

Handbook for the computation of electromagnetic forces in a continuous medium

Abstract — An energy-based theory for electromagnetic forces in continuous media is proposed, aiming at providing a guide and a complete toolbox for their numerical computation. In an Euclidean space, the electromechanical coupling is shown to be realised by a stress tensor, in terms of which the classical electromagnetic force formulae can be re-interpreted, unified, and sometimes generalised.

I. INTRODUCTION

Prof. Dave Lowther maintained in his excellent communication at the EMF Conference in Stratford-upon-Avon [1] that the question of the computation of forces in computational electromagnetics had been solved for long within a sufficient accuracy. One may however notice, by going through the references he mentions, that the techniques he refers to assume very simple materials and do not consider the problem with a sufficient generality. In modern computational electromagnetics, one needs indeed more and more to compute local forces in material exhibiting not only saturation and anisotropy but also magnetostriction and hysteresis. The issue of forces comes then into play entangled with energy and multiphysics considerations.

Whereas the profound theoretical issue and the fundamental understanding of this difficult topic have been tackled with by Alain Bossavit in the previous edition of the newsletter [2] and in former publications [3, 4, 5], we have found it useful in this paper to provide a more operative formalism, i.e. a set of general rules and formulae that lead straightfully from the very statement of the electromechanical problem up to a practical implementation of a solution method for it.

The existence of such a long controversy about the computation of electromagnetic forces and of so many uncertainties among the practitioners is certainly to ascribe to the fact that the issue can not be completely clarified with the concepts of Vector analysis. The mathematical analysis of this problem requires to consider a deforming body, and to apply energy conservation rules to it. The background required to perform such operations is Differential geometry (See e.g. [6]).

The paper begins therefore with a presentation, which assumes no prior knowledge, of Differential geometry. In just four pages, the concepts required to the discussion are presented in a way that is intended to be didactic and understandable to non-specialists.

The second part is an energy-based formulation, entirely based upon the notions introduced in the first part, of electromagnetism and of the electromechanical coupling. Casting the famous Maxwell equations into a coherent energy framework is indeed essential to our purpose and the discussion of electromagnetic forces ensues then naturally. Continuous media and no relativistic effects are however assumed.

The third part is the transcription of the results of the second part in terms of vector and tensor fields. This reveals that the coupling can be expressed in general in terms of a stress tensor. The procedure to determine that stress tensor is described.

Finally, it is shown in the last part of the paper that the wide range of forces formulae and methods that are found in literature and applied in numerical simulations, can be unified quite straightforwardly thanks to this coupling stress tensor, which is not only a backwards result but also a solid departure point to advisedly tackle with more complex materials.

II. DIFFERENTIAL GEOMETRY

A. Manifold

Differential geometry is the branch of Mathematics that studies functions and fields defined on continuous sets, and their mappings. A manifold M of dimension n is a set of points of which any neighbourhood can be mapped by a differentiable 1-1 mapping onto a subset of \mathbb{R}^n . Each such mapping determines by the way a valid local coordinate system on that neighbourhood of P in M . By assuming only the existence of the mapping, without requiring it to be specified, the manifold is solely endowed with differentiability properties (it actually inherits those of \mathbb{R}^n) without getting already equipped with a preferred global coordinate system or even the notion of distance or angle.

In summary : By definition, a manifold is endowed with the differentiable structure of \mathbb{R}^n but not with its metric structure (lengths, angles, scalar product, norm, ...). This gives the mathematical definition we need for a deformable continuous medium.

B. Vector

Let us first introduce the notation

$$u : x \in D \subset M \mapsto y \in E \subset N \quad (1)$$

for a function (or a map). In this exhaustive notation, u stands for the name of the function, x and y are respectively the name of the variable and the name of the value of the function, so that we can write $y = u(x)$. Finally, $D \subset M$ and $E \subset N$ are respectively the domain and the codomain of the function. The alternative notations $D \equiv u^{-1}(E)$ and $E \equiv u(D)$ can be used if necessary. All elements of the complete definition (1) are of course not always relevant, and shortened notations are used whenever no confusion is possible. It will prove to be essential in the following, for the correct definition of electromagnetic forces, to clearly make the difference between the function (here u) and its value (here y).

After this preamble, let the map $u : t \in \mathbb{R} \mapsto C \subset M$ describe a differentiable curve C in M . Let $f : M \mapsto \mathbb{R}$ be a scalar smooth function defined on M . The curvilinear derivative of the function f at a point $P \in M$, which we shall note

$$\partial_t f \equiv \lim_{\Delta t \rightarrow 0} \frac{f(u(t + \Delta t)) - f(u(t))}{\Delta t} \quad (2)$$

is an intrinsic quantity as, by definition, it is determined without the need to refer to any system of coordinates on M . But this quantity is composite in the sense that it mixes up something depending on the curve u with something depending on

the function f . In order to take these two components apart, keeping on however with intrinsic quantities, one resorts for a moment to a coordinate system $\{x^i\}$ defined on M . The map u is now a function $u : t \in \mathbb{R} \mapsto \{x^i\} \in \mathbb{R}^n$, whose value is not any more a point in M but the coordinates referring to this point. The curvilinear derivative writes now

$$\partial_t f = (\partial_{x^i} f) \partial_t x^i. \quad (3)$$

As (3) holds for any function f , one may consider

$$\partial_t = (\partial_t x^i) \partial_{x^i}. \quad (4)$$

as being what depends specifically on the curve in the curvilinear derivative, i.e. the vector tangent to the curve C at P . One indeed recognises in $(\partial_t x^i)$ the *components* of that vector. Hence, the ∂_{x^i} 's, which are the vectors tangent to the coordinate lines, play the role of a basis for vectors anchored at P , and ∂_t is the notation for the vector itself.

