Technical Article

An educational approach on FEM applied to Electromagnetic Field Computation

Abstract **– An educational approach on FEA of electromagnetic phenomena exhibiting plane symmetry is developed. The proposed method leads to the same formulation as the one derived by classical approaches, as Variational Principle or Weighted Residuals, provided first-order triangular meshes are considered. The elemental matrices are derived by applying Maxwell's equations in integral form to suitably chosen contours in the FE mesh. The method is suitable for teaching FEA in Electromagnetics at undergraduate level.**

I. Introduction

The use of Finite Element Analysis (FEA) in Electrical Engineering has continuously grown in the last 30 years since the pioneer work due to Silvester [1].

Besides the search for robust formulations, as well as efficient computational techniques, more efforts might be done to sparkle the interest of undergraduate students.

The main challenge is to find a proper "language" to this first contact. In the authors' opinion, the most straightforward way is to start directly from the Maxwell's equations in their integral form, applied to domains with plane symmetry discretized in finite elements. The main advantage of this approach is to avoid the usual difficult, advanced mathematical concepts of Variational Principle and Weighted Residuals.

The proposed methodology, when applied to two-dimensional geometries subdivided to form a first order triangular mesh, leads to the same algebraic system of equations that result from variational (or Galerkin) formulation. Moreover, a physical meaning can be associated to the stiffness matrix.

The assembly process is accomplished by computing the circulation of a field vector along closed contours suitably chosen in the FE mesh, leading to a simple analytical procedure to derive the system of algebraic equations.

The original idea was first established in 1987 [2], and has been applied successfully to teach FE concepts to undergraduate students of Electrical Engineering at the University of *São Paulo*, Brazil, since 1989. Also, many articles have published since then [3-5].

In the next sections the methodology is briefly outlined.

II. Field Vectors and Integration contours

In order to directly apply Maxwell's equations in integral form to a given domain, one needs to properly select supports for integrations, namely, contours and surfaces. In two dimensions, this is limited to the definition of contours, only.

A numerical solution for the integral equations can then be obtained by subdividing the domain in triangular finite elements. Each node **N** of the FE mesh must be enclosed by a polygonal contour ***C***, as depicted in Fig. 1. This contour is built by linking the mid points of those triangle edges, connected to node **N**.

The field vectors $\vec{E}$ and $\vec{H}$ will then be integrated along contour ***C***. The procedure must be repeated for contours enclosing all nodes of the mesh.

The FEA of several electromagnetic phenomena with plane symmetry leads to formulations with scalar function as unknowns. Vector fields are then obtained by differentiating such scalar functions $U(x, y)$.

By the hypothesis of first order discretization, the potential function $U(x, y)$ is defined inside the element, which has linear variation, as seen in Fig. 2, yielding constant field vectors inside the elements. Therefore, these vectors can be written as a linear combination of vectors which are parallel to the triangles edges. Fig. 2 illustrates the "vectorial sides or edges" of a generic element.

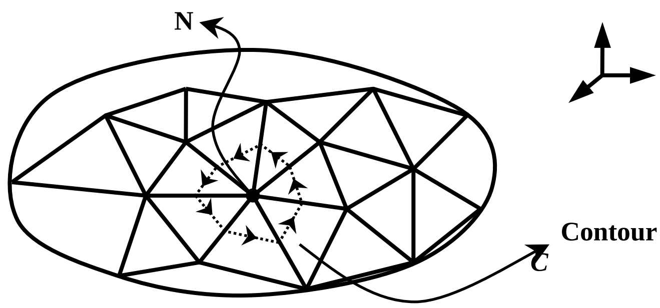

Fig. 1 Triangular mesh, generic node **N** and closed contour ***C***.

The vectorial sides are then defined by:

$$\vec{L}_{1,2,3} = c_{1,2,3}\,\vec{u}_x - b_{1,2,3}\,\vec{u}_y \qquad (1)$$

where $\vec{u}_x$ and $\vec{u}_y$ are the unit vectors of the cartesian coordinate system. Quantities b and c in (1) are computed from the coordinates of nodes $\mathbf{P_1}$, $\mathbf{P_2}$, $\mathbf{P_3}$, as follows:

$$\begin{aligned} b_1 &= y_2 - y_3; \qquad & c_1 &= x_3 - x_2; \\ b_2 &= y_3 - y_1; \qquad & c_2 &= x_1 - x_3; \\ b_3 &= y_1 - y_2; \qquad & c_3 &= x_2 - x_1; \end{aligned} \qquad (2)$$

