Technical Article

Generation of Tangential Vector Finite Elements

Introduction

Edge-based vector finite elements have a number of advantages over scalar elements for modeling electromagnetic field vectors. First and foremost, they provide accurate and reliable solutions free from the spurious modes and numerical instabilities that plague conventional scalar elements. Second, they have a simple form with which it is easy to enforce the tangential continuity of field components. And third, they form a consistent numerical approximation for tangentially continuous field quantities such as **A**, **E** and **H** when scalar finite elements are used to approximate the associated scalar potential functions. This third advantage allows the particular solution to be computed easily for with magnetostatic and eddy current problems [1]. They have, however, one major disadvantage: their rate of convergence is low. The rate of convergence with edge elements is only first-order since only constants are approximated with these elements in the range space of the curl operator.

To achieve higher rates of convergence, one must employ higher-order polynomials in the approximation. However, the procedures required to do this are not obvious. One must be careful to maintain the tangential continuity of the field, to generate consistent polynomial approximations, and to ensure the completeness of the resulting polynomial spaces, all while increasing the polynomial order. While all of these requirements have been met by polynomials in the literature, the methods for generating these polynomials are complicated. As a result, although high-order tangential vector finite elements are now widely used, the procedures used to generate them are not well understood.

This paper presents a new, more easily understood derivation of vector elements than is currently found in the literature [2,3,4]. Our approach is to begin by writing a general two-component, two-dimensional vector as a linear combination of scalar elements. By enforcing tangential continuity of the vector at element boundaries, we generate elements having the proper interelement continuity conditions but not necessarily the correct, complete polynomial order. To determine the correct order, we focus on the order of the polynomials in the resulting range space of the curl operator, rather than considering just the order of the polynomials in the approximation itself. Additional basis functions are added, each generating a higher-order polynomial in the range space, to insure the completeness of the high-order space.

An interesting byproduct of this approach is a new expression for zero-order edge elements. It turns out that zero-order elements can be expressed in terms of two types of functions: one which has zero curl, the other a constant curl. These new basis functions provide a natural way to separate the nullspace of the curl operator from its nontrivial range. Employing tree-cotree methods then allows us to determine a unique solution much as is done with ordinary edge elements.

First we begin with the new derivation of the H^1(curl) element that is complete to first-order in the range space of the curl operator. Next, we apply this procedure to generate a new zero-order element H^0(curl). Use of the resulting partitioned edge element space is then illustrated by an wave propagation example. Lastly, using the lessons learned from the two lower order spaces, we derive a new second-order complete H^2(curl) element.

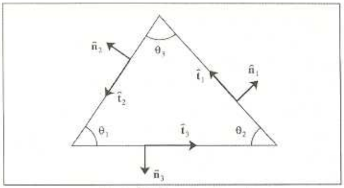


Fig. 1 2D triangular element

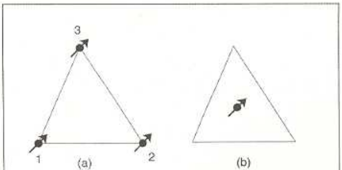


Fig. 2 (a) Linear vector element (b) Constant vector element

A General Procedure

Let $\mathbf{F}=(F_x,F_y)$ be a two-dimensional vector field over a triangular element Δ. In the following, we use the labelling in Figure 1, where θ_i denotes the local vertex angles, $\mathbf{n}_i$ the unit normals, and $\mathbf{t}_i$ the unit tangent vectors. As is well known, a scalar field of complete polynomials of integer order is correctly represented by nodal elements. We will therefore attempt to represent each Cartesian component of **F** individually by nodal elements. Each nodal variable thus becomes a vector variable $\mathcal{F}_i$ as pictured in Figure 2(a) and the vector **F** becomes

$$\mathbf{F} = \sum_{n=1}^{3} \mathcal{F}_n \zeta_n \qquad (1)$$

where ζ_n, n = 1,2,3 are the homogeneous coordinates in the triangle. If the vector nodal variables $\mathcal{F}_i$ were set to have the same value in two adjacent elements across their common boundary, the resulting field **F** would have both tangential and normal continuity across this boundary. This is too much continuity both physically and mathematically. Physically, a field variable such as the magnetic vector potential **A**, or the electric field **E**, or the magnetic field **H** has only tangential continuity and may have discontinuous normal components.

