Perfectly Matched Layers in Static Fields

1. Introduction

Perfectly matched layers (PMLs) have recently become the focus of much attention among analysts of high frequency fields. Originally proposed by Berenger for the Finite Difference Time Domain Method (FDTD) [1], they have been successfully utilised in conjunction with the Finite Element Method (FEM), too [2], [3]. The basic idea of these layers is that they have fictitious material properties ensuring that the electromagnetic field is fully absorbed in them. Hence, they can be used to model absorbing boundary conditions on surfaces truncating the problem region, an absolute necessity if differential methods such as FDTD or FEM are applied to wave propagation problems in free space.

In the static case, the problem of modelling unbounded space by FEM is less important. It is usually sufficient to assume an artificial boundary far enough from any sources or material interfaces and to set homogeneous boundary conditions on them. Nevertheless, much effort has been put in the past into devising methods that avoid the necessity of meshing large, empty regions where the solution is usually not really desired. These range from ballooning methods and infinite elements to inversion techniques and to coupling FEM with the Boundary Element Method. A recently published technique involving PMLs [4] modifies the differential equation in the absorbing layers by coordinate stretching. This means that its FEM implementation requires the inclusion of different differential operators in various subregions thus complicating the satisfaction of the otherwise natural interface conditions on the normal derivatives. The aim of this note is to point out that the use of PMLs with material properties specially constructed for static fields is a viable alternative to these methods, requiring no new FEM code aside from the ability of taking anisotropic materials into account.

2. How to construct a PML for static fields

The discussion that follows assumes a planar, two dimensional static electric field problem extending to infinity. It can involve any number of electrodes with given potential (V) or line charge density (σ), dielectrics of arbitrary permittivity (ε) as well as a given surface charge density (ρ) with the overall charge equal to zero (Fig. 1). The results to be obtained can readily be generalised for three dimensional and/or magnetic problems.

The infinite region will be truncated to a finite problem region, Ω_{int}, wherein the solution of the problem is sought. The boundary of this problem region, Γ, is assumed to be a convex polygon. (Circular arc sections could also be allowed, this will, however, not be detailed). The region external to this problem region is denoted by Ω_{ext}. Within Ω_{ext}, rectangular layers of finite width with special material properties will be assumed (Ω_{PML1}, Ω_{PML2}, ...). The external boundaries of these layers as well as arbitrary curves connecting these make up the boundary Γ_{PML} (Fig. 1).

A Perfectly Matched Layer for static electric fields must have two properties in order to efficiently simulate free space extending to infinity:

- *A* Its interface, Γ, to the problem region, Ω_{int}, must not introduce any change in the field quantities within Ω_{int} with respect to the values that would prevail there in the absence of the layer.
- *B* The field within the layer must decay much more rapidly than it would in free space, so that homogeneous boundary conditions on Γ_{PML} have negligible effect on the field within Ω_{int}, even if the thickness of the layer is small.

The property *B* can be ensured by setting the permittivity of the layer to be much lower than that of free space. This permittivity, however, cannot be isotropic, since then it would be impossible to attain the property *A*. It is, however, sufficient for the field in the layer to strongly decay in the direction normal to the interface (i.e. in the direction towards infinity). Therefore, an anisotropic material has to be assumed wherein the relative permittivity in the normal direction is much lower than 1. This is clearly a nonphysical assumption (the relative permittivity cannot be less than 1), since, however, the field within the PMLs needs not have any physical significance, this does not introduce any difficulty.

It will be shown in the following section that the property *A* can be satisfied by selecting the value of the relative permittivity in the cross direction to be *the reciprocal of the normal relative permittivity*. The relative permittivity in the sections of the region between Γ and Γ_{PML} outside the PMLs is assumed to be 1.

