Technical article

Electromagnetic energy and multiphysics modelling

***Abstract* — This paper presents an energy-based theory of electromagnetism. The fundamental postulate is presented under the form of a diagram with interconnected energy reservoirs. The completely covariant equations obtained by stating energy conservation in this diagram are shown to be a combination of Maxwell's equations with the constitutive laws of the material. They are established in this paper under the assumption of an absolute time, but a relativistic extension can be established as well. This energy-based formulation clarifies several issues related to dissipative and coupled phenomena in magnetic materials, dielectrics and conductors. The last part of the article is devoted to show how the energy-based approach can be exploited in numerical computations.**

I. Introduction

Opening a textbook on electromagnetism, it is likely that the first set of equations presented will be Maxwell's equations

$$\begin{aligned} \operatorname{curl}\mathbf{h} - \partial_t \mathbf{d} &= \mathbf{j} && (1)\\ \operatorname{curl}\mathbf{e} + \partial_t \mathbf{b} &= 0 && (2)\\ \operatorname{div}\mathbf{b} &= 0 && (3)\\ \operatorname{div}\mathbf{d} &= \rho^Q && (4) \end{aligned}$$

complemented by a set of constitutive relations of the form

$$\mathbf{b} = \mu\,\mathbf{h} \quad , \quad \mathbf{d} = \varepsilon\,\mathbf{e} \quad , \quad \mathbf{j} = \sigma\,\mathbf{e} \qquad (5)$$

with the mention that the first set are universal (always valid) and the second one contains any relation one would need to 'close the system' and be able to solve it. Electromagnetism is seen this way as a matter of fields whose evolution in time and distribution in space are ruled by partial differential equations (PDE) and constitutive relations. There is no place in this usual setting for any energy considerations.

Further in the same book, some energy related notions are however likely to be introduced. The magnetic energy, for instance, is usually defined as a functional of $\mathbf{b}$ or $\mathbf{h}$ (or even both). Different materials will be considered, starting with the simplest medium (vacuum) and proceeding in a bottom-up fashion towards more complex materials : linear, anisotropic, nonlinear, etc. Not for long however, because the definitions become quickly rather technical and fall outside the scope of a general monography.

Classical presentations of the theory of electromagnetism leave thus the impression that energy aspects are by-products of the field theory, somehow accessory and difficult to exploit. The principles of Thermodynamics are however universal and they must apply to electromagnetic phenomena also. Maxwell's equations say actually something yet about energy conservation, but they do so in a way that makes is impossible to disentangle the different energy flows in presence. Moreover, classical presentations of the theory leave unanswered fundamental questions like

- What are the state variables in an electromagnetic system ?
- How are magnetic and electric energy defined in the general case ?
- What are the possible dissipation mechanisms ?
- How is magnetic energy converted into electric energy ?
- How is electromagnetic energy converted into other forms of energy ?
- etc.

Those shortcomings are particularly hampering when one deals with the modelling of problems involving the computation of local electromagnetic forces (energy conversion), magnetic hysteresis (energy dissipation), magnetostriction (both), multiphysics couplings in general. In such problems, it is necessary to dispose of a theory of electromagnetism where energy aspects are involved from the beginning and throughout.

After pursuing theoretical investigations in those domains, and accumulating along the way pieces of knowledge about how energy behaves in electromagnetic systems, a big picture has eventually, and somewhat unexpectedly, formed that gives rise to an energy-based theory of electromagnetism [1]. This theory can be presented pictorially under the form of an energy flow diagram. Stating conservation of energy in this diagram entails more governing equations than the classical theory and gives clear answers to the questions listed above. Being expressed in integral form, instead of by a set of PDE's, those governing equations can be derived straightforwardly in arbitrary coordinate systems, i.e. they are completely covariant. The theory is established in this paper under the assumption of an absolute time, which is sufficient for engineering purposes, but a relativistic extension can be established as well and will be presented elsewhere.

From the point of view of numerical computations, the energy-based approach clarifies issues like e.g. electromagnetic forces and vector hysteresis modelling. It gives all terms in weak formulations a clear physical interpretation, and provides a sound framework where the notions of duality and complementary formulations inscribe naturally. Finally it provides operative concepts to deal in a consistent way with coupling terms in multiphysics problems and lumped parameters in reduced models.

The article is organized as follows. The mathematical ingredients are introduced in Sect. II. A basic knowledge of Differential geometry is assumed. Notions like manifold, tangent space, co- and contravariant quantities, pull-back, metric, etc... are indeed used but not defined extensively. Since those notions can be understood intuitively, the unfamilar reader should be able to follow the developments anyway. In Sect. III, the energy-based theory is developed with an absolute time. The mathematical implications of the postulated energy diagram are derived systematically and some physical interpretations are discussed in Sect. IV. Finally, Sect. V gives a few examples how the energy-based theory can be exploited in numerical computations.

II. Theoretical setup

A. Function, map, operator

The notation

$$f : X \in M \subset N \mapsto x = f(X) \in D \subset E \tag{6}$$

for a function f (or a map, or an operator) is first introduced. The sets $M \subset N$ and $D \subset E$ are respectively the domain and the codomain of the function. The generic point X of the domain is the variable of the function. The point x of the codomain is the image of X, also called the value of the function at X. The notations $D \equiv f(M)$ and $M \equiv f^{-1}(D)$ can be used when useful. If f is regarded as an operator rather than a function, one prefers writing $x = f\,X$ instead of $x = f(X)$. All elements of the exhaustive definition (6) are not always necessary, and shortened notations are used whenever no confusion is possible.

B. Differential forms

When the theory of Electrodynamics took its present form around the beginning of the XXth century, electromagnetic fields were represented by tensors, i.e. array of components related to a fixed (local or global) reference frame. Modern theories of Electromagnetism, however, tend to view electromagnetic fields as being by nature differential forms of various degrees.

To start with, let us call $p-$sub-domains any smooth $p-$dimensional subset of a given domain: 0–sub-domains are points, 1–sub-domains are smooth curves, 2–sub-domains are smooth surfaces, etc... Let $\Lambda_p(M)$ denote the set of all $p-$sub-domains in a domain M.

Differential forms are not directly defined as fields but more fundamentally as linear maps from the $p-$sub-domains to the real numbers. A differential form of degree p, or a $p-$form for short, is a map

$$\tilde{\alpha} : C \in \Lambda_p(M) \mapsto \mathbb{R} \tag{7}$$

with $\tilde{\alpha}(C_1+C_2) = \tilde{\alpha}(C_1)+\tilde{\alpha}(C_2)$. The intuitive understanding of this map is the integration over the $p-$sub-domain C of a field α associated with the map $\tilde{\alpha}$. In order to make the link between the map and the associated field, the case of a 1–form is first considered. Let P be a point in M, and $V \in T_PM$ an arbitrary vector of the tangent space at P. The map

$$\Gamma : \lambda \in [-\epsilon, \epsilon] \subset \mathbb{R} \mapsto C_\epsilon^{P;V} \subset M,$$

with $\Gamma(0) = P$ and $(\partial_\lambda \Gamma)(0) = V$ is the parameterization of a family of curves $C_\epsilon^{P;V} \in \Lambda_1(M)$, with parameter ϵ, going through P and having V as tangent vector at P. Thanks to the linearity of $\tilde{\alpha}$, it can be shown that the limit

$$\lim_{\epsilon\to 0} \frac{1}{2\epsilon}\tilde{\alpha}(C_\epsilon^{P;V}) =< \alpha(P) ; V > \tag{8}$$

exists for all $V \in T_PM$ and does not depend on the parameterization. Its value is the duality product of V with the value at P of a covector field α that is uniquely defined by the map $\tilde{\alpha}$.

The case of a 2–form is treated similarly. Let

$$\Sigma : \lambda, \mu \in [-\epsilon, \epsilon] \subset \mathbb{R} \mapsto S_\epsilon^{P;V,W} \subset M$$

with $\Sigma(0) = P$, $(\partial_\lambda \Sigma)(0) = V$ and $(\partial_\mu \Sigma)(0) = W$ be the parameterization of a family of surfaces $S_\epsilon^{P;V,W}$ going through P and having V and W as tangent vector at P. The limit that defines the associated field β writes

$$\lim_{\epsilon\to 0} \frac{1}{4\epsilon^2}\tilde{\beta}(S_\epsilon^{P;V,W}) =< \beta(P) ; V, W > . \tag{9}$$

Since permuting λ and μ amounts to swapping the vectors V and W, and hence to revert the orientation of the surface, one has $S_\epsilon^{P;V,W} = -S_\epsilon^{P;W,V}$ and consequently $< \beta ; V, W >_P = - < \beta ; W, P >_P$.

This result generalizes as follows. The field α associated with the map (7) is a (multi)linear real-valued completely antisymmetric operator acting on p vector fields arguments, i.e. a $p-$covector field. For this reason, $p-$forms are natural arguments for $p-$fold integrals. Given a $p-$form $\tilde{\alpha}$ and a domain M, one proceeds as follows. The domain M is partitioned into N_ϵ small parallelepipeds $C_\epsilon^{(k)} \in \Lambda_p(M)$ of characteristic dimension ϵ. To each parallelepiped, the $p-$form associates a number $\tilde{\alpha}(C_\epsilon^{(k)})$ and the limit

$$\int_M \tilde{\alpha} = \lim_{\epsilon\to 0} \sum_{k=1}^{N_\epsilon} \tilde{\alpha}(C_\epsilon^{(k)}) \tag{10}$$

exists. It defines the integral of $\tilde{\alpha}$ over M.

Due to their equivalence, both the field α and the map $\tilde{\alpha}$ are called differential forms in practice and the tilde can be dropped. The set of all $p-$forms defined on a domain M is denoted by $\Lambda^p(M)$.

Of course, $p-$forms need specific intrinsic antisymmetry-preserving operators. The antisymmetry-preserving tensor product is the exterior product $\wedge$ (11) and the antisymmetry-preserving spatial derivative is the exterior derivative d (12).

$$\wedge \quad : \quad \Lambda^p(M) \times \Lambda^q(M) \mapsto \Lambda^{p+q}(M) \tag{11}$$

$$\mathrm{d} \quad : \quad \Lambda^p(M) \mapsto \Lambda^{p+1}(M) \tag{12}$$

Complete definitions can be found in any Differential geometry treatise, e.g. [2]. One has in particular

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge \mathrm{d}\beta \tag{13}$$

for all $p-$forms α and β. Exterior derivative obeys also Stokes' theorem

$$\int_M \mathrm{d}\alpha = \int_{\partial M} \alpha \tag{14}$$

where ∂M denotes the boundary of the integration domain M.