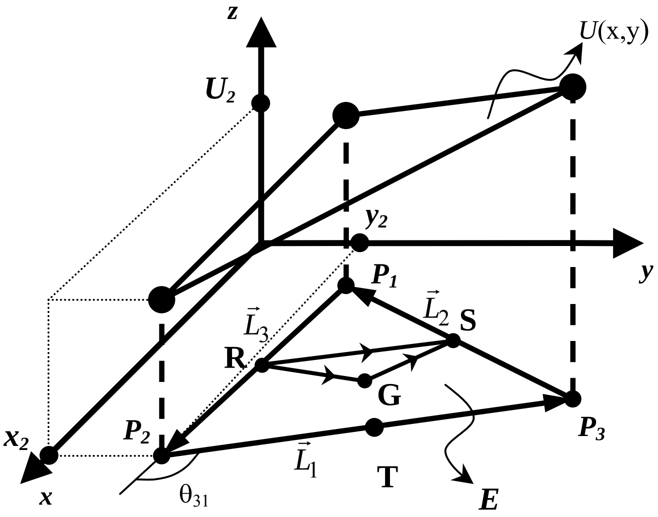

Fig. 2 A generic triangular element showing vectors as its sides, as well as the potential function $U(x,y)$.

It should also be noted that the sides of polygon ***C*** in Fig. 1 are parallel to the sides of the triangles with node **N** as vertex. According to Fig. 2**,** segment **RS** represents one side of the contour involving node 1 (point $\mathbf{P_1}$), that is, $\frac{\vec{L}_1}{2} = \overrightarrow{\mathrm{RS}}$.

On the other hand, the line integration over first-order elements does not depend upon the contour, since the vector field is constant; therefore, the integration can be computed over any other contour with the same end points, for instance **RGS** in Fig. 2, where **G** is the barycenter of the triangle. Note that in all cases points **R** and **S** have to be the mid points of the corresponding sides. The choice of this contour yields the same element matrix obtained by applying the classical approach [1].

The aim of FE simulations is calculating the nodal values of the potential function in the entire mesh. A scalar function $U(x,y)$ is introduced, which is defined inside one element as a function of its values at the triangle vertices (U_1, U_2, U_3), yielding:

$$U(x,y) = \left(\frac{d_1}{D_1}\right)U_1 + \left(\frac{d_2}{D_2}\right)U_2 + \left(\frac{d_3}{D_3}\right)U_3, \tag{3}$$

with
$$N_i(x,y) = \frac{d_i}{D_i} = \frac{a_i + b_i x + c_i y}{2\Delta}, \tag{4}$$

where D_i is the height associated to node i, d_i is the distance between a generic point (x,y) and the side $\vec{L}_i$ and Δ is the area of the triangle. Quantities b_i, c_i, are defined in (2) and $a_i = x_j y_k - x_k y_j$.

Functions $N_i(x,y)$ are the usual nodal shape functions. In the triangular element, the side opposite to node i represents a line of constant value for the shape function $N_i(x,y)$. Then, the side $\mathbf{P_2P_3}$ corresponds to the null isovalue of $U(x,y)$, and along segment **RS,** potential at node 1 is

$$U_1(x,y) = U_1/2.$$

Equation (3) means that $U_i(x,y)$ has 3 components, for i = 1,2,3. Hence, the field vectors have also 3 components, either normal or parallel to the element sides, depending on the field operator used.

In Magnetostatics, the curl operator is used to derive the magnetic field $\vec{\boldsymbol{H}}$ from magnetic vector potential $\vec{\boldsymbol{A}}$, yielding 3 components of $\vec{\boldsymbol{H}}$ which are parallel to the element sides. In Electrostatics, the grad operator applied to the electric scalar potential yields 3 components of the electric field $\vec{\boldsymbol{E}}$ which are normal to the sides of the triangle, as will be shown next

III. Magnetostatics

The second Maxwell's equation can be written as

$$\oint_C \vec{\boldsymbol{H}} \cdot \vec{d\ell} = \int_S \vec{\boldsymbol{J}} \cdot \vec{dS} \tag{5}$$

The final system of algebraic equation can be assembled by applying this equation to the contours enclosing all nodes of the FE mesh.