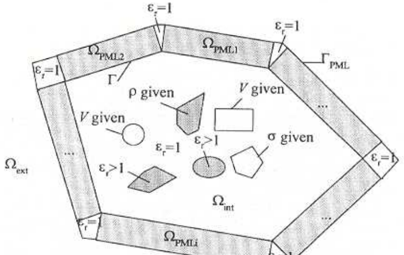

Fig. 1. Typical static electric field arrangement surrounded by perfectly matched layers

In summary, the proposition is as follows:

Let us solve the static electric field in the region bounded by Γ_{PML} with all sources, material properties and boundary conditions in the interior of Γ (i.e. within the problem region Ω_{int}) the same as in the original unbounded problem, with the PMLs in the regions Ω_{PMLi} (i=1,2,...) filled by anisotropic material so that the product of the relative permittivities in the directions normal and tangential to Γ is equal to 1 and with the rest of the region filled by vacuum. The relative permittivity in the PMLs in the direction normal to Γ is (much) lower than 1. The potential is set to zero on Γ_{PML}.

3. *Why do these PMLs work?*

In order to show that, within Ω_{int}, the solution of the problem proposed above does in fact closely approximate the static field arising if the free space region extends to infinity, let us consider the property *A* in the preceding section, i.e. the requirement that the introduction of the PMLs in Ω_{ext} does not alter the field in Ω_{int} with respect to the field that would arise there in the absence of the PMLs. Assume that the problem in free space has been solved. Let us denote the potential function of this solution by V_0 in Ω_{int} and specify this potential on Γ. Let us now proceed by solving the exterior problem in Ω_{ext} with this boundary condition once with Ω_{ext} filled by vacuum (this solution, V_1, coincides with the free space solution), and then with the PMLs present and extending to infinity, yielding the potential function V_2. The PMLs have property *A* if both V_1 and V_2 yield the *same normal component*, D_n, *of the electric flux density* on Γ. Indeed, if this is the case, then the potential function equal to V_0 in Ω_{int} and to V_2 in Ω_{ext} can be obtained by simply solving the field problem simultaneously in Ω_{int} and Ω_{ext} filled by the PMLs with the continuity of both V and D_n satisfied. This is obvious, since V_0 and V_2 satisfy the proper differential equations in the relevant regions, they have the same value on Γ and also satisfy the continuity of D_n provided V_1 and V_2 yield the same D_n on Γ (the continuity of D_n yielded by V_0 and V_1 is trivial). The simultaneous solution of the field problem in Ω_{int} and Ω_{ext} filled by the PMLs with the continuity of both V and D_n satisfied can be readily obtained by ordinary FEM just using the material characteristics of the PMLs in Ω_{ext}.

Since the permittivity of the PMLs is much less than 1 in the normal direction, V_2 decays rapidly, and the truncation error resulting in setting the potential to zero on Γ_{PML} is negligible.

The choice of the elements of the permittivity tensor in a PML that ensures that D_n yielded by V_1 and V_2 coincide can be established as follows. Let us consider the subregion Ω_{PMLi}. Let us choose a coordinate system so that the section of Γ separating Ω_{int} and Ω_{PMLi} coincides with the line $x=x_0$. The length of this section of Γ be denoted by h (Fig. 2). The differential equations for the potential for the two cases of Ω_{PMLi} being filled by vacuum and by an anisotropic material are also indicated.

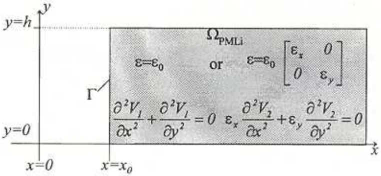

Fig. 2. The differential equations in a PML filled by vacuum or by anisotropic material

Let $V_0(y)$ be the solution of the problem in infinite free space along the section of Γ $x=x_0$, $0<y<h$. This can be expanded in a Fourier series as

$$V_0(y)=\sum_k a_k \sin(k\frac{\pi}{h}y). \qquad (1)$$

Using this, the solution of Laplace's equation with vacuum in Ω_{PMLi} is

$$V_1(x,y)=\sum_k a_k \sin(k\frac{\pi}{h}y)\frac{\exp(-k\frac{\pi}{h}x)}{\exp(-k\frac{\pi}{h}x_0)}. \qquad (2)$$

