Technical article

About Sensitivity Analysis using High Order Derivatives

Introduction

The finite element method is now well known, largely used, and able to solve a wide set of partial differential equation problems. Nevertheless, there still remains an important restriction: it is unbearably time expensive when you want to do sensitivity analysis, or, in other words, when you need the solutions for several values of parameters. In consequence, since the finite element method is increasingly used in design rather than as a simple validation tool, sensitivity analysis must be performed efficiently.

An interesting way of investigation could be the use of high derivatives to construct a polynomial approximation of the finite element solution. The idea was already presented [1], [2], [3], but here, there is a new aspect: the entirely analytical calculus of the derivatives [4], [5], [14]. This approach is more powerful and flexible than a simple automation of successive re-analyses to perform the sensitivity analysis. This method will be presented for two dimensional linear magnetostatic and linear magnetodynamic problems using vector potential formulation. The knowledge of high order derivatives for the "solved variable" (the vector potential) allows the deduction of any dependent quantities (forces, losses, ...).

The high derivatives can be done with respect either to physical parameters (frequency, permeability, ...) [6], or to geometric parameters [7]. This method opens, moreover, some possibilities in automatic optimization, particularly the coupling with genetic algorithms. This is often impossible because of the cost of the test function, but here it is not the case thanks to the Taylor development.

Formulation and Parameters

A. The FE solving

With the magnetic vector potential $\vec{A}$ such as $\vec{B} = curl\ \vec{A}$ and the magnetodynamic formulation, Maxwell equations lead to

$$curl\left(\nu\, curl\ \vec{A}\right) + \sigma\frac{\partial \vec{A}}{\partial t} = \vec{J}_s. \qquad (1)$$

where ν is the given magnetic reluctivity, σ is the given electric conductivity and $\vec{J}_s$ is the given current density.

For a linear problem and with a harmonic excitation current of pulsation ω, we can use a complex representation to separate the space and the time dependencies. Moreover, in a two dimensional problem, we can define only the components of the magnetic vector potential A and the current density J_s which are normal to the section of the device

$$\vec{A} = \begin{bmatrix} 0 \\ 0 \\ real\left(A(x,y)e^{j\omega t}\right) \end{bmatrix},\ \vec{J}_s = \begin{bmatrix} 0 \\ 0 \\ real\left(J_s(x,y)e^{j\omega t}\right) \end{bmatrix}. \qquad (2)$$

Then, equation (1) becomes

$$-\nabla\left(\nu\nabla A\right) + j\omega\sigma A = J_s. \qquad (3)$$

Using the finite element method and the Galerkin process, we obtain an approximate solution by solving a complex matrix equation

$$M.\{A\} = S, \qquad (4)$$

where $\{A\}$ is the set of the N unknown nodal values. By introducing real part P and imaginary part Q of the $N \times N$ complex matrix M, we write

$$M = P + jQ. \qquad (5)$$

Global matrixes P, Q and global vector S are assembled from local element terms P_e, Q_e and S_e

$$P = \sum_{elements} P_e = \sum_{elements} \iint_{\Omega_e} \nu\{\nabla\alpha\}\{\nabla\alpha\}^T dxdy, \qquad (6)$$

$$Q = \sum_{elements} Q_e = \sum_{elements} \iint_{\Omega_e} \omega\sigma\{\alpha\}\{\alpha\}^T dxdy, \qquad (7)$$

and

$$S = \sum_{elements} S_e = \sum_{elements} \iint_{\Omega_e} J_s\{\alpha\} dxdy, \qquad (8)$$

where $\{\alpha\}$ are the trial functions.

B. The high order derivative parameters

It may seem curious to speak about high order derivatives, since the finite element method is based on low order polynomial trial functions. In fact, the space interpolation is only a first or second order approximation, but we speak here of high order derivatives with respect to the design parameters. Our FE problem is continuous with respect to the physical parameters. Even for geometric parameters, it is possible to derive at high order, because we use a derivation of the equilibrium equations.