The magnetic field can be calculated by:

$$\vec{\boldsymbol{H}} = \nu\nabla \times \vec{\boldsymbol{A}} = \frac{\nu A_1}{2\Delta}\vec{\boldsymbol{L}}_1 + \frac{\nu A_2}{2\Delta}\vec{\boldsymbol{L}}_2 + \frac{\nu A_3}{2\Delta}\vec{\boldsymbol{L}}_3, \tag{6}$$

or

$$\vec{\boldsymbol{H}} = \vec{\boldsymbol{H}}_1 + \vec{\boldsymbol{H}}_2 + \vec{\boldsymbol{H}}_3.$$

It can be seen by (6) that the three components of $\vec{\boldsymbol{H}}$ are parallel to the sides of the triangle. The magnetic vector potential has a unique component, normal to the domain Ω, i.e.:

$$\vec{\boldsymbol{A}} = A(x,y)\ \vec{\boldsymbol{u}}_z$$

where the scalar function $A(x,y)$ is written as in (3). It is then easy to derive the field vector, for instance $\vec{\boldsymbol{H}}_1$, as follows:

$$\vec{\boldsymbol{H}}_1 = \nu\nabla \times \vec{\boldsymbol{A}}_1 = \nu\left(\frac{\partial A_1(x,y)}{\partial y}\vec{\boldsymbol{u}}_x - \frac{\partial A_1(x,y)}{\partial x}\vec{\boldsymbol{u}}_y\right)$$

$$\vec{\boldsymbol{H}}_1 = \nu A_1\left(\frac{c_1}{2\Delta}\vec{\boldsymbol{u}}_x - \frac{b_1}{2\Delta}\vec{\boldsymbol{u}}_y\right) = \nu\frac{A_1}{2\Delta}\vec{\boldsymbol{L}}_1$$

As it can be noticed in (6), each nodal potential only affects its own component of field vector, i.e. that with same node index. On the other hand, if the three nodal potentials are equal, the potential in the element will be an isovalue and the resulting field is null, since $\vec{L}_1 + \vec{L}_2 + \vec{L}_3 = 0$. The left hand side of (5) must be evaluated along the oriented contour ***C***, a polygon with "vectorial" edges. These edges are the vectors $\vec{\boldsymbol{L}}_i$ of the triangular finite elements, divided by 2. Then, the circulation of the field vectors along ***C*** yields a summation of dot products of "edge vectors" of a triangle having segments of ***C***. As an example, segment **RS** in Fig. 2 is a section of a contour that encloses node 1; the line integration of magnetic field evaluated in **RS** is:

$$\int_{RS} \vec{\boldsymbol{H}} \cdot \vec{\boldsymbol{dl}} = \vec{\boldsymbol{H}} \cdot \left(\frac{\vec{\boldsymbol{L}}_1}{2}\right)$$

By using (6), it follows that

$$\int_{RS} \vec{H} \cdot \vec{dl} = \frac{\nu}{4\Delta}(A_1\vec{L}_1 + A_2\vec{L}_2 + A_3\vec{L}_3) \cdot (\vec{L}_1) . \quad (7)$$

Then, with the aid of (1) and (2), (7) can be written in terms of nodal coordinates, leading to the following analytical expression for the stiffness matrix of 3-node triangular elements:

$$\int_{RS} \vec{H} \cdot \vec{dl} = \frac{\nu}{4\Delta}[c_1^2 + b_1^2 \quad c_2c_1 + b_2b_1 \quad c_3c_1 + b_3b_1] \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} .$$

This matrix can be obtained by evaluating the three line integrations of an element, namely, **RS**, **ST** e **TR** of Fig. 2, as follows:

$$\begin{bmatrix} \int_{RS} \vec{H} \cdot \vec{dl} \\ \int_{TR} \vec{H} \cdot \vec{dl} \\ \int_{ST} \vec{H} \cdot \vec{dl} \end{bmatrix} = \frac{\nu}{4\Delta}\left[\vec{L}_i \cdot \vec{L}_j\right]_{3x3} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} . \quad (8)$$

The matrix of dot products above can also be expressed in terms of the internal angles of the element. For $i \neq j$, it can be written as:

$$\frac{\vec{L}_i \cdot \vec{L}_j}{2\Delta} = \frac{\vec{L}_i \cdot \vec{L}_j}{\left\| \vec{L}_i \times \vec{L}_j \right\|} = \frac{\cos\theta_{ij}}{\operatorname{sen}\theta_{ij}} ,$$

where θij is the angle formed by edges i and j, as shown in Fig. 2. Then, the same expression for the stiffness matrix, as presented in [1], can be achieved by doing:

$$\frac{\vec{L}_i \cdot \vec{L}_j}{2\Delta} = \cot\theta_{ij} = -\cot\theta_k ,$$

that leads to

$$\frac{\nu}{4\Delta}\left[\vec{L}_i \cdot \vec{L}_j\right]_{3x3} = \frac{\nu}{2}\begin{bmatrix} * & -\cot\theta_3 & -\cot\theta_2 \\ -\cot\theta_3 & * & -\cot\theta_1 \\ -\cot\theta_2 & -\cot\theta_1 & * \end{bmatrix} . \quad (9)$$

