Error Bounds in Computational Electromagnetics

I. Introduction

Most engineering problems must be solved by numerical methods that in general are able to provide only approximate solutions. The error analysis is therefore a theme of great interest in computational engineering. It is nowadays of common use to make error estimation which leads to the definition of the error in the field equations (residual) or in the constitutive relations (constitutive approach). These a posteriori error estimations provide a basis for a selective mesh refinement, and an indication about the convergence of the solution process. However, in some cases it is important to assess the accuracy of the numerical solution with error bounds.

II. Global Bounds

The criteria for the determination of upper and lower bounds for global quantities were established in the thirties in the frame of the elasticity problem [1], while the first applications to computational electromagnetics can be found in the eighties in the papers of Rikabi et al. [2-3].

The Ligurian approach for the magnetostatic problem is based

- on the introduction of a pair of potentials, **A** and Ω, each of them able to satisfy automatically one only of the Maxwell equations $\nabla\times\mathbf{H}=\mathbf{J}$ and $\nabla\cdot\mathbf{B}=0$ by imposing $\mathbf{B}=\nabla\times\mathbf{A}$ and $\mathbf{H}=\mathbf{T}-\nabla\Omega$ with **T** selected such that $\nabla\times\mathbf{T}=\mathbf{J}$;
- on the definition of a local error density whose expression in the linear case is
  $\lambda = (\mathbf{B}-\mu\,\mathbf{H})^2/2\mu = [\nabla\times\mathbf{A}-\mu\,(\mathbf{T}-\nabla\Omega)]^2/2\mu$;
  in the non-linear case the error density can be defined as well, as shown in [2] and Fig.1.

The interface conditions on the discontinuity surfaces can be automatically verified by adopting the edge-element based shape functions for the numerical approximation of **T** and **A**.

This allows for the general definition of a global error functional:

$$\Lambda = \int_V \lambda(\nabla\times\mathbf{A},\mathbf{T}-\nabla\Omega)\,dV = \Gamma(\mathbf{A},\Omega) + \Xi_v(\mathbf{A}) + \Theta_v(\Omega) \geq 0,$$

with strictly definite inequality for each pair of potentials that do not verify the constitutive relationships.

Selecting **A** and Ω so as to verify the essential boundary conditions, the cross term $\Gamma(\mathbf{A},\Omega) = \int_V \nabla\times\mathbf{A}\cdot(\mathbf{T}-\nabla\Omega)\,dV$ splits in two independent functionals and the global functional can be written as $\Lambda(\mathbf{A},\Omega) = \Xi(\mathbf{A}) + \Theta(\Omega) \geq 0$.

By denoting with "0" the actual quantities, we have therefore upper and lower bounds:

$$-\Xi(\mathbf{A}) \leq -\Xi(\mathbf{A}_0) = \Theta(\Omega_0) \leq \Theta(\Omega)\ \forall\ \mathbf{A},\Omega.$$

For linear media and homogeneous boundary conditions $-\Xi(\mathbf{A}_0) = \Theta(\Omega_0)$ is the magnetic energy stored in the domain. Therefore, as well known, there exist upper and lower bound for self-inductance coefficients.

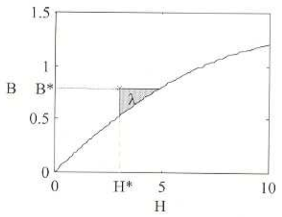

Figure 1.

Graphical representation of the local error in nonlinear magnetostatics. H^* and B^* are the numerical field estimates. The local error , given by the area of the shaded surface, is zero if (H^*,B^*) is on the *B-H* curve and positive otherwise.

III. Local Bounds

By using an extension of the error based approach it is possible to establish upper and lower bounds for local field quantities, namely the average value of a field component in an arbitrarily small region [4-5]. The basic idea relies on the possibility to determine upper and lower bounds for the mutual inductance between a test coil and the source currents.