This shows the set of all vectors tangent to the curves going through a point $P \in M$ has the structure of a linear space of dimension n . This linear space is called tangent space and noted $T_P M$. But the point is this: Whereas the components of the vector $(\partial_t x^i)$ are not intrinsic, as they explicitly involve the coordinates, the entire vector ∂_t is, as suggested by the fact that the differentials of the coordinates cancel each other out in (4). More precisely, this cancellation means that the quantity noted ∂_t and called vector, is independent of the chosen coordinates. Indeed, if a change of coordinates $x^i \mapsto y^J$ is performed, one has by application of the chain rule

$$\begin{aligned} \partial_t &= (\partial_t x^i) \partial_{x^i} = (\partial_{y^K} x^i) (\partial_t y^K) (\partial_{x^i} y^J) \partial_{y^J} \\ &= (\partial_t y^K) \partial_{y^K}, \end{aligned}$$

since $\Lambda_i^J \equiv (\partial_{x^i} y^J)$ and $\Lambda_K^i \equiv (\partial_{y^K} x^i)$ are the coefficients of Jacobian matrix of the change of coordinates and its inverse, $\Lambda_K^i \Lambda_i^J = \delta_K^J$.

A vector field is a rule that associates a vector, i.e. an element of $T_P M$, to each point $P \in M$. The set of all vector fields defined on M is called tangent manifold and noted TM .

In summary: By considering the curvilinear derivative of a function, one finds that the vectors form, at each point of a manifold, a linear space of the same dimension as the manifold itself.

C. Covector

Similarly, one may recognise in the $(\partial_{x^i} f)$'s in (3) the components of the gradient of f in the coordinate system $\{x^i\}$. Let us agree to note df for the gradient of the function f ¹. Since the curvilinear derivative $\partial_t f$ is a scalar, (3) shows that the gradient of f acts as a real-valued intrinsic (again the cancellation of the coordinates's differentials) operator on the tangent vector ∂_t , which we shall write

$$\partial_t f = df(\partial_t) = \langle df, \partial_t \rangle. \quad (5)$$

This operator is (bi-)linear, as a consequence of the linearity of the chain rule of partial derivatives in \mathbb{R}^n . The set of all gradients at P has therefore also the structure of a linear space of dimension n . It is noted $T_P^* M$ and called cotangent space. Its elements are called covectors. The gradients dx^i of the coordinates form a basis for the covectors at P , so that

$$df = (\partial_{x^i} f) dx^i \quad (6)$$

¹The operator d will later on be given a more general definition.

is the intrinsic representation of the gradient.

A covector field is a rule that associates a covector, i.e. an element of $T_P^* M$, to each point $P \in M$. The set of all covector fields defined on M is called cotangent manifold and noted $T^* M$.

In general however, vectors and covectors behave differently under a change of coordinates :

$$dy^J = (\partial_{x^i} y^J) dx^i, \quad \partial_{y^J} = (\partial_{y^J} x^i) \partial_{x^i}. \quad (7)$$

The reason why covectors are however ignored in vector analysis is that they cannot be distinguished from the vectors in that particular framework. As vector analysis exclusively works in an Euclidean space and with orthonormal coordinate systems, all change of coordinates are rotations and the associated Jacobian matrices are orthogonal. Vectors and covectors have then all the same properties.

In summary: When applying a change of coordinates, the chain rule of derivatives induces naturally a fundamental distinction between covariant and contravariant quantities. If by *convention* the coordinates, e.g. $\{x^i\}$, are represented with an upper index, covariant quantities have a lower index and contravariant quantities have an upper index. Whereas the *components* of a covector and the basis vectors ∂_{x^i} are covariant quantities (an upper index in a subscript counts as a lower index), the *components* of a vector and the basis covectors dx^i are contravariant quantities, see (4) and (6). Intrinsic quantities are those for which all upper indices are involved in an implicit summation with a corresponding lower index.

D. Tensor

By a direct generalisation of (5), a tensor at a point $P \in M$ is defined as a real-valued multilinear operator acting on an ordered set of vectors $v, \dots \in T_P M$ and covectors $a, \dots \in T_P^* M$:

$$A(v, \dots; a, \dots) = A_{i\dots}^{j\dots} v^i \dots a_j \dots \quad (8)$$

$$A = A_{i\dots}^{j\dots} dx^i \otimes \dots \otimes \partial_{x^j} \otimes \dots \quad (9)$$

with \otimes the tensor product.

In summary: Tensors are (multi-)linear operators on vectors and covectors. Their coordinate representation is throughout consistent with the upper (contravariant) and lower (covariant) index notation. Change of coordinates are performed by applying the Jacobian matrix to all indices according to the implicit summation rule.

E. Integral

The trajectory C of a particle in M is the codomain of the map $u : t \in [t_A, t_B] \subset \mathbb{R} \mapsto x^i \in C \subset M$. It is a curve. The velocity is the vector tangent to that curve. According to the previous section, it can be simply noted ∂_t . On the other hand, the force F applied to that particle is a covector. The scalar result of the application of F to ∂_t is the instantaneous mechanical power $\dot{W} = F(\partial_t)$. Given a force field, say a covector at each point $P \in M$, which is still noted F , the mechanical work W can be expressed by selecting on the trajectory C a discrete set of points $P_r, r = 0, \dots, N$, corresponding with the values $r \frac{t_B - t_A}{N} = r \Delta t$ of the parameter t . One has then for W the intrinsic expression

$$W = \int_C F = \lim_{N \rightarrow \infty} \sum_{r=1}^N F(\Delta t \partial_t)|_{P_r} \quad (10)$$

where $\Delta t \partial_t$ represents at P_r a finite walk along C . This is the definition of the (curvilinear) integral of F over C . It shows the role played by the covector as an operator on vectors.