The diagonal terms (marked as *) are given by:

$$\frac{\vec{L}_i \cdot \vec{L}_i}{2\Delta} = \frac{\vec{L}_i}{2\Delta} \cdot (-\vec{L}_j - \vec{L}_k) = \cot\theta_k + \cot\theta_j .$$

The element stiffness matrix has all the contributions to evaluate the circulation of $\vec{H}$ along each contour C (around each node **N**). This procedure, extended to all mesh nodes (all contours C), corresponds to the assembly process.

As an example, by taking node **N** in Fig. 1, as being the same as node **1** of Fig. 2, its contour is C. It can be noted in Fig. 1 that seven triangles contribute to this contour, and the evaluation of Ampere's circuital law along such contour leads to:

$$\oint_{C_1} \vec{H} \cdot \vec{d\ell} = \sum_{m=1}^{7}[\vec{H}_m \cdot \left(\vec{RS}\right)_m] . \quad (10)$$

The summation in (10) is equivalent to the summation of each first row of the m elemental matrices, associated with the m finite elements which share node **1**. Therefore, the integration in (10) has been replaced by an algebraic expression.

The right hand side of (5) still remains to be evaluated, which yields:

$$I_1 = \int_{S_1} \vec{J} \cdot \vec{dS} = \frac{1}{3}\sum_{m=1}^{7}(J_m \Delta_m) , \quad (11)$$

where S_1 is the surface defined by C_1, assuming that the current density is constant inside each element. The total current flowing in an element is then equally divided among the nodes.

Finally, the finite element approach applied to the problem formulated by the continuous equation (5) leads to the following algebraic equations:

$$\sum_{m=1}^{M}[\vec{H}_m \cdot \left(\vec{RS}\right)_m] = \frac{1}{3}\sum_{m=1}^{M}(J_m \Delta_m) , \quad (12)$$

where (10) is evaluated around node **N**, which is shared by M elements.

IV. ELECTROSTATICS

The following equation holds for the Electrostatics:

$$\oint_{\Sigma} \vec{D} \cdot \vec{dS} = \int_{\tau} \rho \, dv . \quad (13)$$

The closed surface Σ defines the volume τ, which contains the electric charges that are distributed according to density $\rho(x,y,z)$.

The plane symmetry is obtained by considering domain Ω in Fig. 1 as a cross section of the problem's original domain. Surface Σ then becomes the surface of a prism that is orthogonal to Ω and has its base surface S defined by contour C. Equation (13) can therefore be rewritten as:

$$\oint_{C} \vec{D} \cdot \vec{n} \, h \, dl = h\int_{S} \rho(x,y) dxdy , \quad (14)$$

where h is the undefined height of the domain and $\vec{n}$ is the unit vector normal to contour C.

The unknown potential function is represented by the electrostatic potential V. Through equations (3) and (4) the electric field vector can be obtained as follows:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$$

$$\vec{E}_1 = -\nabla V_1(x,y) = \frac{V_1}{2\Delta}(-b_1\vec{u}_x - c_1\vec{u}_y) \; .$$

The following expression can be derived from equation (1) and the vector product:

$$\vec{E}_1 = \frac{V_1}{2\Delta} L_1 \vec{n}_1 \; ,$$

and therefore:

$$\vec{E} = \frac{V_1}{2\Delta} L_1 \vec{n}_1 + \frac{V_2}{2\Delta} L_2 \vec{n}_2 + \frac{V_3}{2\Delta} L_3 \vec{n}_3 \; . \qquad (15)$$

The electric field vector components are perpendicular to the edges of the triangular element. Fig. 5 shows the components of vectors $\vec{E}$ and $\vec{H}$, which are perpendicular and parallel to the equipotential lines, respectively.

Equation (14) should be replaced by a new expression similar to (12). By considering again node **1** in Fig. 2 and equation (15), it follows that:

$$\vec{D} \cdot \vec{n}_1 = \varepsilon\left(\frac{V_1}{2\Delta} L_1 + \frac{V_2}{2\Delta} L_2 \vec{n}_2 \cdot \vec{n}_1 + \frac{V_3}{2\Delta} L_3 \vec{n}_3 \cdot \vec{n}_1\right)$$

$$\int_{RS} \vec{D} \cdot \vec{n}_1 dl = \frac{L_1}{2} \vec{D} \cdot \vec{n}_1 \; .$$

Therefore,

$$\int_{RS} \vec{D} \cdot \vec{n}_1 \, dl = \frac{\varepsilon}{4\Delta}(V_1 \vec{L}_1 \cdot \vec{L}_1 + V_2 \vec{L}_2 \cdot \vec{L}_1 + V_3 \vec{L}_3 \cdot \vec{L}_1) \, , \; (16)$$

since the angle between two vector sides is the same between the unit vectors normal to these sides.