The self-inductance of the series of two circuits is

$$L = L_1 + L_2 + 2\,M,$$

where L_1 and L_2 are the self-inductance of the two circuits separately, and M is their mutual inductance.

The procedure of the previous section allows to obtain bounds for L, L_1 and L_2. Therefore, we can get upper and lower bounds for $M=(L-L_1-L_2)/2$:

$$M_L=(L_L-L_{1U}-L_{2U})/2$$
$$M_U=(L_U-L_{1L}-L_{2L})/2,$$

where the subscript "L" (respectively, "U") denotes the lower (respectively, upper) bound for the corresponding quantity.

In this context, local bounds were already discussed in the forties. Here, we recall the main points of a paper by Greenberg [6], who analyzed the Dirichlet problem in a homogeneous domain:

$$\Delta w = 0 \text{ in } V, \quad w = f \text{ on } S$$

The starting point is a classic theorem yielding the following inequalities:

$$D(u) \leq D(w) = D^*(w) \leq D^*(v)$$

where

$$D(u) = -\frac{1}{2}\iint_V \nabla u\cdot\nabla u\,dV$$

$$D^*(v) = \frac{1}{2}\iint_V \nabla u\cdot\nabla u\,dV - \int_S f\frac{\partial v}{\partial n}dS$$

$$D^*(w) = -\frac{1}{2}\int_S f\frac{\partial w}{\partial n}dS$$

u is any smooth function satisfying $u=f$ on S, $v\in V$ is any function satisfying $\Delta v=0$ in V. Let us consider three different problems:

1. $\Delta w = 0$ in V, $\quad w = f$ on S

2. $\Delta \overline{w} = 0$ in V, $\quad \overline{w} = -P\,\mathrm{F}$ on S

3. $\Delta \omega = 0$ in V, $\quad \omega = f - P\,\mathrm{F}$ on S

where f is the boundary condition on $S=\partial V$, P is a real constant, and F is the fundamental solution in the free space.

In 2D $\mathrm{F}=P\,log(1/r)$, where $r=\sqrt{(x-x_0)^2+(y-y_0)^2}$ is the distance from (x_0, y_0).

The first one is the original problem; the second one is an auxiliary problem; the third one is obtained by the first two by superposition.

In the 2D case, using the above inequalities, it is possible to get lower and upper bounds for the three quantities:

$$W_1 = D^*(w)$$

$$W_2 = \pi\, P\, \overline{w}\,(x_0, y_0)$$

$$W_3 = W_1 + 2\pi\, P\, w(x_0, y_0) + W_2$$

which provide bounds for the local value $w(x_0, y_0)$:

$$W_{3L} - W_{1U} - W_{2U} \le 2\pi\, P\, w(x_0, y_0) \le W_{3U} - W_{1L} - W_{2L}$$

Similar expressions can be found in the 3D case. Greenberg also defined an iterative procedure able to obtain successively improved upper and lower bounds without recalculating the solution of the Dirichlet problem in the whole domain. It would be extremely useful to transfer this approach to the numerical methods currently used.

IV. Results

Two numerical solutions, i.e. the complementary **A** and (**T**,Ω) solutions, have been determined for the magnetostatic problem shown in Fig. 2 with each of three different meshes having 960, 3840, and 15360 elements, respectively.

The method proposed in [4-5] has been applied to evaluate the average magnetic field in the measurement region. The comparison with the exact value of the average horizontal field $<Bx>$, reported in Table I, clearly shows bounds and convergence. Table I also shows that the complementary formulations do not provide, by themselves, bounds for local field quantities.

Table I

Magnetic field in the measurement region normalized to the exact value of the average horizontal field $<Bx>$

Field / Mesh	$<Bx>_L$	$<Bx>_U$	Bx at the center (A)	Bx at the center (T,Ω)
Coarse	0.88034102	1.11799689	0.99896084	0.96410090
Average	0.96168844	1.03785246	0.99175017	1.01760113
Fine	0.98449695	1.01529388	1.00392486	1.00866121

V. Status and perspective

Error estimators are essential tools in computational electromagnetics. The estimate of a local error distribution provides a basis for a selective mesh refinement, and the evaluation of a global error parameter can be used to check the convergence of the solution process.

