The Idea of an Electromagnetic Field, Numerical Electromagnetism, and the Geometric Perspective.

Percy Hammond's Paper and an Urge to Elaborate.

Prof. Hammond's historical insights into the idea of an electromagnetic field presented so eloquently in this newsletter[H1], teach timeless lessons and speak to current issues in both teaching and research. Being the author of books on electromagnetism addressed to students [H2], practitioners [H3], and researchers [BH] of electrical engineering, he has clearly honed his craft. Given such a wonderful starting point for further dialogue, I could not resist steering it in the direction of several of Prof. Hammond's interests: teaching, geometric and variational underpinnings, and the computational view of electromagnetism.

The concept of an electromagnetic field is not only an important signpost in the history of ideas, but also an organizing principle on which we regularly lean when pushing electromagnetic design in new directions. The geometric underpinnings of electromagnetism can be found in Maxwell's exposition of "tubes and slices" which, for the uninitiated, seems like a qualitative and conceptual framework rather than a quantitative one. I would like to develop the following thesis: Once one separates the concept of an electromagnetic field from the recipes that have evolved for extracting quantitative information in specific problems, one is free to choose the most "natural" mathematical formalism. The choice of mathematical formalism used to describe the electromagnetic field has profound consequences for engineers. The link between Maxwell's Equations and the integral theorems of vector calculus (e.g. Stokes' Theorem and the Divergence Theorem), point to Stokes' Theorem on Manifolds and the formalism of exterior differential forms as the language which doesn't constrain the choice of constitutive laws or any other metric considerations. This view has proven to be the most elegant approach to electromagnetism, but what is not so clear is that this view has also had a profound impact on computational electromagnetism through the adoption of Whitney forms. The thesis can be motivated by means of an easily understood analogy which, incidentally, is a formal consequence of the thesis: When one studies circuit theory, one separates Kirchhoff's laws from the other equations such as those governing branch relations and dependent sources. Kirchhoff's laws summarize topological aspects of circuit theory such as Telegen's theorem and enable us to establish such topological results regardless of whether the branch relations are linear. Furthermore, Kirchhoff's laws dictate the data structures we use in a circuit analysis program. In a similar way, it is necessary to have a formalism to separate the topological aspects of the electromagnetic field which dictate the data structures we use, from the details of constitutive laws.

To support this thesis, I need to strip away the idea of an electromagnetic field from much of what is considered to be engineering electromagnetism. To help with this difficult task, we will consider how solution techniques are often not directly related to the essential electromagnetic aspect of a particular practical problem. This will help us focus on the mathematical tools which have proven essential to harnessing increasingly powerful computational resources for visualizing electromagnetic fields as only we imagined possible in the past. In considering historically significant solution strategies, I hope to make the point that increased computational power has helped us exploit a geometric description of the electromagnetic field to reconcile our understanding of Maxwell's equations with our interaction with computers. Lack of space requires us to give minimal consideration to some general problems in classical electromagnetism. However, to stimulate further discussion, we conclude with some isolated instances of how our students grapple with old mathematical tools in a radically new computational environment.

Preaching to the Choir:

Given how dramatically Maxwell's theory has changed our culture in less than a century and a half, it is futile to summarize its applications and implications. However, as educators and researchers, we are obliged to clarify the fundamental concepts and chip away at new and exciting applications. It is clearly a challenge to present Maxwell's theory as a practical tool for answering design questions, while stressing the underlying geometric and topological insights which were apparent in the 19th century. The latter was articulated mathematically a century ago, but can still seem esoteric. As Einstein discovered, Maxwell's equations in empty space and the definition of mass lead us logically into general relativity and cosmology! We can also marvel at conceptual elegance by studying superconductivity in the context of abelian gauge theories. On the other hand, Maxwell's equations cannot be presented as a self contained theory since constitutive laws governing material media are not part of Maxwell's equations; they rely on materials science and other branches of physics such as solid state physics, thermodynamics and quantum mechanics. The gap between conceptual elegance and practical applications is a challenge: If Maxwell's theory was as easy to use as it is conceptually beautiful, engineers would have no need for intermediate models such as geometric optics, physical optics, microwave circuit theory, transmission line models, or circuits modeled by Kirchhoff's laws. Clearly, simple models, sub-disciplines and interdisciplinary topics are here to stay.