Equation (16) is similar to equation (7); the element matrix and the global matrix are obtained in the same way as in the section III. The integration in (14), performed along contour ***C*** of Fig. 1 (which encloses node **1** of Fig. 2), is replaced by:

$$\oint_C \vec{D} \cdot \vec{n} \, dl = \int_S \rho(x,y) dx dy \; ,$$

and also by the algebraic equation:

$$\sum_{m=1}^{M} [(\varepsilon \vec{E}_m \cdot \vec{n}_1) \| RS \|_m] = \frac{1}{3} \sum_{m=1}^{M} (\rho_m \Delta_m) \, , \qquad (17)$$

assuming constant charge inside the element.

The element matrix for Electrostatics is nearly the same as in Magnetostatics. The physical property must be changed accordingly:

$$[\Omega_e] = \frac{\varepsilon}{4\Delta} \begin{bmatrix} \vec{L}_1 \cdot \vec{L}_1 & \vec{L}_2 \cdot \vec{L}_1 & \vec{L}_3 \cdot \vec{L}_1 \\ \vec{L}_1 \cdot \vec{L}_2 & \vec{L}_2 \cdot \vec{L}_2 & \vec{L}_3 \cdot \vec{L}_2 \\ \vec{L}_1 \cdot \vec{L}_3 & \vec{L}_2 \cdot \vec{L}_3 & \vec{L}_3 \cdot \vec{L}_3 \end{bmatrix} .$$

V. SETTING THE SOURCE TERMS

The sources of electromagnetic fields, which are distributed across the finite elements, are then replaced by an equivalent distribution which is concentrated in the mesh nodes. The integration contours ***C*** enclose these nodal sources.

One possible criterion for concentrating the sources in nodal values is to preserve the total value within the element. In equations (12) and (17) densities $\boldsymbol{J}_m$ and ρ_m are constant in the M elements that are crossed by the integration contour. Each vertex is associated with 1/3 of the total value of current or electric charge.

If the source density varies along the element, as the potential function $U(x,y)$ does, it can be approximated through equations (3) and (4).

In case of a linear variation of ***J*** inside the element, the total current crossing the element is:

$$\boldsymbol{I}_E = \int_S \boldsymbol{J}(x,y) \cdot dS = \boldsymbol{J}(G) \cdot \Delta \qquad (18)$$

$$\boldsymbol{I}_E = \left(\frac{\boldsymbol{J}_1 + \boldsymbol{J}_2 + \boldsymbol{J}_3}{3}\right) \cdot \Delta \, , \qquad (19)$$

since, according to Fig. 2, **G** is the barycenter of the triangle.

The distribution of the total current of an element in three symmetrical nodal values is given by:

$$[\boldsymbol{I}]_{3x1} = \frac{\Delta}{3} \cdot \frac{1}{(\boldsymbol{p}+2)} \cdot \begin{bmatrix} \boldsymbol{p} & 1 & 1 \\ 1 & \boldsymbol{p} & 1 \\ 1 & 1 & \boldsymbol{p} \end{bmatrix} \cdot [\boldsymbol{J}]_{3x1} \, . \qquad (20)$$

The proof that the total current is preserved after the distribution is straightforward, that is,

$$\boldsymbol{I}_E = \sum_{i=1}^{3} \boldsymbol{I}_i \; .$$

Fig. 3 and 4 illustrate the concept of this equivalent source distribution.

