures the elimination of spurious modes. This property is considered the trademark of edge-based vector elements and guarantees the correct topological and geometrical representation of fields. The constraints that this property imposes to the shape functions affect the terms of highest order in the shape function expansion. Generally, for an n-th order tangential vector finite element, the field variation is not fully n-th order, but of order n to the direction normal to the field and n-1 to the parallel direction. This is often referred to as the property of mixed order approximation and is directly related to the correct representation of the curl operator and the irrotational fields. An example of a third-order tetrahedron is shown in Figure 1.

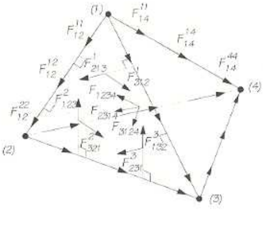
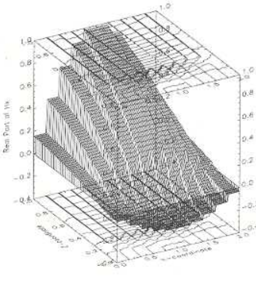


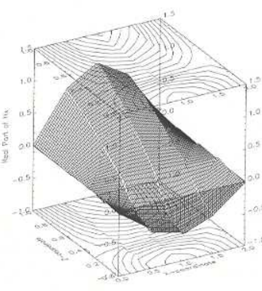
Figure 1.

A third-order tetrahedron with the appropriate degrees of freedom shown only on the first face.

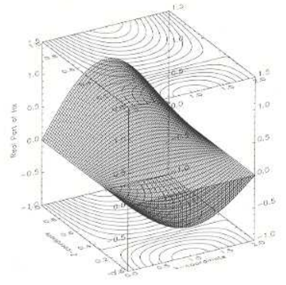
A representative example indicating the efficiency of higher-order edge elements is given below



(a)



(b)



(c)

Figure 2.

3-D surface plot of the real part of Hx magnetic field component with a magnetic permeability equal to 2.0–j2.0. (a) First-order elements 10×5×12×5, (b) second-order elements 4×2×5×2, and (c) third-order elements 4×2×5×2.

Hierarchal Edge Elements

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In general, higher order finite elements give a greater accuracy for a given computational cost, than using a larger number of low order elements. For that reason, higher order elements have been widely used for both scalar and vector field problems.

However, most high order elements are non-hierarchal in nature, i.e. the set of basis functions for the element of order p is not built, hierarchally, from those of lower orders but consists of all new functions. Consequently, if you try to connect an element of order p and an element of order p–1 along a common edge, say, it is not possible (or, at least, not easy) because the basis functions along the common edge do not match up. Effectively, this prevents the mixing of different orders within the same finite element mesh.

On the other hand, the ability to use different orders in different parts of a mesh is very useful. For example, where small elements are needed because of the geometric complexity, low orders can be chosen; where large elements are possible, high orders are selected to support the required field variation. This leads naturally to a kind of adaptive method called p-adaption: from the first (low-order) field solution, the errors in the elements are estimated, the order increased in the worst elements, the problem re-solved, and so on. Even more interesting is the possibility of combined p- and h-adaption, which involves either increasing the order or subdividing the element, depending on which is predicted to have the greater impact. This can lead to very fast convergence rates.

To do p-adaption what is needed is a hierarchal element. Figure 3 shows, on the left, an ordinary Whitney edge element with three degrees of freedom, each associated with one of the edges of the triangle. On the right is a hierarchal element with six degrees of freedom. The six basis functions of this element consist of the three basis functions of the Whitney element, plus three new basis functions (indicated by the heavier arrows). On the common edge, then, the righthand element has contributions from two basis functions, whereas the lefthand element only has a contribution from one basis function. Still, it is easy to impose (tangential) continuity across the common edge. All that has to be done is to set to zero the coefficient of the higher order basis function of the righthand element, and equate the coefficients of the lower order basis functions of the two elements.

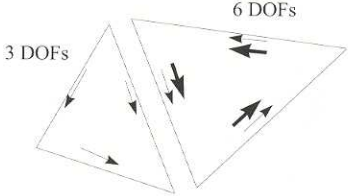

Figure 3.

Two edge elements of different order can share a common edge without violating continuity provided they are hierarchal elements.

In the case of edge elements, as opposed to scalar elements, there is a further advantage to the hierarchal approach. In vector electromagnetics it is frequently necessary, or at least desirable, to split the vector space into a subspace of gradient (irrotational) functions and a subspace of rotational functions (i.e. a subspace that contains no gradients), and to work with a set of basis functions each of which belongs to one subspace or the other. This has application in gauging the magnetic vector potential [3], in a robust edge-element version of the T-Ω method [4], and, recently, in faster solvers for the vector wave equation [5]. With the lowest order (Whitney) edge elements, the split is achieved by defining a spanning tree from the graph of edges of the finite element mesh, removing the degrees of freedom associated with the tree edges and replacing them by the gradient of scalar basis functions associated with the nodes of the mesh (Figure 4) [6]. The rotational basis is then just the set of remaining edge functions. The question naturally arises, how can this be generalized to higher order elements? With non-hierarchal elements, the Whitney edge functions are thrown away and replaced by new basis functions, and a new splitting has to be constructed for the new set. With the hierarchal approach, the splitting that exists at the lowest level is retained; all that has to be done is to ensure that the higher order functions added are also split, and it turns out that this is a purely local matter, i.e. it does not involve the spanning-tree [7].