If we strive to present electromagnetism as a unified subject will we have an audience? Put differently, if we show our students the big picture and include the practical details, will they get neck cramps and little more? There are many good presentations of the modern geometric approach to classical electromagnetism. For example, the recent book of Hehl and Obukhov [HO] is particularly useful in terms of presenting the foundation and modern developments while pulling together literature from both sides of the former iron curtain. Many authors, including Prof. Hammond [BH], have definite ideas on how the modern formalism can be put to work for engineers, and this author is no exception [GK]. One cannot appreciate electromagnetism without having applications or a concrete method for performing calculations and in the three references mentioned above, the mathematical model appears when multivariable calculus is recast in the formalism of differential forms. It is important to stress that this mathematical description of electromagnetism is implicit in the first chapter of Maxwell [JCM] and leads to profound practical, pedagogical, and conceptual uses which transcend the models on which we rely for "answers". Differential forms also yield the tools for handling topological aspects which were first being articulated in Maxwell's time [L], [M]. The rumblings of the modern geometric approach can be found in many inspiring sources [RH], [We], [Wh]. More recently, ties between mathematics and physics have made it easier for students to learn basic mathematical tools and the foundations of classical electromagnetics[B],[F],[D].

How computation defines Engineering Electromagnetics.

As mentioned above, geometric intuitions which underlie the notion of an electromagnetic field and Maxwell's view of electrodynamics fit beautifully with the modern reformulation of multivariable calculus in terms of differential forms. In the context of computational electromagnetics and the finite element method, Whitney forms enable the properties of the electromagnetic field to be mimicked in the discrete setting [AB]. However, in the engineering community it should be stressed that the hard earned success enjoyed by Whitney forms has come about by finding the "proof in the pudding" (e.g. not having spurious modes in resonator calculations), rather than having acceptance based on mathematical truth and beauty arguments associating differential forms to the electromagnetic field, foliations to "tubes and slices", and Tonti-like diagrams (elliptic complexes) to models involving partial differential equations. Regardless of the rationale that drives engineers back to the geometric and/or topological roots of classical electromagnetism, few will dispute that the finite element method has been around for half a century yet the notion of edge interpolation was unheard of a quarter century ago. The "big picture" rationale for Whitney forms still seems to lie in a variational approach to combinatorial Hodge theory via a discrete notion of differential forms. The fact that Whitney forms have become an essential tool for the analysis of electric motors, 3-d eddy current nondestructive testing, photonic crystal fibers, and other band gap structures [GNZL], reinforces the idea that the notion of an electromagnetic field is an unifying theme in engineering electromagnetism, and that the underlying mathematical formalism is apparent in the software developed in seemingly unrelated applications.

Historically, solutions to Maxwell's equations were hard to come by. The mathematical techniques used for finding solutions were those which engineers found useful in a variety of contexts but were not necessarily those which best capture our understanding of the electromagnetic field. This is important to remember as we consider the relative importance of mathematical techniques engineers have accumulated. Like it or not, we try to turn the art of electromagnetic design into a science. We count on computers to help automate design processes and present us with answers which correlate immediately with our experience, senses, and geometric/topological intuitions. As we use computers and spatial reasoning to communicate more directly with our designs and let go of certain well established techniques, it is interesting to make our peace with why certain analytic techniques became so prominent in the days before digital computers became an engineering tool.

The 1930s and 40s can be seen as the golden age of complex variable methods in electrical engineering. Not only did phasors become a necessity for solving large linear circuits, filter design required the introduction of the frequency domain. Linear systems theory grew up around amplifier design, and the constraint of causality brought complex variable methods to the fore via the Payley-Wiener Theorem. Amplifier designers also used the Nyquist stability criterion, justified by the principle of the argument. Starting with Bode plots, Kramers-Kronig dispersion relations and single sideband modulation schemes, Hilbert transform pairs appeared everywhere and anywhere electrical engineers found a function analytic in a half plane. Moving past analog signal processing, the sampling theorem stood on the shoulders of the Poisson resummation formula. In addition to being used for solving 2-d potential problems, electrical engineers found use for conformal mappings in the Nichols chart and the Smith chart. Potential theory reinforced both the understanding of analytic functions in terms of poles and zeros, and kept students busy computing the lumped parameters associated with certain transmission line cross sections. Conjugate analytic functions made perfect sense when one verified that the electromagnetic disturbance on an ideal transmission line traveled at the speed of light. Furthermore, conjugate analytic functions are the basis of graphical solution methods (curvilinear squares) and in 2-d are the analogs of tubes and slices! Before the days of cheap computation and visualization, the conceptual framework given by complex variable methods served the electrical engineer well. Everything was understood in terms of 2-d theory even when it made little sense to do so. For the most part, the use of complex variable methods has atrophied and has been supplanted by numerical algorithms. Occasionally, they outperform the obvious numerical approaches, but they eventually get absorbed into the world of numerical algorithms; just as Bode's gain and phase margins in stability theory have evolved into $\mathbf{H}^{\infty}$ control theory. Today, analytic function techniques still can defy numerical analysts in the context of asymptotic expansions. Be that as it may, the geometric techniques come to the fore when we need to visualize a 3-d electromagnetic field and escape from the traps that our 2-d tools have set for us.