Mathematically, requiring both tangential and normal continuity of low-order vector polynomials makes incorrect approximations of the nullspace of the curl operator. These incorrect nullspace approximations result in spurious modes in wave problems and mesh dependent instabilities in low frequency deterministic problems.

Now let us consider the "connection rules" required to assemble individual elements into a mesh. To do so, consider two elements and resolve each nodal vector in (1) into its components along the two adjacent triangle sides. For example, $\mathcal{F}_1$ is the interpolated value of **F** at vertex 1, which is also the intersection of sides 2 and 3. Simple geometry yields for $\mathcal{F}_1$

$$\mathcal{F}_1 = \mathcal{F}_1^{(2)} \csc\theta_1 \hat{n}_3 - \mathcal{F}_1^{(3)} \csc\theta_1 \hat{n}_2 \qquad (2)$$

where $\mathcal{F}_1^{(2)}$ and $\mathcal{F}_1^{(3)}$ are the tangential components of $\mathcal{F}_1$ along sides 2 and 3. Similar expressions to (2) are obtained for $\mathcal{F}_2$ and $\mathcal{F}_3$. Substituting these results into equation (1), **F** is expressed within Δ as a linear combination of six linearly independent functions $\mathbf{e}_1^{(1)}, ..., \mathbf{e}_6^{(1)}$ where the $\mathbf{e}_6^{(1)}$ are given in Table 1a, along with their curls. The symbols $\alpha_i^{(2)}$, i=1, ..., 6 in this table denote the familiar quadratic shape functions discussed in [5] and l_i denotes the length of side i.

$\mathbf{e}_1^{(1)} = -\alpha_2^{(1)} \csc\theta_2 \hat{\mathbf{n}}_3$	$\nabla \times \mathbf{e}_1^{(1)} = h_1^{-1} \hat{\mathbf{z}}$
$\mathbf{e}_2^{(1)} = \alpha_3^{(1)} \csc\theta_3 \hat{\mathbf{n}}_2$	$\nabla \times \mathbf{e}_2^{(1)} = h_1^{-1} \hat{\mathbf{z}}$
$\mathbf{e}_3^{(1)} = -\alpha_3^{(1)} \csc\theta_3 \hat{\mathbf{n}}_1$	$\nabla \times \mathbf{e}_3^{(1)} = h_2^{-1} \hat{\mathbf{z}}$
$\mathbf{e}_4^{(1)} = \alpha_1^{(1)} \csc\theta_1 \hat{\mathbf{n}}_3$	$\nabla \times \mathbf{e}_4^{(1)} = h_2^{-1} \hat{\mathbf{z}}$
$\mathbf{e}_5^{(1)} = -\alpha_1^{(1)} \csc\theta_1 \hat{\mathbf{n}}_2$	$\nabla \times \mathbf{e}_5^{(1)} = h_3^{-1} \hat{\mathbf{z}}$
$\mathbf{e}_6^{(1)} = \alpha_2^{(1)} \csc\theta_2 \hat{\mathbf{n}}_1$	$\nabla \times \mathbf{e}_6^{(1)} = h_3^{-1} \hat{\mathbf{z}}$

Table 1a. Linear basis functions and curls for first-order complete element.

By examining the expressions for $\mathbf{e}_1^{(1)}$ and $\mathbf{e}_2^{(1)}$ we see that these basis functions have non-zero tangential components along side 1 and zero tangential components along the other two sides. We say that these two functions "interpolate the tangential component of **F** along side 1". Similarly, $\mathbf{e}_3^{(1)}$ and $\mathbf{e}_4^{(1)}$ interpolate the tangential component of **F** along side 2, and $\mathbf{e}^5_{(1)}$ and $\mathbf{e}^6_{(1)}$ interpolate the tangential component of **F** along side 3. We also note that all six basis functions have non-zero normal components along the three sides.

Using this fact, we employ the following connection rule for a multiple element mesh: set the scalar parameters associated with the two basis functions in Table 1a in adjacent elements equal along the common side. The result is tangential continuity of **F** throughout the mesh without enforcing normal continuity. In fact, the space described by these basis functions includes every tangentially continuous function that is element-wise linear in x and y. A subset of these functions are functions of the form $\mathbf{F}=\nabla\varphi$. Since $\nabla\times\nabla\varphi=0$, this subset of functions provides the nullspace of the curl operator. It can be shown that this subspace is spanned by $\nabla\Psi_i$ where $\nabla\Psi_i$ are global, piecewise quadratic basis functions [6].