Similarly, a smooth surface is defined by a mapping $u : s, t \in D \subset \mathbb{R}^2 \mapsto S \subset M$. By analogy with (10), the surface integral of a tensor field A should have an expression like

$$\int_S A = \lim_{N \rightarrow \infty} \sum_{r=1}^N A(\Delta s \partial_s, \Delta t \partial_t)|_{P_r} \quad (11)$$

where the P_r 's are a set of points defining the vertices of a convenient regular grid on the surface S . This means that the tensor should have two vector-arguments and that $A(\Delta s \partial_s, \Delta t \partial_t)$ should represent the flux of A through the finite grid cell underlain by the vectors $\Delta s \partial_s$ and $\Delta t \partial_t$. The tensor A must therefore be such that the flux is reversed whenever the surface itself is reversed (i.e. the vectors are permuted) and the flux is zero whenever the surface vanishes (i.e. the vectors are parallel). This amounts to require A to be antisymmetric. Further on, an antisymmetrised tensor with 3 vector-arguments represents a volume density. The assertion is actually also valid for higher order tensors, so that it is worth giving those tensors a special name. A p -covector is a tensor with no covector-argument and p vector-arguments, verifying

$$A(\dots, v, \dots, w, \dots) = -A(\dots, w, \dots, v, \dots) \quad (12)$$

for any pair of arguments.

Similarly, a p -vector is a completely antisymmetrised tensor with no vector-argument and p covector-arguments. In a 3D space, antisymmetrisation only leaves p -(co)vectors for which $p = 0, 1, 2, 3$, the other ones are identically zero.

In summary : Antisymmetrisation of covariant indices is the operation that selects the particular tensors that play a role for integration.

F. p -forms

A p -form, or a differential form of degree p , is a field of p -covectors. It is fundamentally an argument for an intrinsic p -fold integral. Let $\Lambda^p(M)$ be the set of all p -forms defined on M . In a 3D space, it is easily deduced from (12) that 0-forms and 3-forms have 1 component at each point, whereas 1-forms and 2-forms have 3. Electromagnetic fields can be seen as paradigms for the p -forms. The electric potential is a 0-form; the magnetic vector potential, the magnetic field and the electric field are 1-forms; the induction field, the electric displacement and the current density are 2-forms; the charge density finally is a 3-form. Of course, p -forms need specific intrinsic antisymmetry-preserving operators. The antisymmetry-preserving tensor product is the exterior product \wedge (13) and the antisymmetry-preserving spatial derivative is the exterior derivative d (14).

$$\wedge : \Lambda^p(M) \times \Lambda^q(M) \mapsto \Lambda^{p+q}(M) \quad (13)$$

$$d : \Lambda^p(M) \mapsto \Lambda^{p+1}(M) \quad (14)$$

Complete definitions can be found in any Differential geometry treatise, e.g. [6]. One has in particular

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta \quad (15)$$

for all p -forms α and β .

Exterior derivative obeys also the Stokes theorem

$$\int_{\Omega} d\alpha = \int_{\partial\Omega} \alpha \quad (16)$$

where $\partial\Omega$ denotes the boundary of the integration domain $\Omega \in M$. It reveals the duality pairing between p -forms and p -dimensional integration domains, and it shows how p -forms carry topological information from the boundary of the domain inwards through their exterior derivative. From their being essentially arguments for p -fold integrals, p -forms are endowed with specific continuity properties. If A is a p -form defined on a 3D domain Ω , the continuous part of A across any surface $\Sigma \subset \Omega$ is by definition its trace $A|_{\Sigma}$ on that surface. The trace of a 0-form is determined by its value at the points of Σ , i.e. the trace is the 0-form itself. The trace of a 1-form is determined by its circulation over all curves defined in Σ , i.e. the trace is the tangential component. The trace of a 2-form is determined by the fluxes through all subsets of Σ , i.e. the trace is the normal component. Finally, the trace of a 3-form is determined by the surface integral of a volume density, which gives always zero and hence no continuity condition for a 3-form. Furthermore, the p -forms inherit of a specific way of interpolation on a finite element mesh. Only the 0-forms have the right to be interpolated in terms of their value at the nodes of the mesh. For the other kinds of p -forms, the adequate interpolation parameters are respectively the integrals over the edges of the mesh for the 1-forms, the integrals over the facets for the 2-forms and the integrals over the elements for the 3-forms. Shape functions corresponding with these node-, edge-, facet- and element-based connectors are called Whitney forms.

In summary : Whitney forms ensure a finite element interpolation that preserves the continuity properties of p -forms at the discrete level.

III. MECHANICAL FRAMEWORK

A. Placement map

The theoretical framework we need to analyse the electromechanical coupling, relies upon two manifolds with distinct functions : the material manifold M of which each point is associated with a material particle of the deforming body (e.g. an atom), and the Euclidean space E which is a manifold where a metric (See further) has been defined and which represents the space where the motion takes place.

