Technical Article

Forces inside a magnet

Abstract **—This is an essay on the question of forces, not a research paper. I think the difficulty with electromagnetic forces is less with theory and the required mathematical formalism—quite demanding, one must admit, than in modelling, that is to say, in the translation of physical assumptions and facts into mathematical statements. Once this is done, a clearcut result should emerge—Mathematics don't lie. The thesis I defend is that ambiguities about the formulation of constitutive laws are responsible for the prevailing confusion, and the persistence of controversies, about forces. More precisely, constitutive relations between fields should be known for all possible deformation states of the material. Under this proviso, there is a definite, demonstrably correct expression for body forces. But in confrontations with experiments, this expression will be worth what the constitutive laws are worth. Hence it all relies on the measurements by which behavior laws are determined: For they ought to be observed, not theorized about. Interestingly, a consistent theory of forces is a prerequisite for such observations, which unavoidably involve coupled problems.**

Index terms **— Magnet, energy, coenergy, virtual work principle, force, differential forms, Lagrangian formalism.**

I. Introduction

Isn't it surprising that forces in electromagnetism are still an issue? After all, Electromagnetism comes after Mechanics, in the genesis of classical physics, and such concepts as work, energy, and force are considered as already there, as parts of the accepted framework, when foundations for electromagnetics are laid out. Best evidence for that, the fact that both electric and magnetic field are defined *in terms of* the (pre-existing) concept of force, just after the introduction of the concept of "electric charge". *Electric* and *magnetic* field are defined as the agencies responsible for forces over *steady* and *moving* test charges, respectively, and described, mathematically, by the same objects that describe forces: The total force, called *Lorentz force*, on a test charge Q (small enough not to perturb the field) is $Q(E + \upsilon \times B)$, where υ stands for the charge's velocity. Since forces are there from the beginning, where does the difficulty lie? Can't we just determine where charges are, compute the fields, and sum up the Lorentz forces on all charges present in some body to know which force acts on it?

We can't, in most situations, because a detailed account of charge motion is not part of the modelling. We use constitutive laws, such as the B–H relation in magnets or ferromagnetic materials, which imply a distinction between conduction charges, or "free" charges, which constitute the current density J, and "bound" charges, whose motion at a microscopic scale, responsible for magnetization, is roughly described by a magnetization vector field, linked with B via the B–H relation one adopts. These laws are the mesoscopic[1] summary of a lot of microscopic phenomena, involving the dynamics of bound charges, which we cannot, and don't want to, know about. How Lorentz forces balance at the microscopic level, what the microscopic fluctuations of fields B and H are (they appear in the B–H law via their meso-scale averages), all that is ignored. Hence the danger inherent in trying to derive the mesoscopic force by summation of *some* of the microscopic ones, based on solid-state physics considerations: Odds are high that the selective ignorance at work in such a process, on the one hand, and the selective ignorance inherent in the design of constitutive laws, on the other hand, be not consistent.

That will be our red thread: In need of a link between Electromagnetics and Mechanics, we replace Lorentz force, in this function, by a statement about energy exchanges, called in what follows the "energetic postulate", that will have the required consistency built in. Then, we shall *derive* the force, by mathematical arguments, from a well-defined set of relations: Maxwell equations, constitutive laws, and the postulate. If we can trust the postulate (and we can, as we shall see at once), the force field will be predicted with just as much certainty and accuracy (but no more) as can be achieved in setting up the constitutive laws. Moreover, it will be apparent that such a consistent theory of forces is just what one needs to determine (via measurements) the constitutive laws in the first place.

The basic energetic statement is derived from the Lorentz force law as follows. Suppose free charges (electrons in a solid metal, ions in a fluid) so tiny and numerous that one may describe their distribution at a given instant t (not explicitly mentioned) by a scalar function $\tilde{Q}(x, \upsilon)$ of position x and velocity υ, and thus defined over "position–velocity space": Namely, the total charge of conduction electrons, etc., present in a small volume dx surrounding a point x, and whose velocity lies in a box of volume $d\upsilon$ around υ in velocity space, is $\tilde{Q}(x, \upsilon)\, dx\, d\upsilon$. Thus, the total free charge present in volume dx around x is the integral $Q(x) = \int \tilde{Q}(x, \upsilon)\, d\upsilon$ with respect to υ, hence the charge density Q. Now, to account for the collective motion of charges, one introduces $J(x) = \int \tilde{Q}(x, \upsilon)\, \upsilon\, d\upsilon$, again an integral over velocity space, but now vector valued, called "current density". In the field $\{E, B\}$, a moving charge carrier feels the force $Q(E + \upsilon \times B)$, and the power it taps from the field is $Q\, E \cdot \upsilon$, since $(\upsilon \times B) \cdot \upsilon = 0$. (Note that, since the field is thus able to yield power, it must also be able to *store* it,