C. Euclidean space, proxy fields

An Euclidean space is a space endowed with an Euclidean metric $g_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. This particular choice has for consequence that covariant and contravariant quantities are rendered indistinguishable. Vector analysis is a particularization of Differential geometry to the case of a three-dimensional Euclidean space E^3. The different kinds of fields encountered in Vector analysis are the scalar fields (1 component), the vector fields (3 components) and the tensor fields (9 components). Let $V^p(\Omega), p = 0, 1, 2$ be the sets of those scalar, vector and tensor fields defined on a domain $\Omega \subset E^3$.

Differential geometry, on the other hand, distinguishes a wealthier set of fields. In particular, since there exist four kinds of sub-domains in a three-dimensional domain (points, curves, surfaces and volumes) there exist also four kinds of differential forms. Table 1 summarizes them and gives the associated geometrical map, their physical interpretation and the examples encountered in this paper.

When mapped into E^3, 0–forms and 3–forms get identified with scalar fields, whereas 1–forms and 2–forms are identified with vector fields. These scalar and vector fields are called the proxy fields of their corresponding $p-$form [3]. But both representations are nevertheless not equivalent. The representation of a field in terms of a 1–form for instance still makes sense when

Table 1: *Differential forms of degree 0, 1, 2 and 3 in a 3D space : associated geometrical map, physical interpretation and examples encountered in this paper.*

0–form	Points $\mapsto \mathbb{R}$	scalar function	u
1–form	Curves $\mapsto \mathbb{R}$	circulation density	$\mathbf{a}, \mathbf{e}, \mathbf{h}$
2–form	Surfaces $\mapsto \mathbb{R}$	flux density	$\mathbf{b}, \mathbf{d}, \mathbf{j}$
3–form	Volumes $\mapsto \mathbb{R}$	volume density	$\rho^{\Psi}_M, \rho^{\Psi}_E$

the domain deforms, due to the fundamental definition (7) of a 1–form as a map on the curves. The number associated with a given curve remains unchanged when the curve deforms. On the other hand, Vector analysis provides no rule to drag a vector field along in the deformation of its domain of definition. This distinction between the $p-$forms and their proxy fields is thus crucial when it comes to the definition of electromagnetic forces.

D. Placement map

As for any thermodynamic analysis, a volume control enclosing a given amount of matter must be defined as a reference for establishing an energy balance. This role is played by the so called material manifold, or any subset of it. One is on the other hand familiar with the fact that a large energy density at a certain point, is associated with a large intensity of the associated field. One therefore needs a way to attribute to the field a local intensity, i.e. a norm, in order to express the energy density. It is the role of the metric to define such norms.

The theoretical framework we need relies therefore upon two manifolds : the material manifold M of which each point is associated with a material particle of the continuous medium (e.g. an atom)[1], and a metric space E^3 which is the space in which the motion takes place.

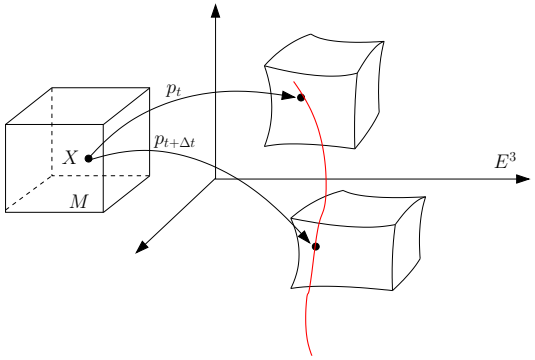

Figure 1: *Placement map at two instants of time and the trajectory of $x = p_t X$ in E^3.*

In order to describe the motion and the deformation of the system, the placement map

$$p_t : X \in M \mapsto x = p_t X \in E^3 \tag{15}$$

is defined, Fig. 1. It attributes a position in E^3 to each material particle $X \in M$ at all instants of time $t \in [t_A, t_B]$. The codomain of the placement map, $\Omega = p_t M$, is the deformed state. On the other hand, the codomain of the map $t \in [t_A, t_B] \mapsto x = p_t X \in E^3$ is the trajectory of a particular material particle X. The velocity field, $\mathbf{v} = \partial_t x$ (vectors in E^3 are denoted with a bold letter), is the field of tangent vectors to all trajectories of the flow at a given instant of time.

The placement p_t is assumed to be regular and invertible at all t. It induces mapping, also noted p_t, of all field quantities defined on M ($p-$forms and other kinds of tensors) to their vector or tensor proxies defined on E^3. Quantities defined on M, i.e. material quantities, are denoted by an uppercase symbol, and their image in E^3 (i.e. their proxy field) by the corresponding lowercase symbol, i.e. $p_t Z = z$. In the following, the symbols Z and z will be used to denote generic fields whose nature is not (or needs not) be specified.

E. Energy functionals

In our theoretical setting, a metric g is available in E^3. Since energy density is related to the local intensity of the field, i.e. to the norm $|z|_x$ of the field z at the point $x \in E^3$, the expression of energies in a system Ω are by definition real-valued functions (functionals) of the fields defined on E^3. Denoting energy functionals with the letter Ψ, one has

$$\Psi : V^p(E^3) \mapsto \mathbb{R}, \tag{16}$$

and

$$\Psi = \int_\Omega \rho^\Psi \quad , \quad \rho^\Psi(z, g) \in V^0(E^3) \tag{17}$$

with the notation ρ^Θ used throughout the paper to represent the volume density of a quantity Θ.

Thanks to the placement map p_t, the norm for fields defined on E^3 induces a norm for the pull-back fields $Z = p_t z$ defined on the material manifold M if one simply states

$$|Z|_X = |p_t z|_X \equiv |z|_{p_t X} = |z|_x. \tag{18}$$

Knowing the expression of the energy of the system $\Psi(z)$ as a function of the proxy fields, the corresponding expression in terms of the associated differential form Z can also be obtained thanks to the placement map. One has

$$\begin{aligned} \Psi(z) &\equiv \int_{\Omega \equiv p_t M} \rho^\Psi(z, g) \\ &= \int_M \left\{p_t \rho^\Psi\right\}(p_t z, p_t g) \\ &= \int_M \left\{p_t \rho^\Psi\right\}(Z, G) \equiv \left\{p_t \Psi\right\}(Z). \end{aligned}$$

This is the formal definition of the function $p_t \Psi : \Lambda^p(M) \mapsto \mathbb{R}$, which is called the pull back of Ψ (Compare with (16)). Note the use throughout the paper of curly braces $\{\}$ for grouping, whereas parenthesis $()$ are reserved to indicate at which point functions are evaluated.

The derivative of the energy density ρ^Ψ with respect to its field argument z is the field $\{\partial_z \rho^\Psi\}(z, g)$ such that

$$\lim_{|\delta z| \to 0} \frac{1}{|\delta z|} \left| \Psi(z + \delta z) - \Psi(z) - \int_\Omega \left\{\partial_z \rho^\Psi\right\}(z, g) \cdot \delta z \right| = 0 \tag{19}$$

where the limit is taken over all sequences of non-zero δz that converge to 0. This is the Fréchet derivative. This derivative allows defining the constitutive laws of the material

$$z^* = \left\{\partial_z \rho^\Psi\right\}(z, g), \tag{20}$$

where the field z^* is the energy dual of z. The induction field $\mathbf{b}$ and the magnetic field $\mathbf{h}$, for instance, constitute a pair of energy dual fields.

[1] The delicate question of the definition of electromagnetic fields in the absence of material support will be discussed elsewhere.

As $\Psi(z) = \{p_t\Psi\}(Z)$ and noting that the integral in (19) can also be written

$$\begin{aligned}\int_\Omega \left\{\partial_z \rho^\Psi\right\}(z,g)\cdot\delta z &= \int_M \left\{p_t\partial_z\rho^\Psi\right\}(p_t z, p_t g)\wedge\delta p_t z\\ &\equiv \int_M \left\{\partial_Z p_t \rho^\Psi\right\}(Z,G)\wedge\delta Z,\end{aligned}$$

one has, by identification, the commutation property

$$\partial_Z p_t = p_t \partial_z. \tag{21}$$

In E^3, $z \in V^p(E^3) \Rightarrow z^* \in V^p(E^3)$ whereas $Z \in \Lambda^p(M) \Rightarrow Z^* \in \Lambda^{n-p}(M)$ in a $n-$dimensional manifold.

F. Co-moving time derivative

Thanks to the concept of $p-$forms, fields can be consistently defined on deforming domains. The time derivative of energy functionals requires however a special care. Considering the functional $\Psi(z)$, with z depending on t, one has

$$\partial_t\Psi(z) = \partial_t\int_\Omega \rho^\Psi(z,g) = \partial_t\int_M \left\{p_t\rho^\Psi\right\}(p_t z, p_t g) \tag{22}$$

by the change of variables induced by p_t. As the material manifold does not depend on time, $\partial_t\int_M = \int_M \partial_t$ holds and the chain rule of derivatives yields

$$\ldots = \int_M \left\{\partial_Z p_t\rho^\Psi\right\}(Z,G)\wedge\partial_t Z + \int_M \left\{\partial_G p_t\rho^\Psi\right\}(Z,G)\,:\partial_t G.$$

Now, the commutation property

$$\partial_t p_t = p_t\,\mathcal{L}_{\mathbf{v}} \tag{23}$$

defines the co-moving time derivative $\mathcal{L}_{\mathbf{v}}$ of any tensorial object. The index $\mathbf{v} = \partial_t p_t X \in V^1(E^3)$ represents the velocity field associated with the placement p_t.