Let the placement map

$$p_t : X \in M \subset \mathbb{R} \mapsto x = p_t X \in E, \quad (17)$$

be the map that associates its position in E to each material particle $X \in M$ at all instants of time $t \in [t_A, t_B]$. The codomain of the placement map, $p_t M$, is the deformed state. On the other hand, the codomain of the map $t \in [t_A, t_B] \mapsto p_t X \in E$ is the trajectory of a particular material particle X . The velocity field, $v = \partial_t x \in TE$, is the field of tangent vectors to all trajectories of the flow at a given instant of time.

The placement p_t is assumed to be regular and invertible at each instant of time. It induces a 1-1 mapping, also noted p_t , of all tensor quantities defined on M to corresponding tensor quantities defined on E . If quantities defined on M are denoted with an uppercase symbol, and if quantities defined on E are denoted with a lowercase, one has $p_t A = a$, $p_t^{-1} a = A$. In index notation, the components of a and A are related by the Jacobian matrix of p_t , e.g. $a^i = \Lambda_J^i A^J$ with $\Lambda_J^i = \partial_{X^J} x^i$. The placement map of p -forms commutes with the exterior derivative d , thus the commutation property

$$d p_t = p_t d. \quad (18)$$

For all p -forms α, β , one has also

$$p_t(\alpha \wedge \beta) = (p_t\alpha) \wedge (p_t\beta) \quad (19)$$

and

$$\int_{p_t M} a = \int_M p_t^{-1} a. \quad (20)$$

B. Material derivative

The material derivative of a tensor field a defined on E is by definition

$$\mathcal{L}_v a = \lim_{\Delta t \rightarrow 0} \frac{p_t p_{t+\Delta t}^{-1} a(t + \Delta t) - a(t)}{\Delta t} \quad (21)$$

Consider now a tensor field A defined on M with $A = p_t^{-1}(a)$. One has

$$\begin{aligned} \partial_t A &= \partial_t p^{-1} a \\ &= \lim_{\Delta t \rightarrow 0} \frac{p_{t+\Delta t}^{-1} a(t + \Delta t) - p_t^{-1} a(t)}{\Delta t} \\ &= p_t^{-1} \lim_{\Delta t \rightarrow 0} \frac{p_t p_{t+\Delta t}^{-1} a(t + \Delta t) - a(t)}{\Delta t} \\ &= p_t^{-1} \mathcal{L}_v a, \end{aligned}$$

which is the fundamental commutation property

$$\partial_t p_t^{-1} = p_t^{-1} \mathcal{L}_v. \quad (22)$$

By (20) and (22), the material derivative verifies

$$\begin{aligned} \partial_t \int_{\Omega} a &= \partial_t \int_{p_t M} a = \partial_t \int_M p_t^{-1} a = \int_M \partial_t p_t^{-1} a \\ &= \int_M p_t^{-1} \mathcal{L}_v a = \int_{p_t M} \mathcal{L}_v a = \int_{\Omega} \mathcal{L}_v a \end{aligned}$$

which means

$$\partial_t \int_{\Omega} a = \int_{\Omega} \mathcal{L}_v a. \quad (23)$$

C. Metric

The notions of length and angle are defined in a manifold Ω by means of the metric g which, at each point $x \in \Omega$, associates a number to any pair of anchored vectors :

$$g : v, w \in T_x \Omega \mapsto g(v, w) = g_{ij} v^i w^j \in \mathbb{R}. \quad (24)$$

In this theory, the metric is only defined in the Euclidean space E , where one has $g_{ij}^E = \delta_{ij}$. The metric can of course be pulled back to M , $g^M = p_t^{-1} g^E$ where it then depends on time. The metric not only determines the lengths but also the intensity of the fields, and consequently the energies, as we shall see further.

IV. ENERGY-BASED FORMULATION OF ELECTROMAGNETISM

A. With Differential geometry

The state variables that represent the electromagnetic fields in our system are two 2-forms defined on the material manifold. Let

$$D, B \in \Lambda^2(M). \quad (25)$$

be respectively the electric flux density and the magnetic flux density. The metric is required in order to attribute them an intensity. The magnetic flux density B , for instance, associates a flux φ (in Weber) to any material surface Σ in M . But one needs

to know the measure of $p_t \Sigma$ in E , hence the metric on E , to determine the intensity of the field $\varphi / \text{measure}(p_t \Sigma)$. As energy is a function of the intensity of the fields, the electromagnetic energy density ρ^Ψ is a function taking as arguments the image by the placement of the state variables :

$$\rho^\Psi : d \equiv p_t D, b \equiv p_t B \in \Lambda^2(E) \mapsto \Lambda^3(E). \quad (26)$$

A particular shortcut notation is adopted for the *value* of the Frechet derivatives of ρ^Ψ :

$$\begin{aligned} (\partial_d \rho^\Psi)(p_t D, p_t B) &\equiv \tilde{e} \in \Lambda^1(E), \\ (\partial_b \rho^\Psi)(p_t D, p_t B) &\equiv \tilde{h} \in \Lambda^1(E), \end{aligned}$$

where the tilde reminds that these 1-forms not only depend on the state variables D and B , but also on the definition of the electromagnetic energy density ρ^Ψ and on the placement map p_t . Note that we do not consider these relations as constitutive laws, but rather as shorthand notations.