Useful 3-d solutions appeared early on and most often were a result of separating variables. Glancing at the last chapter of Stratton[S], we get a glimpse of how electromagnetism was taught on the eve of WW II; the advent of microwave technology and radar. Waveguide analysis and the classification of modes into TE, TM and TEM was performed by Rayleigh in the 1897, (and would become a classified wartime secret 45 years later), Mie's analysis of the scattering of light by a dielectric sphere was well-known and applications to atmospheric physics abounded, as were other wonderful applications of the separation of variables in a vector context. However, separation of variables, as a general method for finding 3-d solutions was identified as a dead end in the late 1890's when Laplace's and Helmholtz' equations were shown to separate in only a limited number of orthogonal curvilinear coordinate systems, where the coordinate surfaces are given by algebraic equations of degree no greater than four. However, because of limited possibilities for solution techniques, "spherical cow jokes" had to wait. With the acceptance of numerical methods exploiting unstructured meshes, it became apparent that analytic solutions in general, and separation of variables in particular, make one numb to the possibility of separating metric and constitutive law information from the geometric insights gained from the distinction between field intensities and flux densities. Although this distinction is apparent in the first chapter of Maxwell's treatise, it is only in the last two decades that Whitney forms have enabled us to appreciate this distinction as a fundamental aspect in the context of numerical solution schemes. In retrospect, the analytic function theory that helped us in 2-d and the special function techniques that were advocated for 3-d left us largely unprepared for "thinking out of the box" when it came to visualizing the electromagnetic field in three spatial dimensions. Fortunately, we can still take refuge in Faraday's and Maxwell's notion of an electromagnetic field within a precise modern language.

Linear algebra and matrices took on a new life with the development of scientific computing in the 1950's but didn't make a splash in the EE curriculum for another decade. Unfortunately learning to compute the Jordan canonical form by appealing to the characteristic polynomial of a matrix, largely kept students in the dark about how eigenvalues were actually computed. Needless to say, the appearance of state-space methods in control theory in the 1960's secured the role of linear algebra in the EE curriculum. Discrete time signal processing and the FFT also taught us the importance of exploiting structured matrices and fast algorithms. The notion of a fast algorithm and the techniques of signal processing in turn, made a great impact on integral equation solvers. Condition numbers and singular value decompositions eventually became important notions to have in one's tool box.

The 1960's also brought a revival of the calculus of variations via optimal control theory. Optimization theory did a lot to focus our attention on duality and analytic details. Unfortunately, for those studying electromagnetism, Lagrangian mechanics now played a secondary role and the description of variational problems posed on manifolds in terms of differential forms, was appreciated by only a small group who were rediscovering the works of Elie Cartan. Texts which reinforced Cartan's view [MTW] certainly didn't seem to be swimming with the flow. Although the generalities of optimization theory offered a framework for inverse problems and electromagnetic design, the abstract framework did not make concrete prescriptions for exploiting the geometric and topological structure of electromagnetism. For members of the COMPUMAG community versed in both classical electromagnetism and effective numerical methods, it is irresistible to meditate on the changing role of and reason for variational methods. Just think of Maxwell's appreciation of Lagrangian mechanics, stress tensors, Hamilton-Jacobi theory, and the Rayleigh quotient, and then consider how direct variational methods and the symmetric eigenvalue problem took center stage early in the twentieth century.

In the 1970's computing power was available for independent research, and the linear algebra, optimization theory, and graph algorithms made the well established integral and differential equation formulations of electromagnetism amenable to solution by programs written by small research groups. The intrinsic challenge of three dimensional vector problems was blissfully obscured by a lack of computing power. Old tricks like using a stream function to represent a 2-d solenoidal vector field couldn't be recycled in three dimensions and one had to learn to deal with the vector potential and honest vector fields. This set the stage for the Whitney form revolution in the 1980's, yet we still struggle to use the coordinate-free geometric language to communicate visually and effectively with computers.

On Professing Maxwell's theory.

The beauty of Maxwell's theory is that four equations (or two in space-time) fundamentally changed physics. Maxwell's theory unified electricity, magnetism, and optics, and in the process, forced us to search for radio waves. In the short time between Maxwell's analyses, Hertz's experiments, and Marconi broadcasting across the Atlantic, our evolving means of communication was revolutionized. As teachers, we are now challenged to articulate modulation techniques, Shannon's theory, cell phone networks, and the terabit capacity of optical fibers to students, within a framework useful for understanding both historic and future developments in engineering electromagnetics.