The second column of Table 1a reveals that the curls of the basis functions $\mathbf{e}_1^{(1)}, ..., \mathbf{e}_6^{(1)}$ are all constants. Thus, using these six functions, and only these six functions, provides an approximation for $\nabla\times\mathbf{F}$ that is complete to only zero order polynomials (constants). To obtain a linear approximation for $\nabla\times\mathbf{F}$, we need to add two basis functions whose curls are linear in ζ_i. The fact that two basis functions must be added to obtain a complete linear polynomial in the range space follows because a linear polynomial requires three basis functions, only one of which (constants) is provided by the first six functions $\mathbf{e}_i^{(1)}$.

$\mathbf{e}_7^{(1)} = \alpha_1^{(2)} \hat{\mathbf{n}}_1$	$\nabla \times \mathbf{e}_7^{(1)} = 4 l_1^{-1} (\zeta_3 - \zeta_2) \hat{\mathbf{z}}$
$\mathbf{e}_8^{(1)} = \alpha_2^{(2)} \hat{\mathbf{n}}_2$	$\nabla \times \mathbf{e}_8^{(1)} = 4 l_2^{-1} (\zeta_1 - \zeta_3) \hat{\mathbf{z}}$

Table 1b. Quadratic basis functions and curls for first-order complete element.

The new basis functions may take a number of forms. Table 1b lists one possibility. In this case, the functions $\mathbf{e}_7^{(1)}$ and $\mathbf{e}_8^{(1)}$ have zero tangential components on all three sides. Thus, these functions do not disturb the above tangential continuity connection rule and provide local basis functions associated only with the one triangle. This is the first-order complete vector element denoted as "H^1(curl)" and originally presented in [7]. It has been shown to produce reliable solutions without spurious modes in electromagnetic field problems.

Edge Elements

Using the above derivation of H^1(curl) as a guide, let us now construct the zero-order element. To do this, let us return to the single triangle case and use constant functions to express each component of **F** in place of the linear functions used before. There are two degrees of freedom assigned to **F** in this triangle, which we symbolize in Figure 2b with an interior node and an arrow. Simple geometry provides

$$\mathbf{F} = F_1 \csc\theta_3 \hat{n}_2 - F_2 \csc\theta_3 \hat{n}_1 \qquad (3)$$

where F1 and F2 are the tangential components of **F** along sides 1 and 2 of the triangle. In this way, **F** is written as a linear combination of two linearly independent basis functions $\mathbf{e}_1^{(0)}$ and $\mathbf{e}_2^{(0)}$ defined along with their curls in Table 2a.

$\mathbf{e}_1^{(0)} = \csc\theta_3 \hat{\mathbf{n}}_2$	$\nabla \times \mathbf{e}_1^{(0)} = 0$
$\mathbf{e}_2^{(0)} = -\csc\theta_3 \hat{\mathbf{n}}_1$	$\nabla \times \mathbf{e}_2^{(0)} = 0$

Table 2a. Low-order nullspace basis functions and curls.

Note that $\mathbf{e}_1^{(0)}$ has unity tangential component along side 1 and zero tangential component along side 2. Correspondingly, $\mathbf{e}_2^{(0)}$ has unity tangential component on side 2 and zero tangential component along side 1. Both basis functions have non-zero tangential components along side 3. Since the curls of both $\mathbf{e}_1^{(0)}$ and $\mathbf{e}_2^{(0)}$ are zero, the range of the curl operator is not complete even to zero-order polynomials. To span the range space to the lowest (zero) order, we must add a third basis function, $\mathbf{e}_3^{(0)}$, which has a constant non-zero curl. The rules of differentiation imply that such a function has linearly varying components. Additionally, to preserve the interpolatory nature of $\mathbf{e}_1^{(0)}$ and $\mathbf{e}_2^{(0)}$, $\mathbf{e}_3^{(0)}$ must be constructed such that it has zero tangential components along sides 1 and 2. The function $\mathbf{e}_3^{(0)}$ defined in Table 2b satisfies these requirements.

$\mathbf{e}_3^{(0)} = \zeta_2 \csc\theta_2 \hat{\mathbf{n}}_1 - \zeta_1 \csc\theta_1 \hat{\mathbf{n}}_2$	$\nabla \times \mathbf{e}_3^{(0)} = (l_1^{-1}\csc\theta_2 + l_2^{-1}\csc\theta_1)\hat{\mathbf{z}}$

Table 2b. Linear low-order basis function and its curl.