We purpose to use high order derivatives of the "solved variable" (the vector potential A). Using state adjoint variables, the process could be generalized to all the

derived quantities of the FE problem [1]. For instance a development of the dissipated thermal energy in a conductor, or these of a magnetic force could be derived. However, just with the main variable, we consider we have the most general approach, but probably not the most efficient in many cases.

We consider the derivatives versus either physical parameters or geometric parameters.

The linear problem under study is constituted of sub-domains D_k where the magnetic reluctivity ν_k, the electric conductivity σ_k and the current density J_{sk} have different values but are uniform. Any of the value ν_k, $\omega\sigma_k$, or J_{sk} inside any sub-domain D_k could be taken as a physical parameter.

As soon as now, we can note that the 3 elementary terms (6), (7) and (8) are linear towards the physical parameters ν, $\omega\sigma$, and J_s. These simple algebraic dependencies will facilitate a direct differentiation. For the physical parameters, the first derivatives are constants and the higher order ones are zeros. Notice however that if a linear dependency on the source term (8) implies a linear dependency on the solved variable, it is not the case for the linear dependencies acting on the matrix terms (6) and (7).

For geometric parameters (for instance air gap thickness, slot dimension, etc.), the dependencies are introduced throughout the nodal co-ordinates which affect both the elementary area Ω_e and the trial function gradients $\{\nabla\alpha\}$. In the general case, there is no obvious simplification and higher order derivatives of P_e, Q_e and S_e versus geometric parameters could exist.

Calculus of High Order Derivatives

A. One parameter study

Thanks to the principle of stationarity [1, 8], equation (4) is true for any value of each parameter. So we can derive (4) with respect to any parameter p, and then find the new equation to solve to obtain the derivatives of the solution vector A

$$M.\frac{\partial A}{\partial p}=\frac{\partial S}{\partial p}-\frac{\partial M}{\partial p}.A, \tag{9}$$

It must be noted that the derivative of the solution is obtained by solving a matrix equation, with the same matrix than in (4). Moreover, we need the solution A to calculate the new right hand side vector and then the derivative.

Recursively, we can calculate an arbitrary high order derivative of A by deriving (4) and solving with a new second member calculated with the first derivatives of A

$$M.\frac{\partial^m A}{\partial p^m}=\frac{\partial^m S}{\partial p^m}-\sum_{i=1}^{m} C_m^i\,\frac{\partial^i M}{\partial p^i}.\frac{\partial^{m-i} A}{\partial p^{m-i}}, \tag{10}$$

where $C_m^i=\dfrac{m!}{i!(m-i)!}$ and $C_m^0=C_m^m=1$.

Finally, with the derivatives of A, we construct a polynomial to evaluate the solution for any value of δp, from an initial solution for the value p_0

$$A(p_0+\delta p)=A(p_0)+\delta p\frac{\partial A}{\partial p}(p_0)+\ldots+\frac{\delta p^{N_p}}{N_p!}\frac{\partial^{N_p}A}{\partial p^{N_p}}(p_0)$$

$$A(p_0+\delta p)=\sum_{i=0}^{N_p}\frac{\delta p^i}{i!}.\frac{\partial^i A}{\partial p^i}(p_0), \tag{11}$$

where N_p is the Taylor development order for the parameter p.

B. Multi-parameter study

In the case of a multi-parameter study, with high order derivatives, we must calculate the crossed derivatives of M and S before calculating the crossed derivatives of A.

Just look at the generalization of (10) and (11) with only two independent parameters p and q

$$M.\frac{\partial^{m+n}A}{\partial p^m\partial q^n}=\frac{\partial^{m+n}S}{\partial p^m\partial q^n}-\sum_{\substack{i=0\\ i+j>0}}^{m}\sum_{j=0}^{n}C_m^iC_n^j\,\frac{\partial^{i+j}M}{\partial p^i\partial q^j}.\frac{\partial^{m-i+n-j}A}{\partial p^{m-i}\partial q^{n-j}} \tag{12}$$

$$A(p_0+\delta p,q_0+\delta q)=\sum_{i=0}^{N_p}\sum_{j=0}^{N_q}\frac{\delta p^i}{i!}.\frac{\delta q^j}{j!}.\frac{\partial^{i+j}A}{\partial p^i\partial q^j}(p_0,q_0). \tag{13}$$

Of course when the number of independent parameters increases, (13) becomes quickly very long because of the crossed terms !