When teaching, an aspect we emphasize about Maxwell's theory is that it left no room for the Galilean invariance of Newton's mechanics, and in doing so it pulled the rug out from under Newton's theory. It is easy to point to the Michelson-Morley experiment and Einstein's 1905 paper on electrodynamics as an isolated stroke of genius, but a cleaner but a less sanitized perspective can be gained by reading a reprint[P] of Poincare's summary, a century ago, of fundamental problems in mathematical physics. Poincare, touches on all the topics of the papers of Einstein's "golden year" (special relativity, Brownian motion, and photoelectric effect). These topics clearly fall in the areas where Maxwell made fundamental contributions and seeing it in this light gives more insight into Einstein's genius. The marvelous development of classical electromagnetics, pointed the way to the General theory of relativity [We]. Long before the Bohm-Aharanov experiment, it laid the foundation for the role of gauge invariance in quantum mechanics, when in the 1920's London reformulated superconductivity in terms of an abelian gauge theory and Dirac discovered a relativistic quantum mechanical theory of the electron. It is clear that deep ideas have to be mastered if one is to extract revolutionary new technology: It took well over a half century to build an optical gyroscope by appreciating the Sagnac effect as a four-dimensional electromagnetic phenomenon. Even the teaching of magnetics is still quite a disgrace; quantum mechanics is required to explain why ferromagnetic materials have domains, and the road from the Dirac equation to the Landau-Lifschitz equation seems untraveled!

The conceptual beauty of electromagnetic theory is appreciated in four dimensions, and the formalism of differential forms and Whitney forms prepare us for a life in n-dimensions, yet when we seek explicit solutions to Maxwell's equations we seem condemned to curse about dimensionality as we pass from two to three dimensions!

Some speculations.

We have separated the truth and beauty description of the electromagnetic field in terms of differential forms from constitutive laws, metric considerations, and solution techniques. This let us focus on the four dimensional aspects which have impacted theoretical physics, as well as Whitney form techniques which have impacted computational electromagnetics. At this point it would be natural to speculate about the metric and the constitutive laws I have chosen to ignore and the interaction with Materials science and thermodynamics. It may also be unwise to do so, as these speculations take time to articulate and there has been more than a century of such speculations. One merely has to look at a recent book[FM] to see how wonderfully these questions can be developed in specific instances yet seem to defy a “one size fits all” treatment. I would like to take the reader on a different path.

Above, we strived to separate the electromagnetic field from the specifics of solution methods in order to focus on the current relevance of the notion of an electromagnetic field. It would seem that the mathematical tools for describing the electromagnetic field are timeless, but that mathematical and algorithmic tools for obtaining solutions constantly adapt to the hardware involved in obtaining solutions. I would now like to turn things around and examine how the algorithmic tools which have evolved since Maxwell’s time can be recycled to fit the needs and interests of our students. I hope this will provoke some further comment.

To fix ideas of how traditional variational methods and eigenvalue problems fit into the modern world, consider the following. After grasping the notions of a condition number, and conditioning, students can easily understand why one can neither compute the characteristic polynomial of a large matrix accurately, nor find the roots of a generic high degree polynomial accurately. Once this knowledge is used to discredit the simple-minded approach to finding eigenvalues, students can turn to Matlab, and experience the irony that roots of polynomials can be computed effectively by forming a companion matrix and finding its eigenvalues by using the QR method with implicit shifting and preprocessing to Hessenberg form! This is just one example of how our teaching of numerical methods has become organized around algorithms that have proven to be “unreasonably effective”, but not necessarily easy to analyze. The conjugate gradient method is probably another good example. Similarly, we have deemphasized the Jordan canonical form in favor of the singular value decomposition which is based on the symmetric eigenvalue problem and its interpretation as a variational problem.

Gone are the days when students get excited about eigenvalue problems by identifying the principal axies of conic sections. Eyes do light up however, when one shows students how the search engine Google finds page rankings through the use of Rayleigh quotients and doubly stochastic matrices[C]! Old timers like me find it mind-boggling that Google depends on finding dominant eigenvectors of matrices with ranks in the billions, while processing thousand of such requests a second. What would Maxwell or Raleigh say? We will never know. What we do know is that the fundamentals of Maxwell’s theory will endure, and equally enduring is the never ending quest to bring the history alive through modern applications, explanations, and uses of algorithms.

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