The centroid ***G*** and the midpoints **R**, **S** and **T** divide the triangular element in three polygons with same surface area, Δ/3. A constant current density value is then set to each surface S_i:

$$\boldsymbol{J}(x,y) = \boldsymbol{J}_{S_i} \, , \; (x,y) \in S_i \, ,$$

where,

$$\boldsymbol{J}_{S_i} = \frac{p\boldsymbol{J}_i + \boldsymbol{J}_k + \boldsymbol{J}_j}{(p+2)} \, .$$

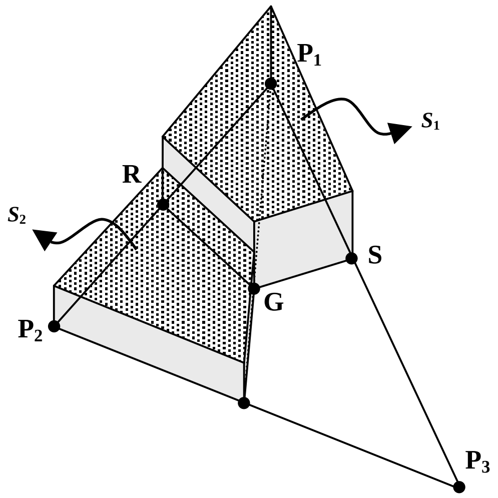

Fig. 3 Distributions of source to nodes 1 and 2.

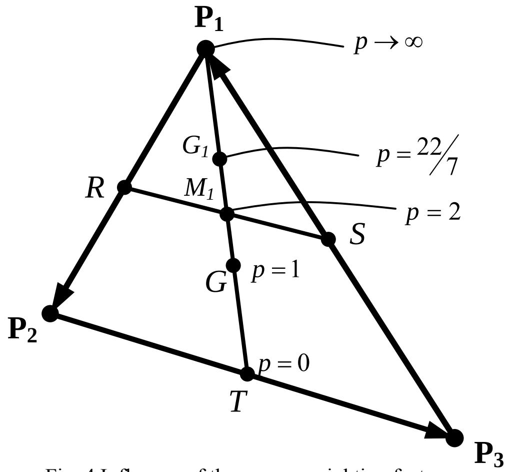

Fig. 4 Influence of the source weighting factor.

The average values J_{Si} are associated to internal points of the surfaces S_i. A particular value of the weighting factor p is associated to a particular point along the median emerging from vertex P_i. As an example, the median $\mathbf{P_1T}$ in Fig. 4 has the sampling points of the current density for node 1. With p=1, the point to be considered is the centroid of the triangle.

The centroid G_1 of surface S_1 gives p = 22/7. Values belonging to the interval [0,1[correspond to points outside surfaces S_i, and therefore are discarded.

The symmetrical distribution of the source value leads to (20), which stands for points along the medians. An additional criterion is to be imposed to define the given point.

In the case of conservation of stored energy W_{mag} inside the element:

$$\boldsymbol{W}_{mag} = \frac{1}{2}\int_{\Omega_E} \vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{J}}\, d\Omega = [\boldsymbol{A}]^T \cdot [\boldsymbol{T}] \cdot [\boldsymbol{J}] \,.$$

The resulting point for vertex P_1 is M_1, as depicted in Fig. 4, since both $\boldsymbol{A}$ and $\boldsymbol{J}$ have linear variation, according to (4). Thus, the weighting factor is p=2. The weighting matrix [$\boldsymbol{P}$] of the source term becomes the same as the matrix [$\boldsymbol{T}$], defined in [1]:

$$[\boldsymbol{P}] = [\boldsymbol{T}] = \frac{\Delta}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

VI. Magnetic induction

In the study of magnetic induction phenomena, the electric field vector possesses a component arising from the time variation of the magnetic field:

$$\vec{\boldsymbol{E}} = -\nabla V - \frac{\partial}{\partial t}\vec{\boldsymbol{A}} \,.$$

The total current density crossing the element in direction $\vec{\boldsymbol{u}}_z$ is obtained by using the constitutive equation $\vec{\boldsymbol{J}} = \sigma\vec{\boldsymbol{E}}$:

$$\vec{\boldsymbol{J}} = \vec{\boldsymbol{J}}_V + \vec{\boldsymbol{J}}_A \,,$$

where $\vec{\boldsymbol{J}}_V$ represents the impressed current density and $\vec{\boldsymbol{J}}_A$ is the element's induced current density.

In the analysis of eddy-current problems the source current density will be interpolated in the same way as the potential function. In Magnetodynamics, for example, the total current density is calculated by:

$$\oint_C \hat{\vec{\boldsymbol{H}}} \cdot \overrightarrow{d\ell} = \int_S \hat{\vec{\boldsymbol{J}}}_V \cdot \overrightarrow{dS} - j\omega\sigma \int_S \hat{\vec{\boldsymbol{A}}} \cdot \overrightarrow{dS} \,. \tag{23}$$

Quantities $\hat{\boldsymbol{H}}, \hat{\boldsymbol{J}}_V, \hat{\boldsymbol{A}}$ are complex in this case. Operator $\partial/\partial t$ has been replaced by $j\omega$.