Then, (21) and (23) gives

$$\begin{aligned}\ldots &= \int_M \left\{p_t\partial_z\rho^\Psi\right\}(p_t z, p_t g)\wedge p_t\,\mathcal{L}_{\mathbf{v}}\, z\\ &+ \int_M \left\{p_t\partial_g\rho^\Psi\right\}(p_t z, p_t g)\,:p_t\,\mathcal{L}_{\mathbf{v}}\, g\end{aligned}$$

and finally, by using the inverse change of variable,

$$\partial_t\Psi(z) = \int_\Omega \left\{\partial_z\rho^\Psi\right\}(z,g)\cdot\,\mathcal{L}_{\mathbf{v}}\, z + \int_\Omega \left\{\partial_g\rho^\Psi\right\}(z,g)\,:\,\mathcal{L}_{\mathbf{v}}\, g.$$

The co-moving time derivatives of the proxy fields associated with $p-$forms, $p = 0,1,2,3$, are :

$$\begin{aligned}\mathcal{L}_{\mathbf{v}} f &= \dot f &(24)\\ \mathcal{L}_{\mathbf{v}}\,\mathbf{h} &= \dot{\mathbf{h}} + (\nabla\mathbf{v})\cdot\mathbf{h} &(25)\\ \mathcal{L}_{\mathbf{v}}\,\mathbf{b} &= \dot{\mathbf{b}} - \mathbf{b}\cdot(\nabla\mathbf{v}) + \mathbf{b}\,\mathrm{tr}(\nabla\mathbf{v}) &(26)\\ \mathcal{L}_{\mathbf{v}}\,\rho &= \dot\rho + \mathrm{tr}(\nabla\mathbf{v})\,\rho &(27)\end{aligned}$$

as can be checked component by component. The definitions of the $\dot{\mathbf{a}}$ time derivative and of the products $\mathbf{a}\cdot(\nabla\mathbf{v})$ and $(\nabla\mathbf{v})\cdot\mathbf{a}$ are given in the Appendix. Alternative expressions in terms of Vector analysis operators are also useful :

$$\begin{aligned}\mathcal{L}_{\mathbf{v}} f &= \partial_t f + \mathbf{v}\cdot\{\mathrm{grad}\, f\}, &(28)\\ \mathcal{L}_{\mathbf{v}}\,\mathbf{a} &= \partial_t\mathbf{a} + \mathrm{grad}\,(\mathbf{a}\cdot\mathbf{v}) - \mathbf{v}\times\mathrm{curl}\,\mathbf{a}, &(29)\\ \mathcal{L}_{\mathbf{v}}\,\mathbf{d} &= \partial_t\mathbf{d} + \mathrm{curl}\,(\mathbf{d}\times\mathbf{v}) + \mathbf{v}\,\mathrm{div}\,\mathbf{d}, &(30)\\ \mathcal{L}_{\mathbf{v}}\,\rho &= \partial_t\rho + \mathrm{div}\,(\rho\mathbf{v}). &(31)\end{aligned}$$

In order to make the link with classical notions, it can be noted that $\mathcal{L}_{\mathbf{v}} \equiv \partial_t + \pounds_{\mathbf{v}}$, where $\pounds_{\mathbf{v}}$ is the Lie derivative [2] ; in the absence of motion, $\mathbf{v} \equiv 0$ and $\mathcal{L}_{\mathbf{v}} \equiv \partial_t$. One recognizes in (24) resp. (28), and (27) resp. (31) material derivatives that are encountered in Fluid dynamics. Equations (25) resp. (29), and (26) resp. (30), could therefore be regarded as the material derivatives of 1– forms and 2– forms respectively. But, as electromagnetic fields do not need material support, the name co-moving time derivative is preferred [2]. Although 1–forms and 2–forms have proxy fields of the same nature (vector fields), their co-moving time derivatives are different. The same remark holds for 0–forms and 3–forms. One may finally recognize in (28)-(31) the Euclidean expression of Cartan's magic formula for differential forms

$$\mathcal{L}_{\mathbf{v}} = \partial_t + \mathrm{d}i_{\mathbf{v}} + i_{\mathbf{v}}\,\mathrm{d}. \tag{32}$$

G. Convex analysis

In order to draw all the benefit from the theory presented in this paper, some concepts from Convex analysis are useful. See e.g. [4] for a sufficient introduction to the subject.

Let X be a set. A **function** $f : \mathrm{dom}\, f \subset X \mapsto \mathbb{R}$ is defined by fixing a **domain** $\mathrm{dom}\, f \subset X$ and a **rule** $x \to f(x)$ that makes sense $\forall x \in \mathrm{dom}\, f$ with $f(x) \in \mathbb{R}$. [2] The **epigraph** of f is the subset of $X\times\mathbb{R}$ defined by $\mathrm{epi}\, f = \{(x,z) : x \in \mathrm{dom}\, f, z \geq f(x)\}$. The function f is upper-bounded iff $\forall x \in \mathrm{dom}\, f$, $\exists\,\alpha \in \mathbb{R} : f(x) \leq \alpha$. The smaller upper bound for f is noted $\sup f$.

Let us suppose now that X is vector space. A subset $K \subset X$ is **convex** iff $\forall x, y \in K, (x+y)/2 \in K$. A function $f : \mathrm{dom}\, f \subset X \mapsto \mathbb{R}$ is convex if its epigraph is convex.

Let us now additionally assume a **norm** $|x|$ is defined on the vector space X. This notion is necessary to express convergence. The set $K \subset X$ is **closed** if it contains the limits of all its convergent suites. The fonction f is **lower semi-continuous** if its epigraph is closed.

Let finally X and Y be two Hilbert spaces with the **scalar product** $(y,x), x \in X, y \in Y$. The **Legendre transform** of a function $\Psi : \mathrm{dom}\,\Psi \subset X \mapsto \mathbb{R}$ is the function $\Psi^* : Q \subset Y \mapsto \mathbb{R}$ defined by the rule

$$y \to \sup_{x\in\mathrm{dom}\,\Psi}\{x \mapsto (y,x) - \Psi(x)\}. \tag{33}$$

and the domain Q that is the set of the points $y \in Y$ for which the function $x \to (y,x) - \Psi(x)$ is upper bounded. It can be shown that the functions Ψ^* defined this way is convex and lower semi-continuous (clsc) and that $\Psi^{**} = \Psi$ if Φ is cslc itself.

The functions $\Psi : \mathrm{dom}\,\Psi \subset X \mapsto \mathbb{R}$ and $\Phi : \mathrm{dom}\,\Phi \subset Y \mapsto \mathbb{R}$, are said to be **dual** iff both

$$\begin{aligned}\Phi(y) &= \sup_{x\in\mathrm{dom}\,\Psi}\{x \mapsto (y,x) - \Psi(x)\} &(34)\\ \Psi(x) &= \sup_{y\in\mathrm{dom}\,\Phi}\{x \mapsto (y,x) - \Phi(y)\}\end{aligned}$$

are true. Dual functions are automatically clsc. Note that a pair of functions Φ and Ψ that are the Legendre transform of each other (i.e. $\Psi^* = \Phi$ and $\Phi^* = \Psi$) are dual by definition but, as Q might be different from a prescribed domain $\mathrm{dom}\,\Phi$, Φ and Ψ might be dual without having $\Phi^* = \Psi$.

It is obvious from the definitions of Ψ and Φ that the inequality

$$\Lambda(x,y) = \Psi(x) + \Phi(y) - (y,x) \geq 0 \tag{35}$$

holds $\forall x \in X$ and $\forall y \in Y$.

[2] Note that the domain $\mathrm{dom}\, f$ might be prescribed as being a subset only of the domain on which the rule $x \to f(x)$ is actually defined.

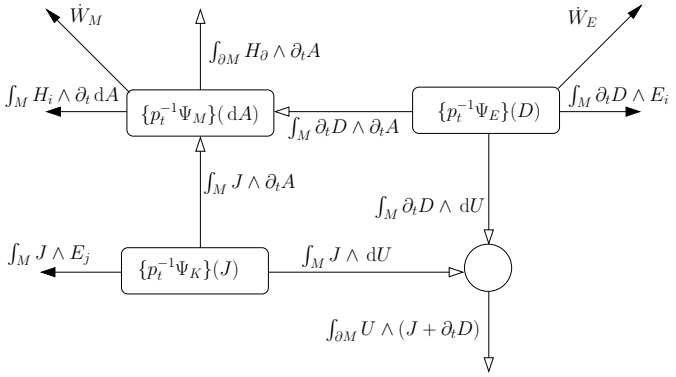

Figure 2: *Electromagnetic energy diagram in the material manifold M.*

The **subdifferential** $\partial_x \Psi$ of the function $\Psi(x)$ is the set

$$\partial_x \Psi = \{y \in Y : \Psi(x') - \Psi(x) \geq (y, x' - x), \forall x' \in \mathrm{dom}\, \Psi\}.$$

The elements of that set are called **subgradients**. If the function $\Psi(x)$ happens to be differentiable at x, its **gradient** is the only element of $\partial_x \Psi$ and $y = \partial_x \Psi$ can be written instead of $y \in \partial_x \Psi$. An important result is that the inequality (35) becomes an equality if either $y \in \partial_x \Psi$ or $x \in \partial_y \Phi$. Finally, the applications $x \mapsto \partial_x \Psi$ and $y \mapsto \partial_y \Phi$ are **monotonous** in the sense that $(y_2 - y_1, x_2 - x_1) \geq 0$ for any given $x_1, x_2 \in \mathrm{dom}\, \Psi$ and $\forall y_1 \in \partial_x \Phi(x_1), \forall y_2 \in \partial_x \Phi(x_2)$.

III. Energy-based theory

A. Energy diagram

Whereas the classical theory of electromagnetism is expressed in terms of the vector fields $\mathbf{h}, \mathbf{b}, \mathbf{e}, \mathbf{d}, \mathbf{j} \in V^1(E^3)$ and the charge density $\rho^Q \in V^0(E^3)$, the set of state variables selected for the energy-based theory is different. It consists of the electric scalar potential $U \in \Lambda^0(M)$, the magnetic vector potential $A \in \Lambda^1(M)$, the electric displacement $D \in \Lambda^2(M)$ and the current density $J \in \Lambda^2(M)$, all defined on the material manifold. The state variables are thus the two electromagnetic potentials, i.e. A and U, and the two fields associated with electric charges, i.e. D and J.