The geometric factors $\csc\theta_2$ and $\csc\theta_3$ are included to provide a unity constant tangential component along side 3. Thus, this function interpolates to the tangential component on side 3.

Many readers will note that $\mathbf{e}_3^{(0)}$ has a familiar appearance. Indeed, simple geometry shows that

$$\nabla\zeta = - \frac{\hat{n}_i}{h_i} \qquad (4)$$

where h_i is the triangle altitude to vertex i.
It follows that

$$\csc\theta_1 = - l_2 \nabla\zeta_3 = - l_3 \nabla\zeta_2 \qquad (5)$$

with similar expressions for $\csc\theta_2$ and $\csc\theta_3$. Thus, the elements of the basis $\{\mathbf{e}_1^{(0)}, \mathbf{e}_2^{(0)}, \mathbf{e}_3^{(0)}\}$ may be written as

$$\mathbf{e}_1^{(0)} = \mathbf{u}_1 - \sin\theta_1 \csc\theta_3 \mathbf{u}_3 \qquad (6a)$$

$$\mathbf{e}_2^{(0)} = \mathbf{u}_2 - \sin\theta_2 \csc\theta_3 \mathbf{u}_3 \qquad (6b)$$

$$\mathbf{e}_3^{(0)} = \mathbf{u}_3 \qquad (6c)$$

where $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is a second basis defined in Table 3.

$\mathbf{u}_1 = l_1(\zeta_2 \nabla\zeta_3 - \zeta_3 \nabla\zeta_2)$	$\nabla \times \mathbf{u}_1 = (l_2^{-1}\csc\theta_3 + l_3^{-1}\csc\theta_2)\hat{\mathbf{z}}$
$\mathbf{u}_2 = l_2(\zeta_3 \nabla\zeta_1 - \zeta_1 \nabla\zeta_3)$	$\nabla \times \mathbf{u}_2 = (l_3^{-1}\csc\theta_1 + l_1^{-1}\csc\theta_3)\hat{\mathbf{z}}$
$\mathbf{u}_3 = l_3(\zeta_1 \nabla\zeta_2 - \zeta_2 \nabla\zeta_1)$	$\nabla \times \mathbf{u}_3 = (l_1^{-1}\csc\theta_2 + l_2^{-1}\csc\theta_1)\hat{\mathbf{z}}$

Table 3. Edge element basis functions and their curls.

The basis functions $\mathbf{u}_1$ are the 2D form of the usual "edge element" functions in the literature [8]. Equation (6) shows that the two sets of basis functions $\mathbf{e}_i$ and $\mathbf{u}_i$ span the same space, which we denote as H^0(curl).

While the functions $\mathbf{e}_1^{(0)}$ nicely separate the range space into the nullspace of the curl operator and its complement, it at first appears to require much more complicated connection rules. The usual edge elements simply require edge values to be set the same along adjacent sides. With the functions $\mathbf{e}_i^{(0)}$ this is not so easy: we need to enforce the tangential continuity of **F** across element boundaries, but the functions are not interpolatory along all three sides. All three are interpolatory along edges 1 and 2, so that two adjacent elements sharing edges 1 or 2 may simply share the common parameter F_1 or F_2. However, for shared edges assigned as local edge 3 for either or both triangles, properly enforcing the connection rule is awkward. The tangential component of **F** on side 3 is a linear combination of the local parameters F_1 and F_2 and tangential continuity requires enforcing a linear relationship between F_1 and F_2 and one or both of the tangential parameters in the second element. While this is possible, a better method is available.

An elegant method of enforcing continuity is obtained by recognizing that

$$\mathbf{e}_1^{(0)} = - l_1 \nabla\zeta_2 \text{ and } \mathbf{e}_2^{(0)} = - l_2 \nabla\zeta_1 \qquad (7)$$

That is, the nullspace basis functions $\mathbf{e}_1^{(0)}$ and $\mathbf{e}_2^{(0)}$ are associated with the gradients of node-based scalar functions within the same triangle.