We can now establish the terms of the energy balance.

a) The electromagnetic energy of the system writes

$$\Psi = \int_{\Omega} \rho^\Psi(p_t D, p_t B), \quad (27)$$

where $\Omega \equiv p_t M \subset E$ is the deformed state of the material manifold.

b) The electromagnetic power exchanged by the system through its surface $\partial \Omega$ writes

$$\dot{S} = \int_{\partial \Omega} \{ \tilde{e} \wedge \tilde{h} \}. \quad (28)$$

We shall assume that all voltage and current sources are left outside the system, so that their contribution is taken into consideration by the term \dot{S} .

c) We have now to describe the interface between the electromagnetic energy compartment and the other energy compartments, i.e. the conversion of electromagnetic energy into other forms of energy. At the macroscopic level, it suffices for that purpose to distinguish the charges that are free to move in the conduction bands of the material from the charges (or electric dipoles, or magnetic moments, ...) that are bound to the rigid crystallographic structure. The electromagnetic power converted in the system has therefore two terms :

$$\dot{R} = \int_{\Omega} \{ \tilde{e} \wedge p_t J + \rho_{em}^{\dot{W}} \}. \quad (29)$$

The first one is the mechanical power delivered by electrostatic forces to the free charges flowing in the material. The current density J , which represents the flow of free charges, is by definition the 2-form $J \in \Lambda^2(M)$ giving for each material surface in M the amount of free charges crossing it per second. In practice, the current density can also be given as a function of the electric field : $p_t J = \sigma \tilde{e} \in \Lambda^2(\Omega)$, with σ the electrical conductivity of the material.

The second term in (29), $\rho_{em}^{\dot{W}} \in \Lambda^3(E)$, is the mechanical power delivered by the electromagnetic forces to the solid structure of the matter. This is the term we want to evaluate.

The energy balance of the system Ω writes

$$\partial_t \Psi + \dot{S} + \dot{R} = 0, \quad (30)$$

and under local form, making use of (16) and (23),

$$\begin{aligned} \mathcal{L}_v \{ \rho^\Psi(p_t D, p_t B) \} + \rho_{em}^{\dot{W}} \\ + \tilde{e} \wedge p_t J + d\{ \tilde{e} \wedge \tilde{h} \} = 0. \end{aligned} \quad (31)$$

Note that this is *not* Poynting's relation.

The application of the chain rule of derivatives gives

$$\begin{aligned} \mathcal{L}_v \{ \rho^\Psi(p_t D, p_t B) \} &= \{ \mathcal{L}_v \rho^\Psi \}(p_t D, p_t B) \\ &+ \tilde{e} \wedge \mathcal{L}_v p_t D + \mathcal{L}_v p_t B \wedge \tilde{h}. \end{aligned} \quad (32)$$

Note the importance of the grouping brackets. One has got at the left-hand side the Lie derivative of the *value* of the function ρ^Ψ , which is a 3-form, whereas one has at the right-hand side the Lie derivative of the function itself, which is another function, with the same domain of definition.

Evaluating the exterior derivative in (31) using (15), and reorganising terms yields

$$\begin{aligned} 0 &= \tilde{e} \wedge \{ \mathcal{L}_v p_t D + p_t J - d\tilde{h} \} \\ &+ \{ \mathcal{L}_v p_t B + d\tilde{e} \} \wedge \tilde{h} \\ &+ \{ \mathcal{L}_v \rho^\Psi \}(p_t D, p_t B) + \rho_{em}^{\dot{W}}, \end{aligned}$$

which has to be verified whatever the choice of Ψ and p_t . It follows

$$0 = \mathcal{L}_v p_t D + p_t J - d\tilde{h} \quad (33)$$

$$0 = \mathcal{L}_v p_t B + d\tilde{e} \quad (34)$$

$$\rho_{em}^{\dot{W}} = -\{ \mathcal{L}_v \rho^\Psi \}(p_t D, p_t B) \quad (35)$$

which are the Maxwell equations (Ampere and Faraday laws) in the Euclidean space E , and the sought definition of $\rho_{em}^{\dot{W}}$.

But the local balance can as well be expressed in material form, i.e. on the material manifold M . One finds by applying p_t^{-1} to (31) and using (23), (18) and (19)

$$\begin{aligned} \partial_t \{ \{ p_t^{-1} \rho^\Psi \}(D, B) \} + p_t^{-1} \rho_{em}^{\dot{W}} \\ + \tilde{E} \wedge J + d\{ \tilde{E} \wedge \tilde{H} \} = 0 \end{aligned} \quad (36)$$

with $\tilde{E} \equiv p_t^{-1} \tilde{e}$ and $\tilde{H} \equiv p_t^{-1} \tilde{h}$. Applying again the chain rule of derivatives gives

$$\begin{aligned} 0 &= \tilde{E} \wedge \{ \partial_t D + J - d\tilde{H} \} \\ &+ \{ \partial_t B + \tilde{E} \} \wedge \tilde{H} \\ &+ \{ \partial_t p_t^{-1} \rho^\Psi \}(D, B) + p_t^{-1} \rho_{em}^{\dot{W}}, \end{aligned}$$

whence finally the Maxwell equations in their familiar (material) form and another definition of $\rho_{em}^{\dot{W}}$:

$$0 = \partial_t D + J - d\tilde{H} \quad (37)$$

$$0 = \partial_t B + d\tilde{E} \quad (38)$$

$$p_t^{-1} \rho_{em}^{\dot{W}} = -\{ \partial_t p_t^{-1} \rho^\Psi \}(D, B). \quad (39)$$

B. Back to Vector analysis

The development of the last section shows that the Maxwell equations in their customary form are valid on the material manifold (i.e. Lagrangian formulation and comoving mesh). The corresponding equations for the Eulerian formulation (fixed mesh) are also given. But most important to our discussion is that the general expression of the mechanical power conveyed from the electromagnetic compartment to the mechanical compartment is obtained, resp. (35) and (39). Since this expression involves the placement map p_t , and therefore the velocity field v , it contains hidden expressions for the electromagnetic forces, which we shall now work out.