Let us state as a postulate that, in an arbitrary material region M, electromagnetic energy flows according to the diagram depicted in Fig. 2. The diagram consists of four energy reservoirs, each one associated with a state variable. The $A-$reservoir (upper left) contains the magnetic energy of the system

$$\begin{aligned} \{p_t \Psi_M\}(\mathrm{d}A) &= \int_M \{p_t \rho_M^\Psi\}(\mathrm{d}A) \\ &= \int_{\Omega = p_t M} \rho_M^\Psi(\mathrm{curl}\, \mathbf{a}) = \Psi_M(\mathrm{curl}\, \mathbf{A}) \end{aligned}$$

which is a function of $\mathrm{d}A = p_t\, \mathrm{curl}\, \mathbf{a}$, i.e. the image in M of the induction field $\mathrm{curl}\, \mathbf{a}$. Similarly, the $D-$reservoir (upper right) contains the electric energy $\{p_t \Psi_E\}(D)$, which is a function of the electric displacement $D = p_t \mathbf{d}$. The $U-$reservoir (lower right) is always empty. The $J-$reservoir, finally, contains the kinetic energy of the charge carriers, which can be expressed as

$$\{p_t \Psi_K\}(J) = \int_M \alpha \frac{|J|^2}{2} = \int_\Omega \frac{|\mathbf{j}|^2}{2} = \Psi_K(\mathbf{j}) \qquad (36)$$

where α is a constant. Except in case of superconductors, the inertia of charge carriers is negligible, and the J-reservoir can then usually be considered empty as well.

The internal flows (the flows connecting two reservoirs of the diagram) depend on the state variables only. The black-headed arrows represent 3 dissipative volume flows associated respectively with magnetic hysteresis, dielectric hysteresis and Joule losses. They involve state variables (U excepted) and empirical dissipative so called generalised forces $H_i = p_t \mathbf{h}_i$, $E_i = p_t \mathbf{e}_i$ and $E_j = p_t \mathbf{e}_j$, whose physical interpretations are discussed below. The surface generalised force $H_\partial = p_t \mathbf{h}_\partial$ is associated with the magnetic energy crossing the surface of the system. The second surface flow, connected to the $U-$reservoir, represents the energy entering the system through the conductors crossing its surface. Finally, the flows $\dot{W}_M$ and $\dot{W}_E$ account respectively for the electric or magnetic energy converted into non-electromagnetic forms of energy (e.g. mechanical, chemical, etc...)

The energy-based theory relies on the structure of this diagram, which tells something fundamental about how electromagnetic fields interact with matter and spacetime. It makes up a framework in which all electromagnetic systems, including dissipative and coupled ones, should inscribe.

B. Conservation equations in M

As the fields A, D, J and U are independent variables, they can be varied freely in order to obtain, following a variational line of argument, the conservation equations implied by the structure of the diagram. By expressing on the one hand energy conservation at node A (the variation of the energy in the reservoir is equal to the sum of incoming flows minus the sum of outgoing flows),

$$\begin{aligned} \partial_t \{p_t \Psi_M\}(\mathrm{d}A) &= \int_M \{J + \partial_t D\} \wedge \partial_t A \\ &- \int_M p_t \mathbf{h}_i \wedge \partial_t\, \mathrm{d}A \\ &- \int_{\partial M} H_\partial \wedge \partial_t A - \dot{W}_M \end{aligned}$$

and applying on the other hand the chain rule of derivatives

$$\begin{aligned} \partial_t \{p_t \Psi_M\}(\mathrm{d}A) &= \int_M \left\{\partial_B p_t \rho_M^\Psi\right\}(\mathrm{d}A, G) \wedge \partial_t\, \mathrm{d}A \\ &+ \int_M \left\{\partial_G p_t \rho_M^\Psi\right\}(\mathrm{d}A, G) : \partial_t G, \end{aligned}$$

two expressions of the variation in time of the magnetic energy are obtained. Note that it has been assumed for the sake of simplicity that $\mathrm{d}A$ is the only argument of $p_t \Psi_M$. Other arguments could be added if necessary, e.g. the strain ε for a magnetostrictive material, with a similar mathematical treatment as the one sketched here.

Identification of the two right-hand sides gives an equation that must be verified, for arbitrary variations of A, i.e. $\forall \partial_t A$. The implied conservation equations are obtained by applying the fundamental lemma of Calculus of variations[3], i.e. by identifying to zero the factors of $\partial_t A$, separately on M and ∂M. Using the commutation property $\partial_t\, \mathrm{d} = \mathrm{d} \partial_t$, (13) and Stokes' theorem (14), to perform an integration by parts, one obtains the Euler-Lagrange equations

$$\begin{aligned} \mathrm{d}\left\{\left\{\partial_{\mathbf{B}}\, p_t \rho_M^\Psi\right\}(\mathrm{d}A, G) + p_t \mathbf{h}_i\right\} &= J + \partial_t D \qquad \text{on } M, \\ \left\{\partial_{\mathbf{B}}\, p_t \rho_M^\Psi\right\}(\mathrm{d}A, G) + p_t \mathbf{h}_i &= H_\partial \qquad \text{on } \partial M, \end{aligned}$$

[3]The arbitrary time derivative $\partial_t A$ plays here the role of the variation δA.

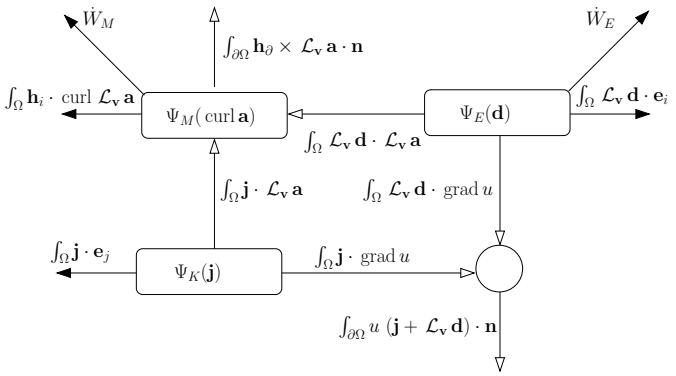

Figure 3: *Electromagnetic energy diagram in the Euclidean space E^3.*

and the remaining of the equation writes

$$-\dot{W}_M = \int_M \left\{\partial_G p_t \rho_M^\Psi\right\}(\,\mathrm{d}A, G)\,:\partial_t G. \tag{37}$$

Conservation relations at the other nodes of the diagram are obtained similarly. One finds :

$$\begin{aligned} \mathrm{d}\bar{H} &= J + \partial_t D && (38)\\ \bar{E} &= -\partial_t A - \mathrm{d}U && (39)\\ p_t\mathbf{e}_j + \alpha\partial_t J &= -\partial_t A - \mathrm{d}U && (40)\\ 0 &= \mathrm{d}\{J + \partial_t D\} && (41) \end{aligned}$$

where the magnetic field $\bar{H}$ and the electric field $\bar{E}$ are not fundamental quantities, as they are in classical electromagnetism, but shorthand for

$$\begin{aligned} \bar{H} &\equiv \left\{\partial_B\, p_t\rho_M^\Psi\right\}(\,\mathrm{d}A, G) + p_t\mathbf{h}_i && (42)\\ \bar{E} &\equiv \left\{\partial_D\, p_t\rho_E^\Psi\right\}(D, G) + p_t\mathbf{e}_i. && (43) \end{aligned}$$

One has also the boundary condition $\bar{H} = H_\partial$ on ∂M, and the equations

$$\begin{aligned} -\dot{W}_M &= \int_M \left\{\partial_G p_t\rho_M^\Psi\right\}(\,\mathrm{d}A, G)\,:\partial_t G && (44)\\ -\dot{W}_E &= \int_M \left\{\partial_G p_t\rho_E^\Psi\right\}(D, G)\,:\partial_t G, && (45) \end{aligned}$$

define the power developed by magnetic and electric forces.

C. Conservation equations in E^3

Equations (38) - (41) can be mapped into E^3, thanks to the placement map p_t, so as to obtain the conservation equations, not in terms of differential forms, but in terms of the corresponding proxy fields. Alternatively, the conservation equations can be derived directly from the diagram in E^3 (Fig. 3) using vector field analysis. One obtains in Ω

$$\begin{aligned} \operatorname{curl}\bar{\mathbf{h}} &= \mathbf{j} + \mathcal{L}_\mathbf{v}\,\mathbf{d} && (46)\\ \bar{\mathbf{e}} &= -\mathcal{L}_\mathbf{v}\,\mathbf{a} - \operatorname{grad} u && (47)\\ \mathbf{e}_j + \alpha\,\mathcal{L}_\mathbf{v}\,\mathbf{j} &= -\mathcal{L}_\mathbf{v}\,\mathbf{a} - \operatorname{grad} u && (48)\\ 0 &= \operatorname{div}\{\mathbf{j} + \mathcal{L}_\mathbf{v}\,\mathbf{d}\} && (49) \end{aligned}$$

with the shorthand

$$\begin{aligned} \bar{\mathbf{h}} &\equiv \left\{\partial_\mathbf{b}\,\rho_M^\Psi\right\}(\operatorname{curl}\mathbf{a}, g) + \mathbf{h}_i, && (50)\\ \bar{\mathbf{e}} &\equiv \left\{\partial_\mathbf{d}\,\rho_E^\Psi\right\}(\mathbf{d}, g) + \mathbf{e}_i. && (51) \end{aligned}$$

The boundary condition writes $\bar{\mathbf{h}} = \mathbf{h}_\partial$ on $\partial\Omega$ and the equations

$$\begin{aligned} -\dot{W}_M &= \int_\Omega \left\{\partial_g \rho_M^\Psi\right\}(\operatorname{curl}\mathbf{a}, g)\,:\mathcal{L}_\mathbf{v}\,g && (52)\\ -\dot{W}_E &= \int_\Omega \left\{\partial_g \rho_E^\Psi\right\}(\mathbf{d}, g)\,:\mathcal{L}_\mathbf{v}\,g && (53) \end{aligned}$$

define, in E^3, the power developed by magnetic and electric forces in terms of the co-moving time derivative of g. Although the latter is fixed in an Euclidean space, its co-moving time derivative is not. It is given by

$$\mathcal{L}_\mathbf{v}\,g = (\nabla\mathbf{v}) + (\nabla\mathbf{v})^T = 2\,\partial_t\varepsilon, \tag{54}$$

where ε is the strain tensor.