The second term in the right-hand side gives the eddy-current density, $\boldsymbol{J}_a = j\omega\sigma\boldsymbol{A}$, inside the first-order element, which has also linear interpolation, as the vector potential $\boldsymbol{A}$ does. $\boldsymbol{J}_V$ stands for the current impressed by the drop of the electrostatic scalar potential inside the element.

By considering that the source current density is constant in the element ($\hat{\boldsymbol{J}}_V(x,y) = \hat{\boldsymbol{J}}_0$), the nodal values of current in the generic element can be expressed as:

$$\begin{bmatrix} \hat{I}_1 \\ \hat{I}_2 \\ \hat{I}_3 \end{bmatrix} = \hat{J}_0 \frac{\Delta}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - j\frac{\omega\sigma\Delta}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \\ \hat{A}_3 \end{bmatrix}.$$

These values represent the contributions of the element to the second member of equation (23). The corresponding first member is developed as in (10).

VII. WaveGuides

In the study of electromagnetic wave propagation in waveguides, functions $\hat{\boldsymbol{H}}_z(x,y)$ and $\hat{\boldsymbol{E}}_z(x,y)$ are taken as unknowns for the TE and TM modes, respectively. These functions are phasors that represent the time variation of field vectors in the propagation direction $\vec{u}_z$.

The wave equations for an incoming wave in lossless dielectric media are as follows [6]:

$$\hat{\boldsymbol{H}}_x = \frac{j}{k^2 - \beta^2} \cdot \left(\omega\varepsilon \frac{\partial \hat{\boldsymbol{E}}_z}{\partial y} - \beta \frac{\partial \hat{\boldsymbol{H}}_z}{\partial x} \right) \quad (24)$$

$$\hat{\mathbf{H}}_y = \frac{-j}{k^2 - \beta^2} \cdot \left(\omega\varepsilon \frac{\partial \hat{\mathbf{E}}_z}{\partial x} + \beta \frac{\partial \hat{\mathbf{H}}_z}{\partial y} \right) \quad (25)$$

$$\hat{\boldsymbol{E}}_x = \frac{-j}{k^2 - \beta^2} \cdot \left(\beta \frac{\partial \hat{\boldsymbol{E}}_z}{\partial x} + \omega\mu \frac{\partial \hat{\boldsymbol{H}}_z}{\partial y} \right) \quad (26)$$

$$\hat{\boldsymbol{E}}_y = \frac{j}{k^2 - \beta^2} \cdot \left(-\beta \frac{\partial \hat{\boldsymbol{E}}_z}{\partial y} + \omega\mu \frac{\partial \hat{\boldsymbol{H}}_z}{\partial x} \right) \quad (27)$$

where $k^2 = \omega^2\mu\varepsilon$.

Mode TM is characterized by the constraint $\boldsymbol{H}_z = 0$, which leads to:

$$\hat{\hat{\boldsymbol{H}}}_{xy} = \frac{j\omega\varepsilon}{k^2 - \beta^2} \cdot \sum_{i=1}^{3} \frac{\left(c_i \vec{u}_x - b_i \vec{u}_y\right)}{2\Delta} \hat{\boldsymbol{E}}_{z_i} \quad (28)$$

$$\hat{\hat{\boldsymbol{E}}}_{xy} = \frac{j\beta}{k^2 - \beta^2} \cdot \sum_{i=1}^{3} \left[\frac{\left(c_i \vec{u}_x - b_i \vec{u}_y\right)}{2\Delta} \times \vec{u}_z \right] \cdot \hat{\boldsymbol{E}}_{z_i} \quad (29)$$

Equations (28) and (29) result from the linear variation of $\hat{\boldsymbol{E}}_z(x, y)$, which is computed from the shape functions (2). These equations suggest the representation of the vector-field components in a generic element, as shown in Fig. 5. Equation (28) shows a vector whose components are parallel to the side of the triangle. The other equations represent the components which are perpendicular to the vectorial sides.

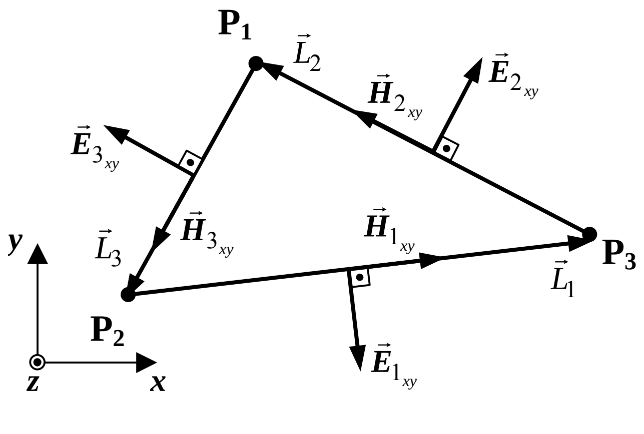

Fig. 5 TM wave: components of both electric and magnetic fields on the sides of a triangular element.