[1] Behavior laws are relations between *averages* of the fields at some definite spatial scale (which depends on the modelling). The prefix "meso" will refer to this scale. This leaves "micro" to connote what happens at a much smaller spatial scale (atomic, for instance), and "macro" for what laboratory instruments are able to record. For instance, a relation such as $V = RI$, as read off an Ohmmeter, is *macroscopic*, and the knowledge of R, for a particular sample, may help determine σ in the *mesoscopic* relation $J = \sigma E$ that holds for this sample's material. Of course, this Ohm's law ceases to be valid at a smaller-scale *microscopic* level.

hence the concept of field's energy, the expression for which will come later.) For a charge distribution $\tilde{Q}(x, \upsilon)$ in position-velocity space, the sum of these powers with respect to υ is $E(x) \cdot \int \tilde{Q}(x, \upsilon)\, \upsilon\, d\upsilon$, i.e., $E(x) \cdot J(x)$. Hence this: *the field yields power, a power that distributes with the density* $E \cdot J$, *to the mechanical system the moving charges are part of*. That will be the foundation for our basic energetic postulate. Note the enormous loss of information when going from $\tilde{Q}$ to Q and J. This loss must be compensated by a constitutive law: Ohm's law, for instance, which says that $J = \sigma E$, for a non-moving[2] conductor (and we shall *not* attempt to derive this from a detailed analysis of the field–charge interaction!).

Where does this energy go? In a non-moving conductor, no mechanical degrees of freedom (DoF) vary, or rather, all DoFs that would account for microscopic motion have been excluded from the modelling, which means that whatever work is associated with their variation must be written off as *heat*. (Nothing new with that: In Mechanics, too, a lot of DoFs are ignored and the associated work, attributed to various undescribed "friction" mechanisms, is reckoned as heat.) Hence the expression for Joule loss, $E \cdot J$, the density of power lost by the field to the "thermal compartment" of the system. This holds whatever the conduction law, irrespective of the details of microscopic motion and friction mechanisms that damp it, and in perfect agreement with the Lorentz law of forces, so this will make a reliable pillar for the theory to be built.

Exception must be taken, however, about conductors belonging to the network that brings electrical power into the system. Conduction charges, there, are not left to the interplay of Lorentz forces and friction forces; they are driven by some electromotive field the details of which, again, we'd rather ignore. This is conveniently done by specifying a source current density, J^s, a given function of space and time, supported in a region of space we consider as "the feeding network", and by taking $\sigma = 0$ there. This way, Ohm's law can be expressed as $J = J^s + \sigma(E + v \times B)$, in one stroke (this time, v is the *material* velocity), valid in all space, and the scalar $-J^s \cdot E$, again in full agreement with the Lorentz force law, is taken as the power density *injected* into the system by the network.

Remark. For the reasons alluded to in Note 2, we need not treat the case of moving conductors, though it would be easy: One readily sees that the sum $\int \tilde{Q}(x, \upsilon)(E + \upsilon \times B)\, d\upsilon$ of Lorentz forces (still an integral over velocity space) is equal to $QE + J \times B$. So the total force on the bundle of conduction charges that happen to be in the volume element dx around x is $[Q(x)\, E(x) + J(x) \times B(x)]\, dx$. These charges transfer the momentum imparted to them by the field to the conductor they belong to, which results in a bulk force $QE + J \times B$. We don't pursue this line of reasoning, because the $J \times B$ force will be recovered in another way later. As for the Coulomb force (the Q-times-E term), Q is so small inside conductors (yet, not on their surface) that only $J \times B$ matters. Coulomb forces will be ignored in this article, as well as the displacement-current term $\partial_t D$ in Maxwell's equations. (The two things are closely linked, but showing how would lead us too far astray.) ◊