The simplest way to do so is to revert to Vector analysis so as to work with vector and tensor fields in the Euclidean space

E . There is finally not much to add to the framework of classical Vector analysis to make the bridge with the results of last section.

One has first to recognise the existence of 2 different kinds of scalar fields, namely the 0-forms and the 3-forms, and the existence of 2 different kinds of vector fields, namely the 1-forms and the 2-forms. Then, Differential geometry provides the expressions of their respective material derivatives, in terms of the gradient of the velocity field:

$$\mathcal{L}_v f = \dot{f} \quad (40)$$

$$\mathcal{L}_v \mathbf{h} = \dot{\mathbf{h}} + (\nabla \mathbf{v}) \cdot \mathbf{h} \quad (41)$$

$$\mathcal{L}_v \mathbf{b} = \dot{\mathbf{b}} - \mathbf{b} \cdot (\nabla \mathbf{v}) + \mathbf{b} \operatorname{tr}(\nabla \mathbf{v}) \quad (42)$$

$$\mathcal{L}_v \rho = \dot{\rho} + \operatorname{tr}(\nabla \mathbf{v}) \rho \quad (43)$$

where $\dot{z} = \partial_t z + \mathbf{v} \cdot \nabla z$ denotes the *total derivative* of $z(t, x^k)$, applied component by component if z is a vector field. Whereas (40) and (43) are classical in fluid dynamics, (41) and (42) are less often encountered and they should certainly be added to the panoply of the differential equations ruling electromagnetism.

Considering now a material for which the energy density is a function of the induction only, $\rho^\Psi : \mathbf{b} \equiv p_t B \in \Lambda^2(E) \mapsto \Lambda^3(E)$, one has successively

$$\begin{aligned} \mathcal{L}_v \{ \rho^\Psi(\mathbf{b}) \} \\ &= \dot{\rho}^\Psi(\mathbf{b}) + \operatorname{tr}(\nabla \mathbf{v}) \rho^\Psi(\mathbf{b}) \\ &= \partial_{\mathbf{b}} \rho^\Psi(\mathbf{b}) \cdot \dot{\mathbf{b}} + \operatorname{tr}(\nabla \mathbf{v}) \rho^\Psi(\mathbf{b}) \\ &= \tilde{\mathbf{h}} \cdot \mathcal{L}_v \mathbf{b} \\ &+ \{ \mathbf{b} \cdot \nabla \mathbf{v} \cdot \tilde{\mathbf{h}} - \operatorname{tr}(\nabla \mathbf{v}) \{ \tilde{\mathbf{h}} \cdot \mathbf{b} - \rho^\Psi(\mathbf{b}) \} \} \end{aligned} \quad (44)$$

by applying (43) and then (42).

A formal identification with (32) assuming $D = 0$ and (35) shows that the second term at the right-hand side of (44) is the mechanical power developed by the magnetic forces, so that one can write

$$\rho_{em}^{\dot{W}} = -\sigma_{em} : \nabla \mathbf{v} \quad (45)$$

with the Maxwell stress tensor

$$\sigma_{em} = \mathbf{b} \tilde{\mathbf{h}} - \{ \tilde{\mathbf{h}} \cdot \mathbf{b} - \rho^\Psi(\mathbf{b}) \} \mathbb{I} \quad (46)$$

defined as the factor of $\nabla \mathbf{v}$. Note the use of the dyadic (undotted) vector product $(\mathbf{v} \mathbf{w})_{ij} = v^i w^j$, the tensor product $a : b = a_{ij} b_{ij}$ and the identity matrix \mathbb{I} .

V. COMPUTATION OF ELECTROMAGNETIC FORCES

Equation (45) represents the transcription into classical terms of the fundamental result (35) found by the energy-based approach. In the last part of this paper, the implications of this result to the definition of electromagnetic forces are listed. It is shown in particular that the variety of existing methods to compute electromagnetic forces in finite element models can be all found back from (45), by considering different virtual velocity fields \mathbf{v} .

a) Maxwell stress: The electromechanical coupling happens to be realised by the work done by a stress tensor on the gradient of a (virtual) velocity field. This is due to the fact that the material derivatives of the p -forms (40–43) involve $\nabla \mathbf{v}$, but *not* \mathbf{v} itself.

b) energy density: Each material has to be given its own expression of the Maxwell stress tensor σ_{em} , by applying the same procedure as above. If, for example, the energy density is also a known function of strain, $\rho^\Psi : \mathbf{b}, \varepsilon \mapsto \Lambda^3(E)$, the Maxwell stress tensor turns out to have just one extra term

$$\sigma_{em} = \mathbf{b} \tilde{\mathbf{h}} + \partial_\varepsilon \rho^\Psi - (\tilde{\mathbf{h}} \cdot \mathbf{b} - \rho^\Psi) \mathbb{I} \quad (47)$$

and this is already enough for magnetostriction. Other examples can be found in [7].

c) **force density**: The link between the Maxwell stress tensor σ_{em} and the electromagnetic force density ρ_{em}^f is found by integrating (45) by part over Ω . One has

$$\int_{\Omega} \sigma_{em} : \nabla \mathbf{v} = - \int_{\Omega} \rho_{em}^f \cdot \mathbf{v} + \int_{\partial\Omega} \mathbf{n} \cdot \sigma_{em} \cdot \mathbf{v} \quad (48)$$

with $\rho_{em}^f = \text{div } \sigma_{em}$ by definition and \mathbf{n} the exterior normal to $\partial\Omega$.