IV. Discussion

The governing equation we have just established in E^3 are now discussed. A similar discussion holds of course for the corresponding equations in M.

A. Magnetic field

Equation (50) shows that the magnetic field is composed of a reversible part $\mathbf{h}_r \equiv \partial_\mathbf{b}\rho_M^\Psi$ that accounts for the magnetization phenomenon (alignment of microscopic magnetic moments), and an irreversible part $\mathbf{h}_i$ that accounts for the local dissipation process. The magnetic field is thus not a fundamental quantity but a composite one representing at the same time two different phenomena.

B. Electric field

One can be puzzled sometimes by the changing visage of the electric field, which can in turn be associated with electrostatic charges ($\mathbf{e} = -\operatorname{grad} u$), the motion of charge carriers ($\mathbf{e} = \rho\mathbf{j}$), electric dipoles ($\mathbf{e} = \varepsilon\mathbf{d}$), or with the magnetic flux ($\mathbf{e} = -\partial_t\mathbf{a}$).

As for the magnetic field $\bar{\mathbf{h}}$ (50), the electric field $\bar{\mathbf{e}}$ is a composite quantity representing phenomena of different natures (51). But the situation is even more confusing for the electric field. Indeed, (47), (48) and (51) give three different and unrelated expressions for $\bar{\mathbf{e}}$.

Equation (48) in particular, which is the conservation equation at node $\mathbf{j}$ of the energy diagram depicted in Fig. 3, is a true equilibrium equation for charge carriers, up to a factor q_c. The term $-\operatorname{grad} u$ is the applied electrostatic force and the term $\mathbf{e}_j = \sigma^{-1}\mathbf{j}$ is the viscous force opposed by the crystal lattice. When the charge carrier accelerates, a certain amount of energy has to be given to increase its kinetic energy and another amount of energy also to increase the magnetic energy of the system. The accelerated charge is indeed associated with a larger current, which in turn generates a larger magnetic field. These two energy transfers are respectively represented by the forces $\alpha\partial_t\mathbf{j}$ and $\partial_t\mathbf{a}$ (up to the factor q_c again), which can be regarded as two inertial forces of respectively mechanical and magnetic origin. The dynamics of charge is thus made, by the energy-based approach, an integral part of the theory.

C. Constitutive laws

Constitutive laws are defined by giving algebraic expressions for the energy density functionals ρ_M^Ψ, ρ_E^Ψ, ρ_K^Ψ and for the dissipation functions $\mathbf{h}_i$, $\mathbf{e}_i$, $\mathbf{e}_j$. The gouverning equations (46) - (53) do not contradict Maxwell's equations, but they are more complete, as they involve the constitutive laws as well. All terms

have a clear physical interpretation in terms of energy or energy transfer, which can be visualized in the diagram Fig. 3.

The definition as fundamental quantities of the magnetic field (50) and of the electric field (51) suffices to eliminate all metric aspects from the conservation equations (46) - (49). The relation between $\mathbf{b}$ and $\bar{\mathbf{h}}$, resp. $\mathbf{d}$ and $\bar{\mathbf{e}}$, can however not be fully represented by the Hodge operator in the presence of the dissipation forces $\mathbf{h}_i$ and $\mathbf{e}_i$ [5, 6].

D. Electromagnetic forces

The conditions $\mathcal{L}_{\mathbf{v}}\,\mathbf{a} = 0$ and $\mathcal{L}_{\mathbf{v}}\,\mathbf{d} = 0$ are the precise mathematical statement of what is usually called "holding magnetic/electric fluxes constant" [7]. By setting $\mathcal{L}_{\mathbf{v}}\,\mathbf{a} = 0$ and $\mathcal{L}_{\mathbf{v}}\,\mathbf{d} = 0$, the $\mathbf{a}-$reservoir and the $\mathbf{d}-$reservoir are isolated from the diagram in Fig. 3, and the variation of energy

$$\dot{W}_M + \dot{W}_E = -\,\partial_t \Psi_M|_{\,\mathcal{L}_{\mathbf{v}}\,\mathbf{a}=0} - \,\partial_t \Psi_E|_{\,\mathcal{L}_{\mathbf{v}}\,\mathbf{d}=0} \tag{55}$$

represents then the power converted into non-electromagnetic forms of energy (mechanical, chemical…).

In particular, if one substitutes (54) in (52) and (53), the Maxwell stress tensor σ_{em} can be defined as the factor of $\nabla\mathbf{v}$ in the right-hand side of (55), i.e.

$$\dot{W}_M + \dot{W}_E = -\int_\Omega \sigma_{em} \,:\nabla\mathbf{v} + \dots \tag{56}$$

The Maxwell stress tensor is the fundamental quantity representing the electromechanical coupling and a unifying concept for all force formulae encountered in literature and used in numerical computations : the virtual work principle of Coulomb and Ren [8, 7], the sensitivity analysis of Lowther [9], Arkkio's formula [10], Kameari [11], and the Eggshell method [12, 13]. Each material has its own Maxwell stress tensor and it has been shown in [14] how its algebraic expression can be derived algebraically from a known expression of the magnetic and electric energy densities of the material.

E. Electric charges

Electric charge are not explicitly in the diagram They are *defined* by

$$\rho^Q = \,\mathrm{d}D = \operatorname{div}\mathbf{d}. \tag{57}$$

The inertia of the charge carrier is also at the root of the definition of the static charges that are present at the surface of current carrying conductors [15]. Identifying the left hand sides of (47) and (49) and assuming $\mathbf{e}_i = 0$, one has

$$\left\{\partial_{\mathbf{d}}\,\rho_E^\Psi\right\}(\mathbf{d}) = \varepsilon_0^{-1}\mathbf{d} = \sigma^{-1}\mathbf{j} + \alpha\,\mathcal{L}_{\mathbf{v}}\,\mathbf{j}. \tag{58}$$

The divergence of the right-hand side is identically zero (div and $\mathcal{L}_{\mathbf{v}}$ commute) inside the conductor, but the term in α has a non-zero contribution on the surface of the conductor, whence the expression $\varepsilon_0\alpha\,\mathcal{L}_{\mathbf{v}}\,\mathbf{j}\cdot\mathbf{n}$ for the surface charges.

F. Superconductors

In practice, the $J-$reservoir can often be considered as empty, because of the very small value of α (negligible inertia of the charge carriers), and the corresponding term in (48) can be disregarded. However, in superconductors, for which σ is infinite ($\mathbf{e}_j = 0$) and $\operatorname{grad} u$ is zero, (48) writes

$$\alpha\,\mathcal{L}_{\mathbf{v}}\,\mathbf{j} = -\,\mathcal{L}_{\mathbf{v}}\,\mathbf{a}, \tag{59}$$

i.e. the first London's equation $\mathbf{a} = -\alpha\mathbf{j}$ for superconductors [16].

G. Poynting's vector

The flux of the Poynting's vector through the boundary of the system is found by noting that the two surface flows combine to form the flow of the Poynting vector (60).

$$\int_{\partial\Omega} \{\mathbf{h}_\partial \times\, \mathcal{L}_{\mathbf{v}}\,\mathbf{a} + u\,\{\mathbf{j} +\, \mathcal{L}_{\mathbf{v}}\,\mathbf{d}\}\}\cdot\mathbf{n} = \int_{\partial\Omega} \bar{\mathbf{e}}\times\bar{\mathbf{h}}. \tag{60}$$

V. Applications

A. Formulations

In many problems encountered in electromagnetism, it is not necessary to solve the complete set of Maxwell equations. According to the dimensions and the time scale under consideration, the materials in presence and the configuration of the system, it happens often that simplifications are possible. Those simplifications consist generally in dropping terms in the full Maxwell's equations and weak formulations are then obtained by applying Galerkine's method to the simplified equations.

The alternative top-down approach, which consists in deriving weak formulations directly from the energy diagram, is not necessarily more straightforward but has nevertheless several advantages. Firstly, the assumptions done take on a physical justification this way, instead of a mathematical one. The different terms in the weak formulation also maintain their interpretation in terms of energy, so that they can be used for establishing a global energy balance of the device or to express coupling terms in multi-physics problems.

Spelling out the wide variety of weak formulations encountered in computational electromagnetism would be fastidious. We are going to consider only electrostatics and magnetodynamics.

Electrostatics

The electrostatic regime is obtained by setting to zero the state variables $\mathbf{a}$ and $\mathbf{j}$ and preventing the system from any energy conversion, i.e. $\dot{W}_M \equiv 0$, and assuming no motion, $\mathbf{v} \equiv 0 \Rightarrow \mathcal{L}_{\mathbf{v}} \equiv \partial_t$. Since dissipative forces act over time, it is also natural to assume $\mathbf{e}_i \equiv 0$ in a static problem. Two conservation equations then remain.

At node u, (49) becomes

$$\operatorname{div}\partial_t\mathbf{d} = \partial_t \operatorname{div}\mathbf{d} = 0, \tag{61}$$

which shows that the quantity $\operatorname{div}\mathbf{d}$ is conserved. The state variable $\mathbf{d}$ is therefore constrained. The vector potential $\mathbf{c}$ is then defined as a new unconstrained state variable, such that $\mathbf{d} = \mathbf{d}_0 + \operatorname{curl}\mathbf{c}$ with $\partial_t\mathbf{d}_0 = 0$, $\operatorname{div}\mathbf{d}_0 = \operatorname{div}\mathbf{d}$.