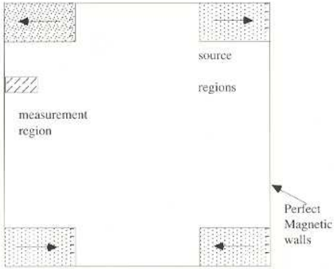

Figure 2.

A 2D magnetostatic problem. Permanent magnets in a hollow iron block

Well-established techniques, borrowed and adapted from other fields of application [7], have been successfully applied to computational electromagnetics. A wide bibliography on these techniques, mainly based on the use of local solvers [8] taking the residual as source term, is given in [9].

The constitutive error approach provides a valid alternative to the Babuška-like error estimators. Its main advantages are its straightforward interpretation and the possibility of providing upper and lower bounds for both global and local quantities.

The local error ("error density") is simply obtained by comparing the different estimates of the fields linked by the constitutive equations. For instance, in magnetostatics, it is given by and ease in obtaining both local and global error by simply comparing the different estimates of the fields linked by the constitutive equations, as shown in Section II. The price to be paid is to double the number of unknowns. However, the splitting of the functional [2] or the use of dual formulations [3, 10] allow to reduce the calculation to the solution of the same problem with two different sets of variables. For instance in magnetostatics a numerical **B**-solution can be obtained by the **A** formulation whereas an approximate **H**-solution can be given by the **T**, Ω method. As shown in [10], there is a means to further reduce the computational effort, by solving the full (linear or nonlinear) problem with one method and determining the other set of variables by simply interpolating the dual unknown via least squares or other linear fitting techniques.

For all stationary problems for which the analogue of a virtual work principle can be established, complementary solutions allow to establish complementary variational principles, and consequently bounds for global quantities. The key point is the possibility of splitting a real-valued error functional in two parts, each depending on a different set of variables. These global bounds are also applicable in nonlinear, inhomogeneous and anisotropic problems. The possible extension of these results to the transient and sinusoidal steady-state problems has been widely discussed. However, the possibility of establishing upper and lower bounds for global parameters has not been demonstrated [2,12,13]. The fact is that so far the splitting has been achieved only for complex functionals for which inequalities cannot be applied except cases of no practical interest.

Most technical articles dealing with the error approach mainly refer to differential formulations and associated finite element codes. Actually, the technical application of the error based approach, although possible using any technique able to provide dual solutions enforcing the canonical equations, became straightforward and widespread when used in conjunction with the edge elements for their well known properties of well representing the continuity behavior of **H**, **E** and **A** vectors [3]. A present limit of the numerical codes based on integral formulations is that they are not accompanied by efficient error estimators. To this respect, an estimator based on the definition of a local error density might probably be applied when using dual integral formulations like those proposed by [14].

It is possible to determine upper and lower bounds for the average values of the fields in arbitrarily small regions. The technique illustrated in Section III can be used for any linear system in stationary conditions for which the analogue of the virtual work principle can be applied. The material media are not necessarily isotropic and homogeneous, but non-linearities are not allowed. It would be interesting to extend the procedure to the nonlinear case. To obtain bounds for field values at given points, it would be interesting to explore the possibility of a combined use of the technique illustrated in [4-5] and the method proposed in [6], which can be applied only in the case of homogeneous media.

In any case, the most critical limitation to the computational accuracy stems from the approximate modeling of the material media and the interaction of electromagnetic fields with other phenomena, e.g. motion, thermal diffusion, MHD, etc.. Therefore, attention should be kept alive on the material modeling, with particular attention to hysteresis, and the formulation and associated numerical codes for the solution of coupled problems.

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