C. Derivatives of M and S versus physical parameters

Obviously in the general case, the main difficulty is to calculate the derivatives of the matrix M and those of the vector S. But if p is one of the previously described physical parameters (ν, $\omega\sigma$, or J_s), this is no longer a difficulty. For the reluctivity ν_k of the sub-domain D_k, we deduce from (5) and (6)

$$\begin{cases}\dfrac{\partial M}{\partial \nu_k}=\displaystyle\sum_{\Omega_e\in D_k}\iint_{\Omega_e}\{\nabla\alpha\}\{\nabla\alpha\}^T\,dxdy\\[2ex] \dfrac{\partial^n M}{\partial \nu_k^n}=0,\ \text{for } n\geq 2\end{cases} \tag{14}$$

and

$$\frac{\partial^n S}{\partial \nu_k^n}=0,\ \text{for n}\geq 1 \tag{15}$$

For the product $\omega\sigma_k$, taken as a whole parameter, we have

$$\begin{cases} \dfrac{\partial M}{\partial(\omega\sigma_k)} = j\sum\limits_{\Omega_e \in D_k} \iint_{\Omega_e} \{\alpha\}\{\alpha\}^T dxdy \\ \dfrac{\partial^n M}{\partial(\omega\sigma_k)^n} = 0, \text{ for } n \geq 2 \end{cases} \tag{16}$$

and

$$\frac{\partial^n S}{\partial(\omega\sigma_k)^n} = 0, \text{ for n}\geq 1 \tag{17}$$

Finally, for the current density J_{sk}, we obtain

$$\frac{\partial^n M}{\partial J_{sk}^n} = 0, \text{ for n}\geq 1 \tag{18}$$

and

$$\begin{cases} \dfrac{\partial S}{\partial J_{sk}} = \sum\limits_{\Omega_e \in D_k} \iint_{\Omega_e} \{\alpha\} dxdy \\ \dfrac{\partial^n S}{\partial J_{sk}^n} = 0, \text{ for } n \geq 2 \end{cases} \tag{19}$$

The summations are restricted to the elements Ω_e belonging to the sub-domain D_k.

As previously mentioned, the higher derivatives are equal to zero and consequently there is no crossed derivative between the physical parameters.

D. Derivatives of M and S versus geometric parameters

The derivatives towards geometric parameters could be obtained using the local jacobian derivative method [8].

First we consider the second member S. Using parametric finite element integration, its expression (8) becomes

$$S = \sum_{all\ elements} \iint_{\Delta_e} J_s\{\alpha\}|G|dudv, \tag{20}$$

where Δ_e is the reference element corresponding to the actual element Ω_e and $|G|$ is the determinant of the jacobian matrix G of the transformation from local (u,v) to global (x,y) co-ordinates.

During the modification of a geometric parameter p, the topology of the mesh is unchanged but some finite elements are distorted. This is due to the moving of the nodes linked to the parameter.

Because in (20) $|G|$ is the only term depending on the actual geometry (Δ_e is the reference element and $\{\alpha\}$ are functions of (u,v)), the first derivative of S is

$$\frac{\partial S}{\partial p} = \sum_{\substack{distorted \\ elements}} \iint_{\Delta_e} J_s\{\alpha\}\frac{\partial |G|}{\partial p} dudv. \tag{21}$$

The high order derivatives can be obtained using the same method. For instance, with two geometric parameters p and q, we have

$$\frac{\partial^{m+n} S}{\partial p^m \partial q^n} = \sum_{\substack{distorted \\ elements}} \iint_{\Delta_e} J_s\{\alpha\}\frac{\partial^{m+n}|G|}{\partial p^m \partial q^n} dudv. \tag{22}$$

The summation is restricted to the elements simultaneously concerned by the parameters p and q. In the general case, if an element is not distorted by at least one parameter, its contribution is equal to zero. This property allows to reduce the amount of computations dramatically.

For the matrix Q, we can apply to (7) exactly the same process as for S.