The conservation equation at node $\mathbf{d}$ in integral form,

$$\partial_t\Psi_E + \int_\Omega \operatorname{grad} u\cdot\partial_t\mathbf{d} = 0 \quad \forall\,\partial\mathbf{d}(t), \tag{62}$$

becomes then

$$\int_\Omega \left\{\partial_{\mathbf{d}}\rho_E^\Psi(\mathbf{d}_0 + \operatorname{curl}\mathbf{c}) + \operatorname{grad} u\right\}\cdot\partial_t \operatorname{curl}\mathbf{c} = 0 \quad \forall\,\partial\mathbf{c}(t), \tag{63}$$

and after an integration by part of the second term

$$\int_\Omega \partial_{\mathbf{d}}\rho_E^\Psi(\mathbf{d}_0 + \operatorname{curl}\mathbf{c})\cdot\operatorname{curl}\partial_t\mathbf{c} + \int_{\partial\Omega} \operatorname{grad} u\times\partial_t\mathbf{c}\cdot\mathbf{n} = 0 \quad \forall\,\partial\mathbf{c}(t).$$

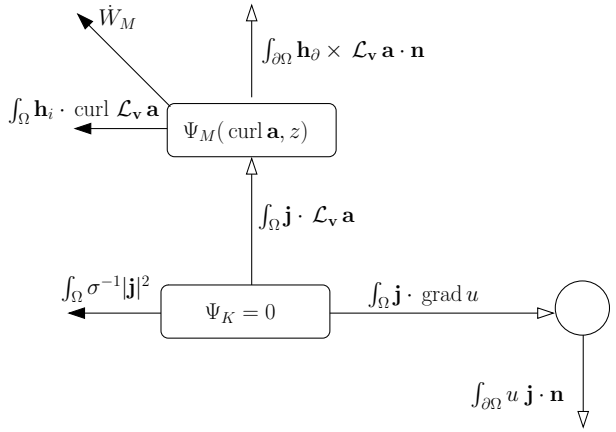

Figure 4: *EM energy flow diagram for the magnetodynamics regime.*

This is the vector potential weak formulation for electrostatics. The arbitrary $\partial_t\mathbf{c}$ can be chosen equal to the shape functions of the field $\mathbf{c}$. At the boundary, either $\mathbf{c}$ (Dirichlet boundary condition) or $-\mathbf{n}\times\mathrm{grad}\,u$ (Neumann boundary condition) must be specified.

The formulation in terms of the scalar potential u, is obtained thanks to the concept of duality introduced above. The dual variables are in this case $x = \mathbf{d} \equiv \mathbf{d}_0 + \mathrm{curl}\,\mathbf{c}$ and $y = -\mathrm{grad}\,u$. Since (63) is a condition stronger than $\partial_\mathbf{d}\rho_E^\Psi \ni -\mathrm{grad}\,u$, the co-energy Φ_E defined by (34) as the dual of the energy Ψ_E satisfies the equality

$$\Phi_E = -\int_\Omega \mathrm{grad}\,u\cdot\mathbf{d} - \Psi_E, \tag{64}$$

so that

$$\begin{aligned}
\partial_t\Phi_E &= -\int_\Omega \partial_t\mathrm{grad}\,u\cdot\mathbf{d} - \int_\Omega \mathrm{grad}\,u\cdot\partial_t\mathbf{d} - \partial_t\Psi_E \\
&= -\int_\Omega \partial_t\mathrm{grad}\,u\cdot\mathbf{d} \\
&\quad -\int_\Omega \left\{\mathrm{grad}\,u + \partial_\mathbf{d}\rho_E^\Psi\right\}\cdot\partial_t\mathrm{curl}\,\mathbf{c} \\
&= -\int_\Omega \partial_t\mathrm{grad}\,u\cdot\mathbf{d} \quad \forall\partial_t u(t)
\end{aligned}$$

by (63). Making now an integration by part, one has

$$\begin{aligned}
\int_\Omega \partial_{\mathrm{grad}\,u}\Phi_E\cdot\mathrm{grad}\,\partial_t u &= \int_\Omega \partial_t u\,\mathrm{div}\,\mathbf{d} \\
&\quad -\int_{\partial\Omega}\partial_t u\,\mathbf{d}\cdot\mathbf{n} \quad \forall\partial_t u(t)
\end{aligned}$$

with $\mathrm{div}\,\mathbf{d} = \mathrm{div}\,\mathbf{d}_0$ the charge density. This is the scalar potential formulation for electrostatics. At the boundary, either u (Dirichlet boundary condition) or $\mathbf{d}\cdot\mathbf{n}$ (Neumann boundary condition) must be specified.

Magnetodynamics

The magnetodynamics regime is obtained by setting $\mathbf{d}\equiv 0$. The corresponding energy diagram is depicted in Fig. 4. Dissipation (Joule and hysteresis) and electromechanical coupling ($\mathbf{v}\neq 0$) are going to be considered in this dynamical formulation, but the kinetic energy of charge carrier is disregarded, i.e. $\Psi_K\equiv 0$. Energy conservation in integral form at node $\mathbf{a}$ and the application of the chain rule of derivatives to the magnetic energy Ψ_M write respectively

$$\begin{aligned}
\partial_t\Psi_M &= \int_\Omega \mathbf{j}\cdot\mathcal{L}_\mathbf{v}\,\mathbf{a} - \int_\Omega \mathbf{h}_i\cdot\mathrm{curl}\,\mathcal{L}_\mathbf{v}\,\mathbf{a} \\
&\quad -\int_{\partial\Omega}\mathbf{h}_\partial\times\mathcal{L}_\mathbf{v}\,\mathbf{a}\cdot\mathbf{n} - \dot{W}_M \\
\partial_t\Psi_M &= \int_\Omega \left\{\partial_\mathbf{b}\rho_M^\Psi\right\}(\mathrm{curl}\,\mathbf{a}, g)\cdot\mathrm{curl}\,\mathcal{L}_\mathbf{v}\,\mathbf{a} \\
&\quad +\int_\Omega \left\{\partial_g\rho_M^\Psi\right\}(\mathrm{curl}\,\mathbf{a}, g)\,:\,\mathcal{L}_\mathbf{v}\,g
\end{aligned}$$

and, after identification of both right hand sides,

$$\begin{aligned}
0 &= \int_\Omega \left\{\partial_\mathbf{b}\rho_M^\Psi(\mathrm{curl}\,\mathbf{a}, g) + \mathbf{h}_i\right\}\cdot\mathrm{curl}\,\mathcal{L}_\mathbf{v}\,\mathbf{a} \\
&\quad -\int_\Omega \mathbf{j}\cdot\mathcal{L}_\mathbf{v}\,\mathbf{a} + \int_{\partial\Omega}\mathbf{h}_\partial\times\mathcal{L}_\mathbf{v}\,\mathbf{a}\cdot\mathbf{n} \\
&\quad +\int_\Omega \left\{\partial_g\rho_M^\Psi\right\}(\mathrm{curl}\,\mathbf{a}, g)\,:\,\mathcal{L}_\mathbf{v}\,g + \dot{W}_M \qquad \forall\,\mathcal{L}_\mathbf{v}\,\mathbf{a}(t).
\end{aligned} \tag{65}$$

Being independent of $\mathcal{L}_\mathbf{v}\,\mathbf{a}$, the last two terms must sum up to zero separately, which defines the power delivered by magnetic forces and gives an already known result (53). The other terms make up the vector potential weak formulation of Magnetodynamics, with an imposed current density.

Now, if the dissipation force $\mathbf{e}_j$ is asumed to be an invertible function of $\mathbf{j}$ (one has for instance $\mathbf{e}_j = \sigma^{-1}\mathbf{j}$ for normal conductors), one can with (48) express $\mathbf{j} = fct(\mathcal{L}_\mathbf{v}\,\mathbf{a} - \mathrm{grad}\,u)$ and substitute this in the weak formulation above in order to obtain the weak formulation of Magnetodynamics with imposed voltages. In practice, the voltage source is simply represented by a discontinuity of u over a given cross section of the conductor.

Motion terms like $\mathbf{v}\times\mathbf{b}$ are naturally present in the formulation by virtue of the co-moving derivative $\mathcal{L}_\mathbf{v}\,\mathbf{a}$, and they must not be introduced on basis of a relativistic argument (Lorentz transformation). This issue is will be discussed in another paper.

B. Magnetic hysteresis

The energy diagram indicates that the state variable of a magnetic material is the induction $\mathbf{b}\equiv\mathrm{curl}\,\mathbf{a}$, which, in the presence of hysteresis, is subjected to a (generalised) force $\mathbf{h}_r = \partial_\mathbf{b}\rho_M^\Psi$ deriving from a potential (the magnetic energy density ρ_M^Ψ) and to a dissipative force $\mathbf{h}_i$. It is remarkable that complying with this observation yields naturally a vector hysteresis model, in contrast to Preisach [17] and Jiles-Atherton [18, 19], which are basically scalar models

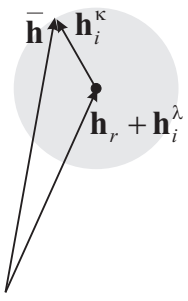

Figure 5: *Equilibrium equation (71). The grey circle represents the subgradient G.*

Starting from the vector potential formulation (66), using $\mathrm{curl}\,\bar{\mathbf{h}} = \mathbf{j}$ and making an integration by part, the conservation

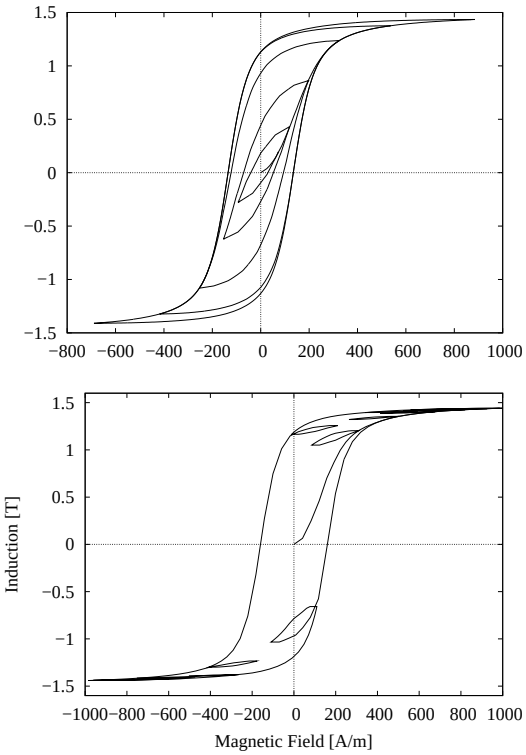

Figure 6: *Internal loops (above) and minor loops (below) are represented by the model.*

equation at node $\mathbf{a}$ in integral form can be put into the form of the First Principle of Thermodynamics $\partial_t \Psi_M = \dot{W} + \dot{Q}$ with

$$\dot{Q} = -\int_\Omega \mathbf{h}_i \cdot \dot{\mathbf{b}} \quad , \quad \dot{W} = \int_\Omega \bar{\mathbf{h}} \cdot \dot{\mathbf{b}} \tag{66}$$

and where $\dot{\mathbf{b}}$ is shorthand for $\mathrm{curl}\ \mathcal{L}_{\mathbf{v}}\, \mathbf{a}$. It follows directly that

$$\int_\Omega \left\{ \mathbf{h}_r - \bar{\mathbf{h}} + \mathbf{h}_i \right\} \cdot \dot{\mathbf{b}} = 0 \qquad \forall\, \dot{\mathbf{b}}(t) \tag{67}$$

so that the conservation equation is $\bar{\mathbf{h}} = \mathbf{h}_r + \mathbf{h}_i$.