In this work, Maxwell's second equation is written as:

$$\oint_C \hat{\hat{\boldsymbol{H}}} \cdot \overrightarrow{d\ell} = j\omega\varepsilon \int_S \hat{\hat{\boldsymbol{E}}} \cdot \overrightarrow{dS} = \hat{\boldsymbol{I}}_D, \quad (30)$$

which produces the global system of equations, when applied to contours in the mesh. As an example, by looking at Fig. 1, the *n*-th equation of the global system will have seven coefficients regarding the seven sides of polygon C.

For a generic element, the corresponding local matrix is obtained by computing the line integration (30) over the partial contours within the element, and by distributing the total source value over the 3 nodes of the element:

$$\left[\int_{\frac{L_i}{2}} \hat{\hat{\boldsymbol{H}}} \cdot \overrightarrow{d\ell} \right]_{3\times1} = \left[\Omega_{e_{TM}}\right] \cdot \left[\hat{\boldsymbol{E}}_z\right] = \frac{j\omega\varepsilon}{(k^2 - \beta^2) \cdot 4\Delta} \cdot \left[\vec{L}_i \cdot \vec{L}_j\right]_{3\times3} \cdot \left[\hat{\boldsymbol{E}}_z\right]_{3\times1}, \quad (31)$$

The matrix of dot products in (31) represents the line integral computed for segments RS, ST and TR in Fig. 2. Equation (32) gives the displacement currents associated with each node, yielded by the symmetrical distribution of the total current crossing the element.

$$\left[\hat{\boldsymbol{I}}_D\right] = j\omega\varepsilon[T] \cdot \left[\hat{\boldsymbol{E}}_z\right]. \quad (32)$$

The general expression for this distribution is as follows:

$$\left[\hat{\boldsymbol{I}}_D\right] = j\omega\varepsilon \frac{\Delta}{3} \cdot \frac{1}{(p+2)} \cdot \begin{bmatrix} p & 1 & 1 \\ 1 & p & 1 \\ 1 & 1 & p \end{bmatrix} \cdot \begin{bmatrix} \hat{E}_{Z1} \\ \hat{E}_{Z2} \\ \hat{E}_{Z3} \end{bmatrix}. \quad (33)$$

The value $p = 2$ corresponds to a symmetrical distribution that preserves the energy stored in the element. The value of the displacement current is calculated and distributed symmetrically among its nodes.

The global system of equations is of the form:

$$[\mathrm{TM}] \cdot \left[\hat{\boldsymbol{E}}_z\right] = 0. \quad (34)$$

Modal analysis is then performed to solve (34) in order to obtain cutoff frequencies and the wave propagation modes.
The same procedure can be applied for mode TE, yielding the following system of equations:

$$[\mathrm{TE}] \cdot \left[\hat{\boldsymbol{H}}_z\right] = 0. \quad (35)$$

In this case, Maxwell's first equation is applied by computing the circulation of the electric field around each node. The total magnetic flux crossing the element is distributed among its nodes, corresponding to the second term in the equation.

VIII. Summary

An approach has been presented, which is suitable to teach the

FEA of Electromagnetic phenomena with plane symmetry to undergraduate students of Electrical Engineering. It is based on the direct integration of the Maxwell´s equations and the use of first-order triangular elements, thereby avoiding the complex mathematical treatment of this theory that is often encountered in the literature.

By applying the integral form of Maxwell equations to electromagnetic phenomena, the developed approach leads to a numerical formulation.

For both first and second Maxwell equations, a suitably chosen polygon $\boldsymbol{C}$, enclosing each node of the finite element mesh, is used as the integration contour for computing field vectors $\boldsymbol{E}$ and $\boldsymbol{H}$.

In case of both third and fourth Maxwell equations, polygon $\boldsymbol{C}$ is the cross section of a closed surface Σ, through which the fluxes of field vectors $\boldsymbol{B}$ and $\boldsymbol{D}$ are computed.

In both cases, the choice of mid points **R**, **S** and **T** of triangles edges for defining polygon $\boldsymbol{C}$ has led to the same stiffness matrix as those of classical FE formulations, when applied to first order triangular meshes.

IX. References

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