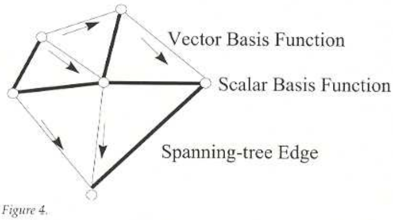

Figure 4.

A spanning tree of the mesh is used to split the basis into a gradient part (gradient of scalar basis functions) and a rotational part (vector functions on cotree edges).

The fall(acy) of edge elements

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Introduction

Over the past decade edge elements have earned an explosive growth in attention in the electromagnetic finite-element community and this rapid development still seems to be continuing unhampered. They gained their first popularity only after the fundamental theoretical paper by Nédélec [8] and the application of these elements, first by Bossavit and Verité [9] and subsequently by many others.

The main reasons for the success of edge elements seem to be the following:

- Edge elements can be used for representing fields with continuous tangential components while leaving the normal component free to jump.
- Edge elements can be, and usually are, designed such that they are free of divergence. Among other reasons, this freedom of divergence has motivated the hope, and even conviction of many, that solutions of field problems obtained by using edge elements will be free of divergence and, consequently, free of spurious solutions.

Nédélec's paper was followed by many other papers proposing ever new types of edge elements. We mention only the types that are relevant in the context of the present paper.

In 1985 Mur and de Hoop [10] introduced the so-called consistently linear edge elements. Contrary to "first-order" mixed edge elements they provide a linear approximation of each component of the field in each Cartesian direction. In [11] Nédélec presented a very general discussion on edge elements of this type.

In the present paper the validity of the various claims that are made regarding edge elements is analysed. A few additional properties of edge elements are also discussed.

Edge elements do allow spurious solutions

A simple and explicit example demonstrating that *edge elements do allow spurious solutions* in driven problems was given by Mur [12] and it is a trivial exercise for the reader to construct a similar example for eigenvalue problems.

About the example of edge elements allowing spurios modes we note the following:

- The example was chosen for the sake of utmost clarity and simplicity. From it it follows that accurate, non-spurious, solutions can only be guaranteed by making the continuity of the normal component of the flux between edge elements a part of the formulation of the finite-element method [13, 14]
- The example is such that *the properties edge elements have by definition* are used for constructing an unambiguous example of their failure in preventing the occurence of spurious solutions.

More complaints about edge elements

We now catalogue a number of additional disadvantages and problems one may encounter when using edge elements:

- Edge elements are known to be *less efficient*, than nodal elements [12,14,15].
- The condition of the representation of a field using vectorial finite elements depends on the bases of the reference frames used in those elements. In edge elements those bases are often related to the vectorial orientations of the faces of the elements. These faces usually are not mutually perpendicular *which will degrade the condition* of the edge element representation [12].
- Most types of edge elements have a *zero divergence*. They can be applied only to solving problems the solution of which is free of divergence.
- Under specific circumstances the use of edge elements may result in *singular stiffness matrices* [16].
- Plots of solutions obtained by using edge elements often seem to be rather *"rough"*. The problem may be "solved" by smoothing the solution in post processing. Some authors [17] claim that post processing yields a reduction of the error in their results. It is clear that such a claim cannot have any mathematical justification.

Edge elements and re-entrant corners

Edge elements are often mentioned as a method to eliminate the large errors that are made when using nodal elements near re-entrant corners in, for instance, a perfectly electrically conducting outer boundary. Obviously edge elements, which are polynomials, cannot be expected to accurately model the singular behaviour of the field near a re-entrant corner and, when using edge elements near re-entrant corners, *the local error term will remain unbounded.*

Do we have alternatives?

The above shopping list of problems encountered when using edge elements cries out for an alternative. Fortunately a number of methods is available that are expected to free us of edge elements.

We mention the following:

- The first alternative is provided by a new class of vectorial finite elements, the generalized Cartesian elements [18]. These elements can accurately model fields that are discontinuous across interfaces as well as fields in homogeneous subdomains.
- A second class of alternatives may be provided by so-called "dual" or "complementary" formulations. In this type of formulations, two "complementary'" vectorial field quantities are chosen that *together* allow a consistent representation of electromagnetic field inside the domain of computation. Dual approaches assume a subset of the Maxwell's equations to be exactly satisfied, i.e. the local curl equations as in [19, 20] or the domain-integrated curl *and* divergence equations as in [21], while imposing the remaining field equations in a weak form.

Conclusions

A critical discussion of the properties of edge elements was presented. A more general and detailed discussion of the subject of the present paper is given in [22].

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