Equation (7) suggests that we may obtain tangential continuity of the nullspace basis functions by simply taking the gradient of a scalar. To do this, consider a continuous scalar function φ defined on a mesh using N node-based finite elements

$$\varphi = \sum_{n=1}^{N} \varphi_n \Psi_n \qquad (8)$$

where Ψ_n are global linear scalar basis functions and φ_n are interpolated values of φ at the nodes. Note that in any particular element, Ψ_n is just ζn. Since φ is continuous, its gradient

$$\nabla\varphi = \sum_{n=1}^{N} \varphi_n \nabla\Psi_n \qquad (9)$$

is tangentially continuous. Also, since Ψ_n are constructed from the patch of ζ_n interpolating to 1 in all the elements surrounding node n, according to (7) the $\nabla\Psi_n$ may be constructed from the functions $\mathbf{e}_1^{(0)}$ and $\mathbf{e}_2^{(0)}$.

Taking the gradient of a first-order scalar finite element approximation therefore creates the desired piecewise constant and tangentially continuous vector field. However, to be precise, we note that (9) contains one too many degrees of freedom. One of the $\nabla\Psi_n$ can be written as a linear combination of the other N-1. To see this, consider equation (9) for the case φ=constant. In this case, (9) yields

$$\sum_{n=1}^{N} \varphi_n \nabla\Psi_n = 0 \qquad (10)$$

Thus the $\nabla\Psi_n$ are linearly dependent. This is the only dependency since no function other than φ=constant has a zero gradient. Therefore, the piecewise constant field is spanned by N-1 independent functions of the form (7).

We remark here for future use that a similar argument to the above shows that the dimension of the gradient space for high-order scalar elements is one less than the number of nodes in the element for a single element, or one less than the number of nodes in the mesh for an assembly of elements.

As first noted by Albanese and Rubinacci [9], the dependent and independent degrees of freedom of the curl operator may be determined by decomposing a zero-order edge element mesh into a tree and a cotree. A tree of the mesh is defined as a set of edges which visits all nodes, but does not contain any closed loops; the cotree space is defined as the subspace of the edge-element space in which all edge variables on the tree are set to zero. Note that the number of branches in a tree is (N-1), which equals the dimension of the nullspace of the curl operator. As explained in [8], the cotree space is disjoint from the nullspace and the nullspace and cotree space together comprise the entire edge element space.

We are thus led to the following prescription for using the basis functions $\mathbf{e}_i^{(0)}$. Let $\mathbf{e}_1^{(0)}$ and $\mathbf{e}_2^{(0)}$, the curl nullvectors, be defined in terms of the scalar variable φ and let $\mathbf{e}_3^{(0)}$ be associated with the edges of the cotree. Continuity of φ is accomplished in the usual manner of first-order scalar finite elements, and continuity of $\mathbf{e}_3^{(0)}$ in the usual manner of zero-order edge elements. However, $\mathbf{e}_3^{(0)}$ is only defined on the cotree edges of the mesh. One then sets up and solves the finite element equations for the variables φ and $\mathbf{e}_3^{(0)}$. The number of variables in this system is identical to the number of variables in $\mathbf{u}_i^{(0)}$ since φ generates a matrix of rank (N-1) which is the same as the number of tree branches.

To illustrate the use of the basis functions $\mathbf{e}_i^{(0)}$, consider the driven waveguide problem in Figure 3. Here we wish to compute the magnetic field **H** as it travels left to right in a TE_1 mode in a parallel plate waveguide. Figure 3(a) depicts a meshed rectangular region. A tree for this mesh appears in Figure 3(b). Solving for the magnetic field **H** using a node-based representation for φ, and local basis functions $\mathbf{e}_3^{(0)}$ for the cotree space gives the magnetic field solution presented in Figure 3(c). Absorbing boundary conditions were enforced on the left and right boundaries to obtain this solution. Solutions obtained by this process are observed to be both accurate and reliable.