In the next Section, we shall formulate these physical premises in mathematical language. This *could* be done in the received formalism: Eulerian setup, with a privileged reference frame in which the physical field is represented by the vector fields E, B, etc. But we need a theory that applies to discrete formulations, with finite elements and all that, and Lagrangian discretizations (with a "comoving" mesh, where each element is linked with a specific chunk of matter, and moves with it) are more popular than Eulerian ones, with a fixed grid. (It's only for a subclass of problems, those with liquid metals in motion especially, that one resigns oneself to use a fixed grid, and the $v \times B$ term then comes in the way.)

Moreover, what the literature has to say about forces strongly suggests a Lagrangian formalism as preferable. For there is this folk theorem that "force is obtained by differentiating the coenergy with respect to the configuration parameter, *while keeping currents constant"*. When such currents are those in threads and coils, one well understands what it means for them to be "constant" in spite of the conductor's deformation. But what about massive conductors, where current lines may and will change with deformation, if we take the deformation field as the (infinite dimensional, then) configuration parameter? "Constant" cannot mean that $J(x)$, at any spatial point x, stays constant in time. Rather, some relation with the underlying matter is implied. What makes perfect sense, in this respect, is for the intensity through a given "material" surface (one that always contains the same grains of matter, irrespective of the deformation), to stay the same as time goes by. Eventually, we shall verify that the old rule is indeed correct, if thus understood, but first, we need to lay down the mathematical formalism by which such concepts as the previous "material intensity", and connected ones ("material" flux, "material" emf) can be expressed. This done, we state the Maxwell equations and constitutive laws within this formalism, formulate the energetic postulate, and from this point on, it's only a matter of formula manipulations and logic to derive the force law. We then interpret this result, in a variety of situations, including the "deformable permanent magnet" one. There are some surprises in store.

II. Electromagnetism in "material form" (Lagrangian formulation).

The very first thing to deal with is *kinematics:* We need to describe the mechanical configuration of the system by an appropriate set of variables, one which retains aspects of motion we are interested about, and ignores others. In many applications, a *single* kinematical parameter $\underline{u}$, which can stand for a translation, a rotation angle, etc., may be enough. In such one-mechanical DoF modellings, the moving part is considered as rigid, and one is willing to ignore stresses inside it. Figure 1 gives a toy example, which it will be useful to keep in mind.

To address the "force-field inside a magnet" issue, for instance when the piece M in Fig. 1 is deformable, we need more: The configuration parameter u must then be a record of positions of all material points, what we describe below as a "placement", thus an infinite-dimensional object. (Finite-

[2] Otherwise, $J = \sigma(E + v \times B)$, where v is the conductor's velocity, because the electric field in the comoving frame is $E + v \times B$, as can be derived from the Maxwell equation. Therefore, knowing the conduction law for immobile bodies is knowing enough. This may help to make more intuitive the idea, developed later on, that stating the energetic postulate for non-moving bodies is enough too.

dimensionality, as for $\underline{u}$, is connoted by underlining.) We shall try, at the price of a few harmless notational abuses, to treat both kinds of situations (rigid bodies moving apart, deformable bodies) in a unified manner.

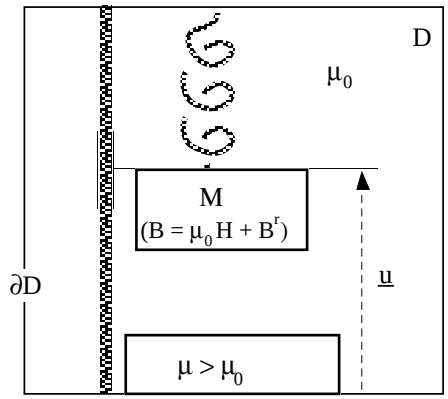

Fig. 1. Typical one-DoF problem. Grey parts (suspension spring and vertical column), electrically neutral, are intended to limit the movement of the mobile part M to a vertical line, hence a single real kinematical parameter $\underline{u}$. M is a magnet (characterized by its remanent induction B^r, roughly vertical), attracted by a fixed ferromagnetic plate at the bottom.