For instance, by applying the useful formula

$$-a^i \frac{\partial b^k}{\partial x^i} + a^i \frac{\partial b^i}{\partial x^k} = (\mathbf{a} \times \text{curl } \mathbf{b})^k, \quad (49)$$

the force density derived from

$$\sigma_{em} = \mathbf{d} \tilde{\mathbf{e}} + \mathbf{b} \tilde{\mathbf{h}} - \{\tilde{\mathbf{e}} \cdot \mathbf{d} + \tilde{\mathbf{h}} \cdot \mathbf{b} - \rho^{\Psi}(\mathbf{d}, \mathbf{b})\} \mathbb{I} \quad (50)$$

is found to be

$$\rho_{em}^F = \text{curl } \tilde{\mathbf{e}} \times \mathbf{d} + \text{curl } \tilde{\mathbf{h}} \times \mathbf{b} + \tilde{\mathbf{e}} \text{ div } \mathbf{d} + \tilde{\mathbf{h}} \text{ div } \mathbf{b}. \quad (51)$$

Substituting Maxwell equations gives

$$\rho_{em}^F = -\mathcal{L}_{\mathbf{v}} \{\mathbf{b} \times \mathbf{d}\} + \mathbf{j} \times \mathbf{b} + \tilde{\mathbf{e}} \text{ div } \mathbf{d} + \tilde{\mathbf{h}} \text{ div } \mathbf{b}. \quad (52)$$

This expression of the electromagnetic force density is applicable to all materials for which the energy density can be represented by $\rho^{\Psi} : \mathbf{d}, \mathbf{b} \mapsto \Lambda^3(E)$, i.e. a function of the fields \mathbf{b} and \mathbf{d} only.

d) **sign convention**: It should be carefully noted that the Maxwell stress tensor σ_{em} is a true mechanical stress, i.e. its work is delivered by the mechanical compartment and received by the electromagnetic compartment. On the other hand, ρ_{em}^f is a magnetic force. Its work is withdrawn from the electromagnetic compartment and received by the mechanical compartment.

e) **continuity**: At material interfaces, the Maxwell stress tensor is in general discontinuous. The force is there defined, in the sense of distributions, as the jump $\sigma_{em} \cdot \mathbf{n}$, as can be seen by applying (48) material domain by material domain, and then summing up the surface contribution on the inner material interfaces.

f) **applied stress**: The Maxwell stress tensor can be used directly as an applied stress in the structural equations and boundary conditions of the system. One has

$$\text{div } \{\sigma + \sigma_{em}\} + \rho^f = 0, \quad (53)$$

which is easier than coupling through the forces

$$\text{div } \sigma + \{\rho_{em}^f + \rho^f\} = 0, \quad (54)$$

since ρ_{em}^f requires a special treatment at material interfaces.

g) **force-free region** Z : Applying (48) to a force-free region, i.e. $\rho_{em}^f = 0$ on Z (cf Fig. 1), yields

$$- \int_Z \rho_{em}^{\dot{W}} \equiv \int_Z \sigma_{em} : \nabla \mathbf{v} = \int_{\partial Z} \mathbf{n} \cdot \sigma_{em} \cdot \mathbf{v}. \quad (55)$$

This shows that the velocity field \mathbf{v} is arbitrary on the *interior* of any force-free region, in the sense that it does not affect the mechanical power exchanged with that region. In all cases, \mathbf{v} needs however to remain continuous.

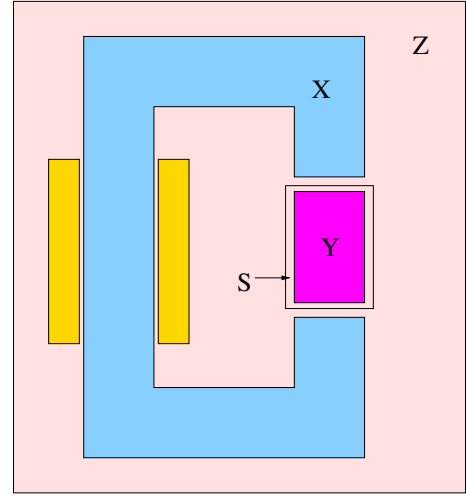


Figure 1: Typical resultant magnetic force problem, Y is the moving rigid region (body), Z is the force-free region, X is fixed, S is the eggshell.

h) **rigid region** Y : Considering a region Y with a rigid body velocity field $\mathbf{v} = \mathbf{v}_0 + \mathbf{w}_0 \times \mathbf{r}$, (48) gives

$$\begin{aligned} \int_Y \hat{\sigma}_{em} \cdot \mathbf{w}_0 + \int_Y \rho_{em}^f \cdot \{\mathbf{v}_0 + \mathbf{w}_0 \times \mathbf{r}\} \\ = \int_{\partial Y} \mathbf{n} \cdot \sigma_{em} \cdot \{\mathbf{v}_0 + \mathbf{w}_0 \times \mathbf{r}\} \end{aligned} \quad (56)$$

with $\hat{\sigma}_{em}^k = \epsilon^{ijk} (\sigma_{em})_{ij}$. The vectors \mathbf{v}_0 and \mathbf{w}_0 being constant, one may define the resultant magnetic force

$$\mathbf{F}_Y = \int_Y \rho_{em}^f = \int_{\partial Y} \mathbf{n} \cdot \sigma_{em} \quad (57)$$

and the resultant magnetic torque

$$\mathbf{T}_Y = \int_Y \{\hat{\sigma}_{em} + \mathbf{r} \times \rho_{em}^f\} = \int_{\partial Y} \mathbf{r} \times (\mathbf{n} \cdot \sigma_{em}) \quad (58)$$

acting on the rigid region Y . The term $\hat{\sigma}_{em}$ is zero when σ_{em} is symmetric.