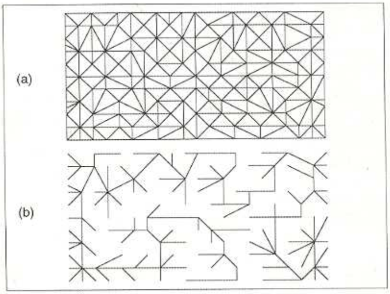


Fig. 3 (a) 2D Delaunay mesh (b) Mesh tree

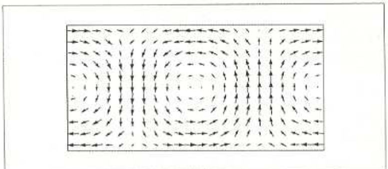


Fig.3 (c) Magnetic field for TE_1 parallel plate waveguide mode

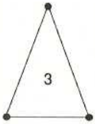
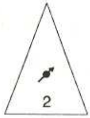
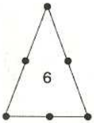
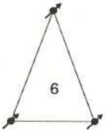
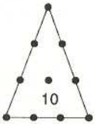
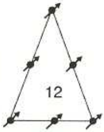
	$P_{q+1}(\Delta)$	dim $\nabla P_{q+1}(\Delta)$	$T_q(\text{curl},\Delta)$	$\dim(\hat{z}P_q(\Delta))$	$\dim(H^q(\text{curl},\Delta)) = \dim(R) + \dim(K)$	number of additional domain variables	additional domain variables
q	$\frac{1}{2}(q+2)(q+3)$	$\frac{(q+1)(q+4)}{2}$	$(q+1)(q+2)$	$\frac{(q+1)(q+2)}{2}$	$(q+1)(q+3)$	$q+1$	
0		3 - 1 = 2		1	2 + 1 = 3	3 - 2 = 1	
1		6 -1 = 5		3	5 + 3 = 8	8 - 6 = 2	
2		10 - 1 = 9		6	9 + 6 = 15	15 - 12 = 3	

Fig. 4 Element construction

The H^2(curl) element

Figure 4 summarizes the previous construction for the H^1(curl) and H^0(curl) finite element spaces. In addition, it provides a basis for constructing the second-order complete space, denoted "H^2(curl)". The development of each order is read from left to right, with the rows q=0, q=1, and q=2 referring to the H^0(curl), H^1(curl), H^2(curl) spaces, respectively. The row following "q" provides the dimension of the associated function space. Several symbols not used earlier in the text appear along the top row of this figure. Here $P_{q+1}(\Delta)$ denotes a single scalar element constructed from the usual (q+1)-th order polynomials in x and y [5]. The multi-element version of this space is spanned by a set of node-based global basis functions Ψ_i, and the gradients of these functions span a tangentially continuous "gradient" space ∇P_{q+1}. $T_q(\text{curl},\Delta)$ denotes the incomplete, tangentially continuous, intermediate space which contain the gradient space, discussed above for q=0 and q=1. Finally, "R" and "K" stand for range and nullspace of the curl operator, respectively. Before discussing H^2(curl), let us summarize the pattern established in constructing H^0(curl) and H^1(curl) as portrayed in Figure 4. In order to adequately model the nullspace of the curl operator, the first step is to create an intermediate space by associating a vector variable with each node of a scalar Lagrangian element. Scalar polynomial spaces and the resulting dimension of each included gradient space are shown in columns 2 and 3 of Figure 4. The intermediate space $T_q(\text{curl},\Delta)$ which contains the gradient space is presented symbolically in column 4 of this figure. Next, each vector variable of the intermediate space is resolved into tangential and normal components along the element sides to enforce tangential but not normal continuity in element assembly. For q=0 and q=1, the resulting spaces are known to contain the gradient space ∇P_{q+1}. The final step is to create additional linearly independent basis functions so that the range of the curl operator is a complete polynomial. These functions are constructed to leave undisturbed the tangential interpolation property of the intermediate space $T_q(\text{curl},\Delta)$. While the polynomials used are not unique, the number required can be deduced by inspecting the curls of the existing basis functions as discussed earlier. Alternatively, the overall required dimension of the H^q(curl) space can be computed as the sum of the dimensions of the curl range space and curl nullspace [10].

The dimension of the complete polynomial range space is shown in column 5 of Figure 4. The resulting calculated dimension for the space $H^q(curl)$ is given in column 6. Thus, the dimension of the entire space is known, as well as the dimension of the intermediate space $T_q(curl,\Delta)$ constructed to provide tangential continuity. We simply need to subtract column 4 from column 6 to find the number of additional basis functions required. This is (q+1) and is given in column 7. The final column of Figure 4 pictures the additional basis functions selected to make each space complete. For $H^0(curl)$, the linear basis function $\mathbf{e}_3^{(0)}$ is added as depicted in the figure. Similarly, quadratic basis functions $\mathbf{e}_5^{(1)}$ and $\mathbf{e}_6^{(1)}$ are added to the q=1 row. These are represented as perpendicular vectors on two of the triangle edges.