For this, consider a reference configuration—the one at time $t = 0$, for instance. Each material point occupies a position x in the computational domain D to which we intend to limit the modelling (the enclosing box, in Fig. 1). At time t, this material point occupies another position, denoted $u(t, x)$, and we shall denote by $u(t)$ the one-to-one mapping $x \to u(t, x)$, from D to itself, called *placement* at time t. Note that $u(0)$ is thus the identity. (This is not essential, as other configurations could be taken as reference.) For non-moving parts (walls of the box, points of the bottom plate), $u(t, x) = x$ for all t. For moving parts, the derivative $v(t, x) = \partial_t u(t, x)$ is the velocity of the material particle that passes at point x at time t. (We assume from now on that v is continuous in x, as required by material continuity.) The placement at time t (which we shall feel free to denote u_t, or even u, omitting the t) can be considered as a point in a large set (an infinite-dimensional manifold, actually), that we denote by U. So much for kinematics, thus described as a continuously differentiable curve, or *trajectory* $t \to u(t)$ in U. We shall not here address *dynamics*, that is, how to determine this trajectory, together with the electromagnetic field's evolution, because this depends on how matter reacts to electromagnetic forces. We are only interested in the latter, and considering a *virtual trajectory*, for values of t near 0, will prove sufficient for that. We may therefore consider the velocity field v only at time 0, and treat it as a steady vector field.

The concept of material surface, and of intensity through it, can now be made precise: Let S be a smooth surface inside D, and consider its image $u(t, S)$ under the placement, at time t, that is to say, the set $\{u(t, x) : x \in S\}$. The "material intensity" through S, at time t, is defined as the flux $\int_{u(t, S)} n(t) \cdot J(t)$ of the current density vector field J, taken as time t. (We shall omit the t in the field of unit normals $n(t)$ from now on.) This intensity will be denoted, for both typographical comfort and deeper reasons on which more later, by $\int_S \mathrm{J}(t)$, with a SMALL CAPITAL J. A similar definition applies to the "material induction flux embraced by S", denoted by $\int_S \mathrm{B}(t)$.

The integration domain is S, in both cases, not $u(t, S)$: If we conceive S as representing a thin membrane of material particles, these very same particles form $u(t, S)$ at time t, so the intensity $\int_S \mathrm{J}(t)$ is indeed taken with respect to a fixed material reference. "Material" may sound like a misnomer when S belongs to the air, rather than hard matter. Yet, since $u(t)$ maps all D to itself, points in the air at time 0 get a specific placement at time t, so it makes sense to speak not only of $\int_S \mathrm{J}(t)$ (which is null), but of $\int_S \mathrm{B}(t)$ (which need not be). In spite of the lack of material support for the family of surfaces $u(t, S)$ in this case, we shall keep referring to $\int_S \mathrm{J}(t)$ and $\int_S \mathrm{B}(t)$ as the "material" intensities and fluxes. One may imagine the air filled out by some flexible putty, compliant enough to make no difference mechanically, the deformation of which is described by u on the same footing as for all other materials.

This raises a question, however, that we should dispose of immediately. Suppose two placements u^1 and u^2 coincide, at all times, for points x which belong to conductors, magnets, iron pieces, etc., in the reference configuration, but may differ at points in the air, and more generally, in electrically irrelevant materials. Is there a risk that our theory could predict *different* forces for u^1 and u^2? Fortunately, this is not the case, as will later be proved, so we shall call *dynamically equivalent* two such placements. Now, we can replace U by the set $\underline{\mathrm{U}}$ of equivalence classes with respect to this relation, which in some cases may well reduce to a one-dimensional manifold. For instance, if we consider M in Fig. 1 as a rigid body, in order to concentrate on the vertical force, there are a lot of dynamically equivalent placements u which correspond to a given value of the height $\underline{u}$: Just set (calling k the vertical unit vector) $u(t, x) = x + (\underline{u}(t) - \underline{u}(0))k$ for all x which belong to M at time 0, $u(t, x) = x$ for points in all other electrically active parts (including the boundary ∂D of the box), and for points that remain, attribute to u_t any value compatible with the one-to-oneness and continuous differentiability requirements. Such flexibility in the choice of placements will prove useful later. This example shows how the effective dimension of U can be reduced to a small number, and we shall treat in a similar way, abusing the notation if required, the full-fledged placement u and the reduced set of parameters $\underline{u}$ (the dynamically meaningful ones).