The solution of the differential equation in Ω_{PMLi} if it is filled by an anisotropic material satisfying the boundary condition (1) is, on the other hand,

$$V_2(x,y)=\sum_k a_k \sin(k\frac{\pi}{h}y)\frac{\exp(-k\sqrt{\frac{\varepsilon_y}{\varepsilon_x}}\frac{\pi}{h}x)}{\exp(-k\sqrt{\frac{\varepsilon_y}{\varepsilon_x}}\frac{\pi}{h}x_0)}. \qquad (3)$$

The condition that both solutions yield the same x-component (i.e. the normal component) of the electric flux density is

$$\frac{\partial V_1}{\partial x}(x=x_0)=\varepsilon_x\frac{\partial V_2}{\partial x}(x=x_0) \qquad (4)$$

Setting the solutions (2) and (3) in (4), one obtains

$$-\sum_k a_k k\frac{\pi}{h}\sin(k\frac{\pi}{h}y)=-\sum_k a_k\varepsilon_x\sqrt{\frac{\varepsilon_y}{\varepsilon_x}}k\frac{\pi}{h}\sin(k\frac{\pi}{h}y) \quad (5)$$

This is identically satisfied, provided

$$\boxed{\varepsilon_x\varepsilon_y=1}, \qquad (6)$$

which is exactly the condition set in the proposition.

4. *Numerical example*

A planar electrostatic problem consisting of two infinitely long parallel cylinders in free space with given potentials on them is considered (Fig. 3). The potential values of the cylinders have been chosen so that the potential in infinity is zero and the sum of the charges of the two cylinders is also zero. The problem has an analytical solution.

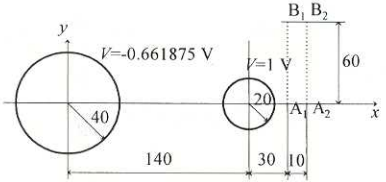

Fig 3. Two infinitely long, parallel cylinders

A finite element mesh shown in Fig. 4 consisting of 8-noded second order elements has been generated. It models a half of the model due to symmetry. The external four finite element layers are filled by PMLs with the relative permittivity equal to 0.005 in the normal and to 200 in the cross directions. The resulting equipotential lines are shown in Fig. 5. The x- and y-components of the electric field intensity along the lines A_1-B_1 and A_2-B_2 shown in Fig. 3 are compared to the analytical solutions in Figs. 6 and 7. The line A_2-B_2 is along the interface to the PML. The agreement is satisfactory, indicating that the PMLs do in fact model free space.

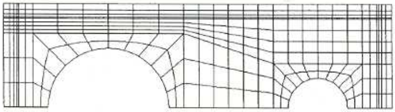

Fig. 4. Finite element mesh

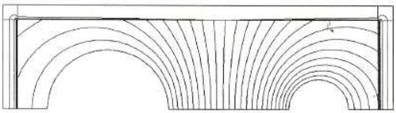

Fig. 5. Equipotential lines obtained from the finite element solution

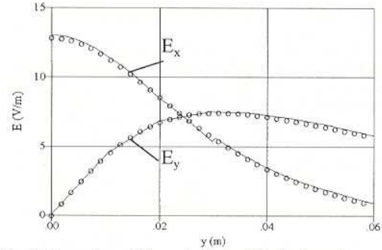

Fig. 6. Comparison of the analytical and finite element solution along the line A_1-B_1.
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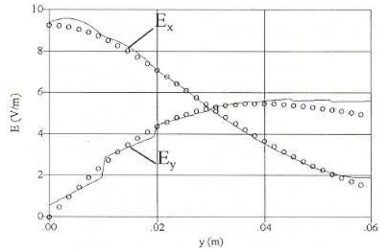

Fig. 7. Comparison of the analytical and finite element solution along the line A_2-B_2.
o o o o o: analytical, __________: FEM

Conclusions

The PML method presented can be used by any FEM code capable of treating anisotropic materials. Further investigations are necessary for the optimal setting of the material properties and the thickness of the PMLs.

References

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