Finally, we study the matrix P. This case is more complicated than the precedent ones due to the gradients $|\nabla\alpha|$ under the integral sign (6). In each element, the gradient $\nabla\alpha_i$ of any function α_i can be expressed as

$$\nabla\alpha_i = \begin{bmatrix} \dfrac{\partial\alpha_i}{\partial x} \\ \dfrac{\partial\alpha_i}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial\alpha_i}{\partial u} \\ \dfrac{\partial\alpha_i}{\partial v} \end{bmatrix} = G^{-1}\nabla_u\alpha_i, \tag{23}$$

where G^{-1} is the inverse of the jacobian matrix G and $\nabla_u\alpha_i$ is the local gradient in the reference element. The gradient expression inside (6) becomes

$$\{\nabla\alpha\}\{\nabla\alpha\}^T = \{\nabla_u\alpha\}G^{-1T}G^{-1}\{\nabla_u\alpha\}^T. \tag{24}$$

Then, (6) can be rewritten as

$$P = \sum_{elements} \iint_{\Delta_e} \nu\{\nabla_u\alpha\}.|G|.G^{-1}.G^{-1}.\{\nabla_u\alpha\}^T dudv \tag{25}$$

In the previous expression, the product $[|G|.G^{-1T}.G^{-1}]$ is the only term depending on the actual geometry, so the derivatives of (6) can be calculated by deriving only it. For instance, high order derivatives relative to two geometric parameters p and q are

$$\frac{\partial^{m+n} P}{\partial p^m \partial q^n} = \sum_{\substack{\text{distorted} \\ \text{elements}}} \iint_{\Delta_e} v\{\nabla_u \alpha\}.\frac{\partial^{m+n}}{\partial p^m \partial q^n}\left[|G|.G^{-1T}.G^{-1}\right].\{\nabla_u \alpha\}^T dudv \quad (26)$$

The expressions of G and $|G|$ depend on the finite element type and on the nodal co-ordinates. If we know the derivatives of the nodal co-ordinates, we can compute the derivatives of G and $|G|$. From the property $G^{-1}G=I$, we can deduce the derivatives of G^{-1} knowing the ones of G. For instance the first derivative of G^{-1} is

$$\frac{\partial G^{-1}}{\partial p} = -G^{-1}.\frac{\partial G}{\partial p}\,G^{-1}. \quad (27)$$

So, any derivative at any order of the product $[|G|.G^{-1T}.G^{-1}]$ can be expressed as a function of G, $|G|$, their derivatives and G^{-1}.

E. Crossed derivatives between physical parameters and geometric parameters

As noted in section C, the crossed derivatives between physical parameters are zero. However, the crossed derivatives between physical parameters and geometric parameters exist and can be obtained using the same process as in section D. Although, they are restricted to the first order for the physical parameters.

For instance, resulting from (14), the high order crossed derivatives for the parameter v_k and two geometric parameters p and q are

$$\frac{\partial^{1+m+n} M}{\partial v_k \partial p^m \partial q^n} = \sum_{\substack{\text{distorted} \\ \text{elements} \in D_k}} \iint_{\Delta_e} \{\nabla_u \alpha\}.\frac{\partial^{m+n}}{\partial p^m \partial q^n}\left[|G|.G^{-1T}.G^{-1}\right].\{\nabla_u \alpha\}^T dudv \quad (28)$$

Similar derivatives arise from (16) and (19)

$$\frac{\partial^{1+m+n} M}{\partial(\omega\sigma_k)\partial p^m \partial q^n} = j \sum_{\substack{\text{distorted} \\ \text{elements} \in D_k}} \iint_{\Delta_e} \{\alpha\}\{\alpha\}^T \frac{\partial^{m+n}|G|}{\partial p^m \partial q^n} dudv \quad (29)$$

$$\frac{\partial^{1+m+n} S}{\partial J_{sk} \partial p^m \partial q^n} = \sum_{\substack{\text{distorted} \\ \text{elements} \in D_k}} \iint_{\Delta_e} \{\alpha\}\frac{\partial^{m+n}|G|}{\partial p^m \partial q^n} dudv \quad (30)$$

Use of High Order Derivatives

A. Steps of the construction of the polynomial

Following the formula given in the previous developments, the first step in the method is to find the derivatives of the matrix M and of the vector S at the central values p_0, q_0, ... for the parameters.