Having thus clarified the meaning of "material", let us focus on the *mappings* $S \to \int_S \mathrm{B}$ and $S \to \int_S \mathrm{J}$, at a given instant, which we shall denote[3] by B and J (t understood). They are obviously *additive* with respect to S (the flux through S_1 plus the flux through S_2 is the flux through the two-piece surface $S_1 + S_2$), and *continuous* (a small deformation of S slightly changes the flux). Such objects—additive continuous mappings from p-dimensional manifolds to real numbers—are called *differential forms* of degree p, with $p = 2$ here. The vector fields B and J by which they were defined are called the "proxy" fields of B and J, which stresses the point that B and J, which give the physically meaningful material fluxes, are the real thing, whereas B and J are mere representations of them. *It is extremely important,* at this juncture, *to notice that the correspondences* B—B *and* J—J *depend on the placement* u_t and hence, depend on time: If J is to be constant, as in

[3]This makes perfect sense: The *function* $x \to f(x)$ that maps x to the value $f(x)$ is denoted by f. Same mechanism.

the above "constant currents" clause about force, the proxy field $J(t)$ must, as a rule, vary with t —which is why it would be so difficult to make sense of this clause in a Eulerian framework. Conversely, when a conductor moves through a *steady* magnetic field B, the material fluxes change, so $\mathfrak{b}$ does depend on t.

What we are up to is a formulation of Maxwell's equations in terms of differential forms (DF) which "live", like $\mathfrak{b}$ and $\mathfrak{j}$, on the "material manifold", so let us define the material counterparts $\mathfrak{h}$ and $\mathfrak{e}$ of H and E.

The "material magnetic strength" at time t is the 1-form $\mathfrak{h}(t)$ whose proxy is $H(t)$. So it assigns to a material curve c the circulation $\int_{u(t,\,c)} \tau \cdot H$, where τ denotes the unit tangent vector along c. This number is the "material magnetomotive force" (mmf) associated with c. As we see by applying the Stokes theorem, Ampère's relation rot H = J takes the very simple form

(1) $\int_{\partial S} \mathfrak{h} = \int_S \mathfrak{j}$ for all material surfaces S,

where ∂S denotes the boundary of S. (Orientations of S and ∂S are matched by the right-hand rule.) Another way to state (1) is $d\mathfrak{h} = \mathfrak{j}$, where d is an operator[4], called "exterior derivative", which is *defined* by the clause $\int_{\partial S} \mathfrak{h} = \int_S d\mathfrak{h}$ for all S. (So d is a kind of adjoint of ∂, as the alternative notation $\langle \partial S\,;\,\mathfrak{h}\rangle = \langle S\,;\,d\mathfrak{h}\rangle\ \forall\ S$ suggests more forcibly.)

Last, what about $\mathfrak{e}$? We want it to capture the physical notion of electromotive force (emf). The "material emf", along a material line c, is the (virtual) work associated with the motion of a unit electric charge along $u_t(c)$ at time t (this is *virtual* motion, during which time is frozen at t), so we define $\mathfrak{e}(t)$ as the map that assigns to any material line c the "material emf" $\int_c \mathfrak{e}(t) = \int_{u(t,\,c)} \tau \cdot (E + v \times B)$. So the proxy of $\mathfrak{e}$ is *not* E, but $E + v \times B$, the Lorentz force. (The asymmetric way in which $\mathfrak{h}$ and $\mathfrak{e}$ are thus treated stems from our decision to neglect displacement currents.) This allows us to state Faraday's law in a very neat way: *For all material surfaces* S,

(2) $\mathrm{d}_t \int_S \mathfrak{b} + \int_{\partial S} \mathfrak{e} = 0$,

that is to say, "the rate of change of the flux through a material surface balances the emf around its rim," irrespective of *why* the flux changes (time-variations of B or movement, or both). So (2) subsumes all "cut flux", "embraced flux", etc., rules. Just as with Ampère's law, there is a differential version of (2), which is $\partial_t \mathfrak{b} + d\mathfrak{e} = 0$. The time-derivative, here, should not be misinterpreted: $\partial_t \mathfrak{b}$ is, like $\mathfrak{b}$ itself, a 2-form, that assigns to a material surface S the rate of change of the induction flux, i.e., the time derivative $\mathrm{d}_t [\int_{u(t,\,S)} n \cdot B(t)]$. So its proxy, as shown by the classical calculation of this derivative (more on this below), is the vector field $\partial_t B - v \times B$, and not $\partial_t B$. This is why the velocity field does not appear explicitly in (1)(2). At this stage, we have expressed the Maxwell equations—but not the constitutive laws, yet—"in material form".