The principle of the dynamic hysteresis model is introduced by making a mechanical analogy. The dissipative phenomenon can be accurately represented by the friction force $\mathbf{h}_i$ obtained from the non-smooth non-negative convex potential

$$\dot{Q}(\dot{\mathbf{b}}) = -\int_\Omega \left\{ \kappa\, |\dot{\mathbf{b}}| + \lambda \dot{\mathbf{b}}^2 \right\} \leq 0. \tag{68}$$

Since the dissipation functional $\dot{Q}$ is a function of $\dot{\mathbf{b}}$, and *not* of $\mathbf{b}$ like Ψ_M is, the relation between $\dot{Q}$ and $\mathbf{h}_i$ is not a differential one (subgradient) but an algebric one, a kind of division of $\dot{\rho}^Q$ by $\dot{\mathbf{b}}$ for which a mathematical definition is needed. Fortunately, for a large class of dissipation functionals, this division can be expressed in terms of the subgradient of the functional thanks to the notion of homogeneous function. A homogenous function of order n is a function such that $f(\xi x) = \xi^n f(x)$. It has the property $x\partial_x f = nf$. This can be written $f/x = (\partial_x f)/n$, which is precisely the sought relation.

The quadratic term in (68) represents a viscous friction force. It stands for microscopic eddy currents induced in the material by the variation with time of induction. Since this term is a homogenous function of order 2 of $\dot{\mathbf{b}}$, one has

$$\mathbf{h}_i^\lambda = \frac{1}{2}\partial_{\dot{\mathbf{b}}}(\lambda \dot{\mathbf{b}}^2) = \lambda \dot{\mathbf{b}}. \tag{69}$$

The pinning phenomenon, which is at the origin of magnetic hysteresis, is on the other hand represented by the dry friction force associated with the term $\kappa\, |\dot{\mathbf{b}}|$. This term is not differentiable at $\dot{\mathbf{b}} = 0$, but, as it is a convex function, it has a subgradient G defined by

$$G = \{ \mathbf{h}_i^\kappa, |\mathbf{h}_i^\kappa| \leq \kappa \text{ if } \dot{\mathbf{b}} = 0, \mathbf{h}_i^\kappa = \kappa\, \mathbf{e}_{\dot{\mathbf{b}}} \text{ if } \dot{\mathbf{b}} \neq 0 \} \tag{70}$$

where $\mathbf{e}_{\mathbf{x}} \equiv \mathbf{x}/|\mathbf{x}|$. Since it is a homogeneous function of degree 1, one has $\mathbf{h}_i^\kappa = \partial_{\dot{\mathbf{b}}} \kappa |\dot{\mathbf{b}}|$, i.e. one can identify $\mathbf{h}_i^\kappa$ with the subgradient G.

The equilibrium equation writes finally

$$\bar{\mathbf{h}} - \mathbf{h}_r - \mathbf{h}_i^\lambda = \mathbf{h}_i^\kappa \in G. \tag{71}$$

The memory effect originates from the non-univocity of the friction force $\mathbf{h}_i^\kappa$ at $\dot{\mathbf{b}} = 0$. The subgradient, i.e. the set of possible forces $\mathbf{h}_i^\kappa$, is represented by the grey circle of radius κ in Fig. 5. If the tip of $\bar{\mathbf{h}}$ is inside the circle, one has $\dot{\mathbf{b}} = 0$ by (70), which implies $\dot{\mathbf{h}}_r = 0$, i.e. no change of the magnetisation. A given magnetisation can thus persist although the applied magnetic field $\bar{\mathbf{h}}$ has decreased, whence the memory effect. If on the contrary the tip of $\bar{\mathbf{h}}$ tends to get out of the circle, $\mathbf{h}_r$ is updated according to the differential equation in time

$$\bar{\mathbf{h}} - \mathbf{h}_r - \mathbf{h}_i^\lambda = \kappa \mathbf{e}_{\dot{\mathbf{h}}_r}, \tag{72}$$

where we have noted that $\mathbf{e}_{\dot{\mathbf{b}}} = \mathbf{e}_{\dot{\mathbf{h}}_r}$. Details on the implementation can be found in [20]. The presence of a non-smooth functionals is essentially a theoretical issue. In the implementation, it amounts to a simple *if* statement (70).

This model is able to represent minor loops, Fig. 6. By combination of several submodels with different values of κ, the number of parameters of the model can be increased for a better accuracy. Fig. 7 shows the agreement obtained with 5 submodels. As this hysteresis model is based on a real physical description of the phenomenon, it makes sense to use it in a 3D model, even when parameter identification has been done on basis of uniaxial quasi-static measurements.

C. Model reduction

It is getting increasingly more important in modern computations to dispose of a concise, computationally tractable and though accurate representation of a given large system, in order to allow real time computation, coupling with other parts of a larger system, etc. There are essentially two ways to create such simplified representations.

The first approach consists in truncating an asymptotical (in some sense) representation of the initial system. These are e.g. the Model Order Reduction (MOR) techniques, which are mostly applicable to linear problems [21]. In this case the initial and simplified representations are of the same nature. The approximation error is measured by the mathematical norm in terms of which the convergence of the asymptotical representation is expressed. As this norm has however scarcely a physical meaning, the neglected terms turn out often to have a significant impact on the physical properties of the reduced model. Therefore, special actions need be taken in order to preserve physical properties like passivity, stability, etc.

The second category gathers Parameter Identification methods, which are often based on energy criteria. When it comes to construct such models, energy turns out indeed very often to be the fundamental quantity to preserve. A good model is therefore a model able to account accurately for the energy stored in the

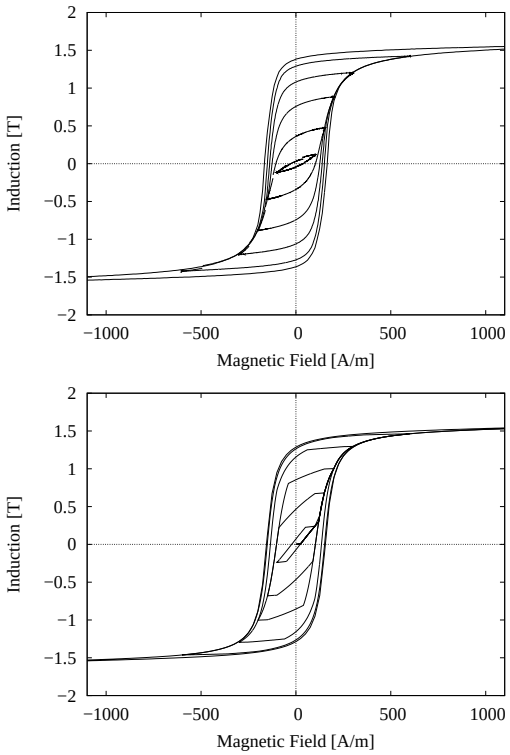

Figure 7: *Measurements (above) and model (below) obtained with 5 submodels for electrical steel.*

system, and for the main energy flows entering the system, being converted inside it, or leaving it.

Various applications of reduction methods implicitly based on energy criteria can be found in the literature, see e.g. [22, 23, 24]. This approach can be seen also as the one that leads to the definition of classical RLC lumped parameters in electrical circuits. Lacking a unifying theoretical background, it is however not regarded as a model reduction method, but the energy-based theory introduced in this paper contributes to providing such a theoretical framework.

Application to a synchronous electrical machine

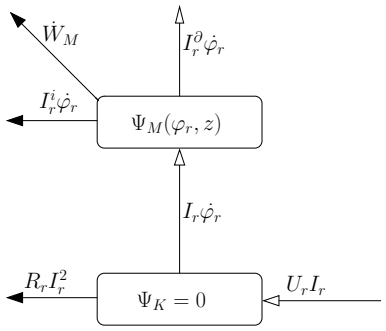

Figure 8: *EM energy flow diagram in scalar representation.*

As an example, a synchronous electrical machine is considered, for which one disposes of a detailed representation (e.g. a finite element model) in terms of the field state variables $\mathbf{a}$, $\mathbf{j}$ and u, and for which one wishes to extract a reduced model in terms of the corresponding scalar state variables φ, I_r and U_r, $r = 0, \ldots, N$, where N is the number of phases of the motor. The energy diagrams of the field representation and the scalar representation are depicted at Fig. 4 and 8 respectively.

The simplifying feature that allows reducing the model is the banal observation that the current density j can be written

$$\mathbf{j} = \sum_r I_r \mathbf{w}_r, \tag{73}$$

where the current shape functions $\mathbf{w}_r$ have support in the conducting regions $C \subset \Omega$. Note that (73) entails no approximation if the $\mathbf{w}_r$'s are allowed to depend on time.

Requiring now that the magnetic work is exactly represented, i.e. the corresponding energy flows in the field and scalar energy diagrams are equal,

$$\int_\Omega \mathbf{j} \cdot \mathcal{L}_\mathbf{v}\, \mathbf{a} \equiv \sum_r I_r \dot{\varphi}_r \quad \Rightarrow \quad \dot{\varphi}_r = \int_\Omega \mathbf{w}_r \cdot \mathcal{L}_\mathbf{v}\, \mathbf{a} \tag{74}$$

a mapping between $\dot{\varphi}$ and $\mathcal{L}_\mathbf{v}\, \mathbf{a}$ is obtained, whereas one needs a mapping between the state variables φ and $\mathbf{a}$. One makes therefore the *assumption* that the $\mathbf{w}_r$'s do not depend on time, so that one obtains the sought mapping

$$\varphi_r = \int_\Omega \mathbf{w}_r \cdot \mathbf{a}. \tag{75}$$

Phase resistances are determined by identification of the dissipation functionals

$$R_r I_r^2 = \int_\Omega \sigma^{-1} |\mathbf{j}|^2 \quad \Rightarrow \quad R_r = \int_\Omega \sigma^{-1} |\mathbf{w}_r|^2. \tag{76}$$

There are two different ways to identify the magnetic energy, i.e. the inductance matrix, of the reduced model. Either one makes a global identification or a linearisation around a given working point. For a global identification, the inductance is defined as the matrix of multiplicative factors such that

$$\Psi_M(\operatorname{curl} \mathbf{a}) = \int_0^{\varphi_r(\mathbf{a})} L_{rs}^{-1} \varphi_s \, \mathrm{d}x \quad \Rightarrow \quad L_{rs}^{-1} \varphi_s = I_r. \tag{77}$$

The inductance is in this case a non-linear function of all state variables, and of the φ_r's in particular. In practice, the magnetic energy of the system or the fluxes are pre-computed by static finite element computations over the state space of the system (i.e for all rotor positions, I_r, $\ldots$) and the computed values are stored in look-up tables [24].