In order to have a general approach, capable of treating big size problems, all these terms are saved on files instead of managing them all in the memory. The finite element matrix M is naturally sparse. Moreover, its derivatives and the those of the vector S are more and more sparse as the derivative order increases. So, it is a good idea to use a specific file format for sparse structures.

Then, the high derivatives of A are calculated recursively, by solving the systems (12). All these systems have the same matrix M but different right hand side vectors. It is worth using a direct method (i.e., a factorization of the matrix M), to solve all these systems. Iterative methods (e.g., preconditioned conjugate gradients) are superior for a single solution, both in terms of storage requirements and computational efficiency. A direct method is more expensive for the first solving (the computation of A), but then all the derivatives only cost a forward and backward substitution each. Finally, the high derivatives have a very low cost when using a factorization.

The storage of the derivatives of the vector A, is done on files again, but with a set of plain arrays. So write accesses, as well for read, may be random.

Finally for any variation δp, δq, ... of the parameters, using (13), a polynomial is constructed allowing the solution to be evaluated.

Because evaluation of a polynomial is instantaneous, it can be used interactively, and it allows a new approach of the sensitivity analysis.

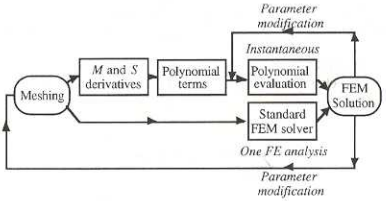

Fig. 1. Compared steps between standard finite element solver and polynomial approach

Fig. 1 shows how new developments are introduced in a finite element package. Calculations of the derivatives of the matrix and vector need to be coupled with the assembling function. Construction of the polynomial is an independent program. Evaluation must be interfaced with the post-treatment module.

To sum up, the initial cost of this method is quite high, but affordable. After an important investment, we obtain a polynomial which provides fast evaluations (Fig. 2.).

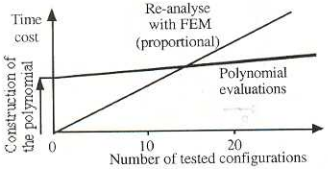

Fig. 2. Compared time costs between re-analyze and polynomial approach.

B. Precision and convergence radius

It is important to distinguish the precision and the convergence radius. Obviously, inside the convergence radius, the more terms are taken, the better the reconstructed solution is. Outside, the more terms are taken, the worst it is. For the current density J_s, the convergence radius is infinite (it is a source term). For the other physical parameters ν and $\omega\sigma$, which act on the matrix M, if ν_0 and $(\omega\sigma)_0$ are the central values of the Taylor development, the domain of convergence is $]0, 2\nu_0[$ and $]0, 2(\omega\sigma)_0[$.

Here is a beginning of a proof for a one parameter study only. As the vector S is independent of the parameter ν and $\omega\sigma$ and only the first derivatives of M are non zero, (10) becomes

$$M.\frac{\partial^m A}{\partial p^m} = -m\frac{\partial M}{\partial p}.\frac{\partial^{m-1} A}{\partial p^{m-1}}. \tag{31}$$

This recursively gives

$$\frac{\partial^m A}{\partial p^m} = (-1)^m m!\left[M^{-1}.\frac{\partial M}{\partial p}\right]^m.A. \tag{32}$$

Then, the Taylor development (11) is

$$A(p_0+\delta p) = \left[\sum_{i=0}^{N_p}\left[-\delta p M^{-1}.\frac{\partial M}{\partial p}\right]^i\right].A(p_0)$$
$$= \left[I+\delta p M^{-1}.\frac{\partial M}{\partial p}\right]^{-1}\left[I-\left[-\delta p M^{-1}.\frac{\partial M}{\partial p}\right]^{N_p+1}\right].A(p_0) \tag{33}$$

To insure the convergence when N_p tends to the infinite, the following condition must be verified at the central parameter value p_0

$$\left\|-\delta p M^{-1}.\frac{\partial M}{\partial p}\right\| < 1. \tag{34}$$

Because we have a linear dependency of $M(p)$ relatively to the parameter p, we can write (when $p_0 \neq 0$)

$$\frac{\partial M}{\partial p} = \frac{M(p_0)-M(0)}{p_0-0} = \frac{M-M(0)}{p_0}, \tag{35}$$

where $M(0)$ is the matrix obtained when the parameter (reluctivity or conductivity) is set to zero inside the parametered sub-domain.