A digression, before moving on. Just as there is a vector calculus, with a lot of involved formulas such as—one example among many— $\mathrm{div}(E \times H) = H \cdot \mathrm{rot}\, E - E \cdot \mathrm{rot}\, H$, there is a formulary to do calculus with differential forms, and a dictionary of correspondences between proxies and vector calculus operations, on the one hand, forms and their specific operations, on the other hand. For instance, d is the counterpart of grad, rot, or div acting on proxies. Dot product and cross product also have their counterparts in the so-called *wedge product*, thus defined: If E_u and H_u are proxies for the 1-forms $\mathfrak{e}$ and $\mathfrak{h}$, then $E_u \times H_u$ is the proxy for a 2-form, denoted $\mathfrak{e} \wedge \mathfrak{h}$, defined at time t as the map $S \to \int_{u(t,\,S)} n \cdot E_u \times H_u$. If H_u and B_u are proxies for the 1-form $\mathfrak{h}$ and the 2-form $\mathfrak{b}$, then $H_u \cdot B_u$ is proxy for a 3-form, denoted $\mathfrak{h} \wedge \mathfrak{b}$, defined as the map $\Omega \to \int_{u(t,\,\Omega)} H_u \cdot B_u$, where Ω is a material volume (i.e., a 3-dimensional manifold, as befits a 3-form). The above formula, recast in terms of DFs, is $d(\mathfrak{e} \wedge \mathfrak{h}) = d\mathfrak{e} \wedge \mathfrak{h} - \mathfrak{e} \wedge d\mathfrak{h}$. Weak formulations such as "$\int_D \mathrm{rot}\, H \cdot E' = \int_D J \cdot E'$ for all E' " rewrite as "$\int_D d\mathfrak{h} \wedge \mathfrak{e}' = \int_D \mathfrak{j} \wedge \mathfrak{e}'$ for all $\mathfrak{e}'$ ", which means that all tricks of the trade in finite element variational techniques are available.

Some tools of functional analysis also can be transposed. For instance, one may define the scalar product of two 1-forms $\mathfrak{h}$ and $\mathfrak{h}'$, in configuration u, by $(\mathfrak{h}, \mathfrak{h}')_u = \int_D \mu_0\, H_u \cdot H'_u$, thus putting a Hilbertian structure, configuration-dependent, on the space of 1-forms. A well-known result, the Riesz theorem, says that a linear continuous functional over a Hilbert space can be represented via a scalar product. It implies here that, given a 2-form $\mathfrak{b}$, there is a 1-form $\mathfrak{h}$ such that $\int_D \mathfrak{b} \wedge \mathfrak{h}'$ (which is linear continuous with respect to $\mathfrak{h}'$) is equal to $(\mathfrak{h}, \mathfrak{h}')_u$ for all $\mathfrak{h}'$. As $\mathfrak{h}$ depends linearly on $\mathfrak{b}$, we may write $\mathfrak{h} = \nu_u \mathfrak{b}$, hence an operator ν_u from 2-forms to 1-forms (a variant of what differential geometers call the "Hodge operator"), and it should be obvious that $\mathfrak{h} = \nu_u \mathfrak{b}$ is equivalent to the relation $B_u = \mu_0 H_u$ between the proxies, in configuration u, of $\mathfrak{b}$ and $\mathfrak{h}$. Notice that $\int_c \mathfrak{h}(t) = \int_{u(t,\,c)} \tau \cdot (\mu_0^{-1} B)$, which illustrates the fact that ν_u sets up a correspondence between material fluxes (accounted for by $\mathfrak{b}$) and material mmf's (accounted for by $\mathfrak{h}$) in configuration u.