Equations (57) and (58) show that the resultant force \mathbf{F}_Y and the resultant torque \mathbf{T}_Y acting on a rigid region Y can both be evaluated by means of a surface integral on its boundary ∂Y . This classical result implies however a surface integration, which requires a specific implementation. Note that the rigid region need not be identified with a material body. It may be larger, provided that the extra domain enclosed is force-free.

i) **eggshell method**: In practice, it is easier to work with volume integrations which are already implemented in the finite element programme. In order to get rid of the surface integration in (57) and (58), one chooses a domain Ω larger than the rigid region Y , i.e. enclosing as well a part of a force-free region (generally air). One chooses a velocity field which describes a rigid motion of Y , decays smoothly outside Y and vanishes on $\partial\Omega$, i.e

$$\mathbf{v} = \{\mathbf{v}_0 + \mathbf{w}_0 \times \mathbf{r}\} \gamma, \quad (59)$$

where γ is any smooth function whose value is 1 on Y and 0 on $\partial\Omega$. Applying now (55) to the force-free region $\Omega - Y$ yields successively

$$\begin{aligned} - \int_{\Omega - Y} \sigma_{em} : \nabla \mathbf{v} &= - \int_{\partial\Omega - \partial Y} \mathbf{n} \cdot \sigma_{em} \cdot \mathbf{v} \\ &= \int_{\partial Y} \mathbf{n} \cdot \sigma_{em} \cdot \{\mathbf{v}_0 + \mathbf{w}_0 \times \mathbf{r}\} \\ &= \mathbf{F}_Y \cdot \mathbf{v}_0 + \mathbf{T}_Y \cdot \mathbf{w}_0, \end{aligned} \quad (60)$$

by the definition of γ , (57) and (58). This gives an alternative way to compute the resultants \mathbf{F}_Y and \mathbf{T}_Y , now by means of a volume integral, with σ_{em} the Maxwell stress tensor of empty space.

In the particular case of a translation velocity field $\mathbf{v} = \mathbf{v}_0\gamma$, one has

$$\mathbf{F}_Y = - \int_{\Omega-Y} \sigma_{em} : \nabla \gamma. \quad (61)$$

In practice, the support of \mathbf{v} is reduced to a minimum, i.e. the rigid region plus a thin shell $S \equiv \Omega - Y$ of air around it. The shell can be defined explicitly by the user, like in Fig. 1, and γ is then an also user-defined analytic function. In general, it is easier to have the eggshell automatically defined. A natural choice is to take one layer of finite elements around the moving region, i.e. all elements touching the boundary ∂Y on the outer side. The function γ is then simply, on that support, the sum of the shape functions associated with the nodes of Y . This alternative method to compute the resultant force on rigid bodies has been proposed in [8] and [9].

j) Arkkio's method: The torque in 2D models of electrical rotating machines can be calculated with the eggshell method by considering the rotation velocity field

$$\mathbf{v} = \gamma \mathbf{w}_0 \times \mathbf{r} = \frac{R_o - r}{R_o - R_i} \{w_0 \mathbf{e}_z\} \times \{r \mathbf{e}_r\} \quad (62)$$

in cylindrical coordinates, where R_o and R_i are respectively the outer and inner radius of any cylindrical air region S contained in the airgap. The gradient of the velocity field is

$$\nabla \mathbf{v} = \begin{pmatrix} \partial_r v^r & \frac{1}{r} \partial_\theta v^r \\ r \partial_r (\frac{v^\theta}{r}) & \partial_\theta (\frac{v^\theta}{r}) \end{pmatrix} = \frac{-w_0 r}{R_o - R_i} \mathbf{e}_r \mathbf{e}_\theta, \quad (63)$$

whence by (60), the formula of Arkkio [10]:

$$\mathbf{T}_Y = \frac{\mathbf{e}_z}{R_o - R_i} \int_S r (\sigma_{em})_{r\theta}. \quad (64)$$

k) Coulomb's method: The Coulomb's formula to compute nodal electromagnetic forces by the local derivative of the Jacobian [11], is obtained by identifying \mathbf{v}_0 with the virtual velocity of one node and the function γ with the shape function of that node in the real axes. One obtains

$$\left\{ - \int_{\Omega} \sigma_{em} : \nabla \gamma \right\} \cdot \mathbf{v}_0 = \left\{ \int_{\Omega} \rho_{em}^f \gamma \right\} \cdot \mathbf{v}_0. \quad (65)$$

where the domain of integration can be limited to the support of γ , which is in general not a force-free region. This allows to define the nodal net force \mathbf{F}_N acting on the node by

$$\mathbf{F}_N \equiv \int_{\Omega} \rho_{em}^f \gamma = - \int_{\Omega} \sigma_{em} : \nabla \gamma, \quad (66)$$

which is identical to the formula proposed in [12]. It can also be shown to be equivalent to the formulae presented in [13] for linear materials and in [11, 14] for non-linear materials. Equation (66) is however more general, as it does not assume any particular form of σ_{em} .

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