Then, for the first member of (34) we have

$$\left\|-\delta p M^{-1}.\frac{\partial M}{\partial p}\right\| = \left\|-\frac{\delta p}{p_0}.M^{-1}.[M-M(0)]\right\| \le \left|\frac{\delta p}{p_0}\right|. \tag{36}$$

The previous inequality is obtained thanks to the properties of finite elements matrixes M, $M(0)$ and $[M\text{-}M(0)]$ which are symmetric and, when meshes are of good quality, with real parts which are definite non negative and dominant diagonal.

So, from (34) and (36), we obtain a sufficient convergence condition on δp

$$|\delta p| < |p_0|. \tag{37}$$

This conclusion is confirmed in [6] on the study of an analytical solution for the physical parameters. The domains of practical use, for a reasonable accuracy of 5% with a 10 to 20 order Taylor development, are $]0.2\nu_0, 1.8\nu_0[$ and $]0.2(\omega\sigma)_0, 1.8(\omega\sigma)_0[$ (see a later section for an example).

Validation on Induction Heating Problem (Physical Parameters Study)

A. Presentation of the studied problem

We choose an induction heating process (Fig. 3) where inductors produce eddy currents in the steel tubes. Such tubes become malleable and can be extruded to the desired form. We do not treat the problem using geometric parameters, but just develop the magnetic solution with the three physical parameters ν, ω, and J_s under a hypothesis of linear magnetization. The first parameter is the characteristic of the steel and the two last are the command parameters for the heating process. The mesh used for this process is presented on Fig. 4. It contains 2686 nodes and 1017 second order finite elements.

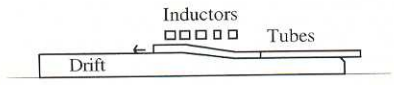

Fig. 3. The 2D induction heating problem.

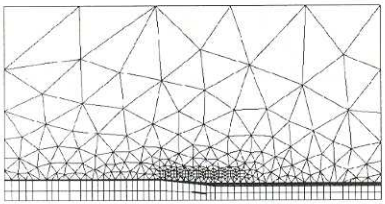

Fig. 4. Mesh of the 2D induction heating problem.

B. Validity of the polynomial approach

This example, shows that the polynomial method works on a realistic multi-parameter problem. We verified the previously cited convergence limits and confirm the conclusion on the order of development given in [6]. The 2D example diverges for values of the reluctivity and frequency superior to twice the initial ones. In any case, reasonable orders (10 to 20) are sufficient to a good precision for the computation of common problems (5% between polynomial solution and standard solution). In conclusion, the polynomial is valid for the domain $]0.2\nu_0, 1.8\nu_0[$ and $]0.2(\omega\sigma)_0, 1.8(\omega\sigma)_0[$. Table I shows the costs for classical finite element analysis (using direct method) and polynomial approach.

This application of the high order derivative method has been implemented within a specific multiplatform and pluridisciplinary finite element program [9].

TABLE I
TIMES FOR EACH STEP :
COMPUTER CLOCK INDICATOR AND MINUTES MEASUREMENTS

Classical FE analysis	Derivatives	Calculus of the polynomial	Evaluation of the polynomial
2500 to 2800	1900 to 2100	27000 to 30000	1 to 3
about 25 seconds	less than 25 seconds	about 5 min	instantaneous

Validation on SMES Optimization (Physical and Geometrical Parameters Study)

A. Presentation of the studied problem

In recent years, Genetic Algorithms [10], [11] have been applied successfully in many fields of optimization. These algorithms present numerous advantages over classical gradient methods. They are able to locate the global optimum and they do not require the use of the derivatives. However, Genetic Algorithms require thousands of evaluations in order to reach the optimum, and thus, they are particularly penalized - in terms of cpu time - when the objective function is calculated by a Finite Element Method. To reduce such computational constraints, we propose to link a Genetic Algorithm with an accurate enough, but less expensive approximation method, based on Taylor development.