This approach has two drawbacks. Firstly, the size of the look-up tables grows exponentially when the number of parameters increases. Secondly, differentiation of the stored values must be approximated numerically by finite differences. The discretisation of the state space must therefore be fine enough, yielding again an increase of the look-up table dimensions. One avoids one differentiation by storing directly the fluxes φ_r's, instead of the energy, but one numerical differentiation is still required to evaluate $U_r = R_r I_r + \dot{\varphi}_r$.

The second approach consists in linearising the magnetic behaviour of the system around a given working point. This approach is very useful when one wants to couple the reduced model of the motor with the a high dynamic model of the supplying inverter. The state variables of the linearised model are denoted by $\delta\varphi_r$, and the governing equations are

$$L_{rs}^{-\partial} \delta\varphi_s + R_r^{-1} (\partial_t \delta\varphi_r - \delta U_r) = 0. \tag{78}$$

The tangent inductance matrix of the reduced system is defined by

$$L_{rs}^{-\partial} = \partial_{\varphi_r} \partial_{\varphi_s} \Psi_M(\varphi^*) \quad \Rightarrow \quad L_{rs}^{-\partial} \dot{\varphi}_s = \dot{I}_r. \tag{79}$$

It can be shown it can be evaluated as follows

$$L^{\partial}_{rs} = W_{ri} J^{-1}_{ij} W_{sj} \quad , \quad W_{ri} = \int_{\Omega} \mathbf{w}_r \cdot \alpha_i \tag{80}$$

where J_{ij} is the jacobian matrix of the non-linear system and α_i denotes the i^{th} edge shape function.

The approximation (75) allows identifying all parameters of the reduced model. It determines therefore also the domain of validity of the reduced model. The reduction is accurate if the actual $\mathbf{w}_r$'s do not vary *too much* in time. This assumption is always true for coils, but it can also be fulfilled in a more restrictive way, e.g. for a given frequency in time-harmonic problems, or on a limited time interval for a linearised model.

D. Equivalent time-harmonic reluctivity

Another interesting application of energy-based parameter identification is the definition of equivalent material characteristics for time-harmonic models. Periodic phenomena are ubiquitous in electromagnetic applications but, due to magnetic saturation or the presence of non-linear electronic components, actual wave shapes are scarcely sinusoidal, which invalidates the phasor representation. Still, the complex formalism is so practical and offers so many useful mathematical properties that it is often worth in practice to seek for approximative phasor representations. In this case again, it is meaningful to adopt energy as identification criterion.

In frequency domain, vector fields are represented by two vectors, e.g. for an harmonic induction field, one has $\mathbf{b}_\omega = \mathbf{b}_r + j\mathbf{b}_i$. The associated time domain vector field

$$\mathbf{b}_\omega(t) = \mathbf{b}_r \ \cos\omega t - \mathbf{b}_i \ \sin\omega t \tag{81}$$

describes an ellipsis in the three-dimensional geometrical space, of which the two axis are given by

$$\left.\begin{array}{c} b_{max} \\ b_{min} \end{array}\right\} = \sqrt{\frac{|\mathbf{b}_r|^2 + |\mathbf{b}_i|^2}{2} \pm \Delta}$$

$$\Delta^2 = \left(\frac{|\mathbf{b}_r|^2 - |\mathbf{b}_i|^2}{2}\right)^2 + (\mathbf{b}_r \cdot \mathbf{b}_i)^2 .$$

The mathematical space of phasors being linear, the most general relation between an induction phasor and a magnetic field phasor, when anisotropy is disregarded, is represented by a complex reluctivity $\nu = \nu_r + j\nu_i$, where ν_r and ν_i are real constants. The corresponding representation in time-domain is the operator

$$\nu = \nu_r + \frac{\nu_i}{\omega}\partial_t . \tag{82}$$

In time domain, considering magnetic hysteresis but disregarding anisotropy, the local relation between the induction vector $\mathbf{b}$ and the magnetic field vector $\mathbf{h}$ can be written formally

$$\mathbf{b}(t) = \mathcal{H}[\mathbf{h}, t], \tag{83}$$

where $\mathcal{H}$ denotes an hysteresis operator. Numerous theoretical and phenomenological representations of hysteresis operators can be found in literature. We use here the one presented above. The principle of the identification is now to determine ν_r and ν_i so that the energy balance of the equivalent material represented by the complex ν matches as closely as possible the energy balance of the hysteretic material represented by $\mathcal{H}$. Since we have two parameters to identify, we may impose two conditions.

Let us prealably assume that one has been able, for a given $\mathbf{b}_\omega(t)$, to determine a field $\mathbf{h}^\star(t)$ such that $\mathcal{H}[\mathbf{h}^\star, t] = \mathbf{b}_\omega(t)$. From this particular hysteresis curve, the model is able to provide the value of the amount of energy dissipated over one period

$$Q^\star \equiv \int_0^T \mathbf{h}^\star \cdot \partial_t \mathbf{b}_\omega \tag{84}$$

and the amplitude of the fluctuation of the magnetic energy density ρ^Ψ_M

$$\begin{aligned} (\Delta\rho^\Psi_M)^\star &\equiv \left\{ \max_{[0,T]} - \min_{[0,T]} \right\} \rho^\Psi_M(\mathbf{b}_\omega) \\ &= \rho^\Psi_M(b_{max}) - \rho^\Psi_M(b_{min}). \end{aligned}$$

On the other hand, the magnetic field in the material represented by the complex ν

$$\mathbf{h}_\omega(t) \equiv \nu \mathbf{b}_\omega(t) = (\nu_r + \frac{\nu_i}{\omega}\partial_t)\ \mathbf{b}_\omega(t) \tag{85}$$

allows to write the energy balance

$$\mathbf{h}_\omega \cdot \partial_t \mathbf{b}_\omega = \partial_t \left\{ \frac{\nu_r}{2} |\mathbf{b}_\omega|^2 \right\} + \frac{\nu_i}{\omega} |\partial_t \mathbf{b}_\omega|^2, \tag{86}$$

where the bracketed term represents the magnetic energy density. One has therefore the two relations

$$\begin{aligned} \int_0^T \mathbf{h}_\omega \cdot \partial_t \mathbf{b}_\omega &= \frac{\omega T}{2} \nu_i \left(|\mathbf{b}_r|^2 + |\mathbf{b}_i|^2 \right) \\ &= \pi \nu_i \left(b^2_{min} + b^2_{max} \right) \equiv Q^\star \end{aligned} \tag{87}$$

$$\left\{ \max_{[0,T]} - \min_{[0,T]} \right\} \nu_r \frac{|\mathbf{b}_\omega|^2}{2} = \nu_r \Delta \equiv (\Delta\rho^\Psi_M)^\star \tag{88}$$

that allow identifying ν_r and ν_i.

VI. Conclusion

Does it make sense to come up like this with another theory of Electromagnetism ? We believe the answer is yes. This energy-based theory is indeed not just a re-formulation. It offers substantial improvements with regard to the classical theory, in particular an explicit and natural link with mechanics and the universal principles of Thermodynamics. It encompasses more Physics also, since the constitutives laws, the dynamics of charges, the behaviour of superconductors and motion-induced effects are now part of the theory. Moreover, it brings clear answers to long-standing controversies like the ones about the computation of local magnetic forces and the vectorization of hysteresis models.

The energy-based formulation relies on two points : the mathematical representation of fields with differential forms instead of vector and tensor fields, and the introduction of the material manifold as the volume control for thermodynamic analysis. The whole theory follows from these statements by simply applying continuity the rules of energy conservation. The mathematics involved is more demanding but all theoretical results can be expressed in the classical frameworks of vector and tensor analysis at the end. From the point of view of Physics, the obtained governing equations are completely covariant, i.e. they are valid in arbitrary coordinate systems, whereas the classical theory shows invariance with respect to a local Lorentz transformation only. The presented theory is a continuum theory but it is the opinion of the author however, that nothing opposes the introduction of singularities (if need be) or quantum mechanics elements in

such a variational formulation by means of e.g. the theory of distributions.

The energy-based formulation has also advantages for numerical modelling. Conservation equations can be read directly form the diagrams in a form that is readily useable by the finite element method, and all terms retain a clear physical interpretation. This helps for the definition of coupling terms in multiphysics modelling and provides meaningful criteria for identifying the lumped parameters of reduced models.

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Appendix

$$
\begin{aligned}
(\nabla \mathbf{G}) \cdot \mathbf{F} &= \sum_{ij} \mathbf{e}^i \frac{\partial G^j}{\partial x^i} F^j \\
\mathbf{F} \cdot (\nabla \mathbf{G}) &= \sum_{ij} F^i \frac{\partial G^j}{\partial x^i} \mathbf{e}^j \quad \neq (\nabla \mathbf{G}) \cdot \mathbf{F} \\
\mathbf{F} \times \operatorname{curl} \mathbf{G} &= (\nabla \mathbf{G}) \cdot \mathbf{F} - \mathbf{F} \cdot (\nabla \mathbf{G}) \\
\dot{z} &= \partial_t z + \mathbf{v} \cdot \operatorname{grad} z \\
\dot{\mathbf{z}} &= \partial_t \mathbf{z} + \mathbf{v} \cdot (\nabla \mathbf{z})
\end{aligned}
$$

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