We will now construct $H^2(curl)$ based on the above principles. A single node-based element of cubic order is pictured in column 1 of row 5 of Figure 4. This element has 10 degrees of freedom, each one associated with a node. The resulting gradient space contains 10-1=9 degrees of freedom. The quadratic intermediate space $T_2(curl,\Delta)$ is presented symbolically in column 4. This element is formed by associating a vector variable with each node in a node-based quadratic Lagrangian element.

This single element is now modified by connection rules to be valid over an entire mesh. The requirement is again to enforce tangential continuity, but not normal continuity. To accomplish this, the three vertex vector variables are resolved along the triangle edges. The degrees of freedom associated with these tangential components are shared between adjacent elements in the same way as those in the $H^1(curl)$ space. However, the $H^2(curl)$ element presents a new feature: the midpoint vector variables contain both normal and tangential components. Here the tangential degrees of freedom along a common edge of adjacent triangles are shared, but the normal degrees of freedom are not. The basis function arising from each midpoint normal component is associated only with the parent triangle.

Next, to achieve a quadratic rate of convergence, the curl range space of the $H^2(curl)$ element must be spanned by a complete z-directed quadratic polynomial. A quadratic polynomial in x and y has 6 degrees of freedom; this is therefore the required dimension for the curl range space.

Since the intermediate space $T_2(curl,\Delta)$ has a nullspace dimension of 9, the total number of degrees of freedom required for $H^2(curl)$ is dim(K)+dim(R)=9+6=15. Since the number of degrees of freedom in the intermediate space is 12, three remaining basis functions need to be added. These three functions may be constructed in many ways. However, they must be linearly independent of the new and existing polynomials, and their polynomial order must be 3 so that their curls will supply quadratic terms to span the curl range space. One possibility is sketched in the last column of Figure 4. In this case, each of the three vector basis functions is associated only with this triangle and is formed by multiplying the $P_3(\Delta)$ scalar basis function associated with the pictured node times the normal vector to that side. This construction provides linear independence and does not disturb the tangential interpolation property of the intermediate space.

The 15 basis functions thus derived, along with the connection rules described above, defines the $H^2(curl)$ space. This space enforces tangential but not normal continuity, avoids the problem of spurious modes, and is a full order more accurate than the published $H^1(curl)$ element in [7].

Conclusion

We have shown that tangential vector finite elements are easily derived by beginning with the usual scalar element, forming a two-component vector from this scalar, and then restricting the vector to have tangential but not normal continuity. The resulting intermediate vector space contains the nullspace of the curl operator, and therefore avoids the problem of spurious modes, but does not provide complete polynomials in the range space of the curl operator. To achieve completeness in the range space, additional domain space variables must be added. The number of additional domain variables equals the dimension of the curl nullspace plus its range minus the dimension of the intermediate space.

A byproduct of this new derivation is a better understanding of zero-order edge elements and their relationship to the trees and cotrees of the finite element mesh. We have shown in a new way that the nullspace of these elements is associated with a first-order finite element scalar function and that the complement of this nullspace is associated with the cotree edges in the mesh. Solutions of waveguide problems demonstrate the validity and accuracy of the new approach.

While many of the results in this paper exist in the literature, it is hoped that the new approach presented here will help to clarify the derivation of high-order tangential vector finite elements. These elements are endlessly fascinating, and much future work depends on our understanding of these complex yet highly important elements.

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XIV EPNC'96

14th Symposium on Electromagnetic Phenomena in Nonlinear Circuits will be held in Poznan, Poland, on 29 May to 1 June 1996. The aim of the EPNC Symposium is to discuss the development in problems of the analysis and synthesis of nonlinear electric and magnetic circuits as well as to provide the forum for presentation of recent applications of nonlinear phenomena in electrical engineering. Topics of interest to the conference include:

- Ferromagnetic and magnetics circuits
  - Theory and application of nonlinear phenomena in ferromagnetic elements and structures containing such elements,
  - Electromagnetic filed calculation for nonlinear problems
  - Modelling of material properties and the numerical treatment of hysteresis, anisotropy and permanent magnets.

- Semiconductors and nonlinear electric circuits
  - Power electronic devices and converters,
  - Modelling and simulation of nonlinear electronic elements and systems with these elements,
  - Chaos in electric circuits,
  - Nonlinear digital circuits.

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