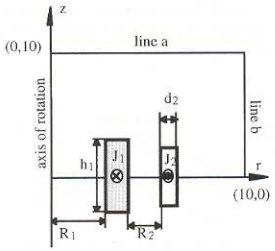

Fig. 5. SMES design parameters.

The test case problem deals with the optimization of a system, consisting of two coils, for a Superconducting Magnetic Energy Storage device (SMES). This problem is a simplification of a more complex one [12] and is schematically shown in Fig. 5. The SMES configuration has six degrees of freedom (four geometrical and two physical) and shall be optimized with respect to these objectives

- Stored Energy should be 180 MJ,
- Magnetic field along two given lines should be as small as possible.

It is also subject to the following constraint:

- Magnetic field within each coil must satisfy a certain Physical condition in order to guarantee a superconductivity behaviour.

The two objectives were coupled with a weighted sum to form a single objective (0.8 for relative stored energy and 0.2 for relative magnetic field). Then the constrained problem was transformed into an unconstrained one by associating a penalty with all the constraints.

B. Polynomial approach

In order to avoid Finite Element solution for each different configuration, an approximate objective function can be obtained with Taylor development [13]. The procedure requires an initial configuration of the design parameters and an initial mesh. In our study, the initial values of the parameters were chosen near the centre of the domain where they are able to evolve (Table II). It must be pointed out that a good mesh should be able to support the greatest variations of parameters. When this is achieved, the Taylor development of the "solved variable" is constructed in about thirty minutes and gives second order derivatives for geometrical parameters. Accuracy has been tested by comparing the energy in the device between polynomial development and a Finite Element re-analysis: worst cases - when parameters are far from initial Taylor point - give reasonable accuracy of 4 %.

TABLE II
PARAMETERS SPACE DOMAIN AND TAYLOR POINT

	R_1 [mm]	R_2 [mm]	$h_1/2$ [mm]	d_2 [mm]	J_1 [A/mm^2]	J_2 [A/mm^2]
min.	600	60	500	100	10	-30
max.	1400	500	850	600	30	-10
Taylor point	1000	300	650	350	20	-20

C. *Optimization using genetic algorithm*

The Genetic Algorithm used has the following main characteristics: Real coding, Linear ranking selection, four crossover operators (1-point, 2-points, Uniform, Arithmetic), three mutation operators (Uniform, Non Uniform, Gaussian) and a roulette wheel, depending on weights, to select operators.

TABLE III
AVERAGE AND BEST TRIAL

	$Bstray^2$ [T^2]	Energy [MJ]
Average	2.54E-9	173.4
Best	0.61E-9	180

Table III and Table IV summarize the results from different trials with different population sizes and operator weights. In these tests the width of the first coil was fixed to 594.3 mm and the length of the second coil fixed to 1418.4 mm. B_{stray} represents the average of the magnetic field along line *a* and line *b*. The genetic approach requires at least 8000 evaluations to reach the optimum but a search with a uniform grid of twenty points in each dimension in parameter space would have required 64,000,000 evaluations.

TABLE IV
BEST SOLUTION

	R_1 [mm]	R_2 [mm]	$h_1/2$ [mm]	d_2 [mm]	J_1 [A/mm^2]	J_2 [A/mm^2]
Best	1298.57	129.52	722.24	258.37	17.70	-11.54

Conclusion

A polynomial representing the finite elements solution in terms of the design parameters can be reached using the high order derivatives. This is a hybrid approach: a continuous solution versus the design parameters through the Taylor development and a discrete solution in space through the finite elements method. This allows a quick evaluation for new values of physical and geometrical parameters and a sufficient precision for design. It provides a good tool to perform physical and geometrical sensitivity analysis in case of linear problems.

The case of non linear problems is a much more complicated one. The main difficulty comes from the B(H) curves which cannot be well approximated by a single low order polynomial. A promising direction of investigation could be to use some Pade's approximations as proposed in [15].

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