Let us proceed, addressing now the constitutive laws needed to close the system (1)(2). Ohm's law, in material form, is

(3) $\mathfrak{j} = \mathfrak{j}^s + \sigma_u \mathfrak{e}$,

where $\mathfrak{j}^s$ is defined like $\mathfrak{j}$ (its proxy is J^s), and σ_u is the operator such that $\int_S \mathfrak{j}(t) = \int_{u(t,\,S)} n \cdot (\sigma E)$. For linear materials, the $\mathfrak{b}$–$\mathfrak{h}$ law has a similar form, $\mathfrak{b} = \mu_u \mathfrak{h}$ or $\mathfrak{h} = \nu_u \mathfrak{b}$, where ν_u is the same as in last paragraph. The operators σ_u and ν_u do indeed depend on the placement, even if the intrinsic conductivity and reluctivity of the materials do not change in time, because of a *shape effect* that will be found again later in relation with forces: Imagine, at a material point x, a small material line c and a small material surface S. Suppose $\mathfrak{b}$ steady, so that $\int_{u(t,\,S)} n \cdot B$ is kept constant in time, and set $H = \nu(u(t, x))\, B$. Unless the placement realizes a rigid motion of the matter around x, the circulation of H along $u(t, c)$ will *not* stay fixed, which means $\mathfrak{h}$ varies in time, while $\mathfrak{b}$ doesn't.

The conduction law plays no role in the question of forces, so we shall leave it at that, but a linear $\mathfrak{b}$–$\mathfrak{h}$ law is too much of a restriction for what we have in view. For enough generality, let us introduce a pair of so-called "convex functions in duality", $\Psi(u, \mathfrak{b})$ and $\Phi(u, \mathfrak{h})$ such that $\Psi(u, \mathfrak{b}) = \sup\{\mathfrak{h}' : \int_D \mathfrak{b} \wedge \mathfrak{h}' - \Phi(u, \mathfrak{h}')\}$ and vice versa. For example, if $B = \mu_0 H$ all over, the simplest case, then

[4] More generally, d maps a p-form to a $(p + 1)$-form, and is the differential geometric expression of grad, rot, or div, when $p = 0$, 1, or 2.

$\Psi(u, B) = 1/2 \int \mu_0^{-1} |B_u|^2$ and $\Phi(u, H) = 1/2 \int \mu_0 |H_u|^2$ are in such a relation. Then, the equality

(4) $\qquad \Psi(u, B) + \Phi(u, H) = \int_D B \wedge H$

establishes a correspondence between B and H which one can take, as the linear example shows, as constitutive law. Adding a function of u alone to Ψ, and subtracting this same function from Φ, doesn't change this B–H relation, obviously, so we make the convention that $\Psi(u, 0) = 0$. Functions Ψ and Φ are called *energy* and *coenergy* of the material fields B and H, respectively, but be careful, that Ψ *deserves* to be called "energy" needs proving! (This will come in due time.) The formulation is general enough to cover the case of linear materials with different permeabilities (anisotropic permeability tensors *are* allowed) and of nonlinear ferromagnetic materials (provided there is no hysteresis). Functions Ψ and Φ need not be smooth, but if they are, the partial derivative $\partial_B \Psi$ is H, by which we mean (this is the concept of "Fréchet derivative") that the linear part of the map $\delta B \to \Psi(u, B + \delta B) - \Psi(u, B)$ is the map $\delta B \to \int_D H \wedge \delta B$. Symmetrically, $\partial_H \Phi$ is B.

It remains to treat boundary conditions. To keep things simple, we shall suppose the wall ∂D is a perfect conductor, so that $\int_c E(t) = 0$, at all times, for any curve c included in ∂D, which we write as

(5) $\qquad E = 0$ on ∂D,

the counterpart of the standard vectorial notation $n \times E = 0$.

Equations (1)(2)(3)(4)(5) fully describe[5] the electrodynamics of the system: Given initial conditions at $t = 0$ and a trajectory $t \to u(t)$, they determine the field. If we can find the force from this, it will be possible to couple these equations with the dynamic ones. As announced earlier, a postulate about energetic exchanges must be proferred (and justified) to replace the Lorentz law. We now have what it takes to phrase it:

Power exchange postulate. *The network brings power* $-\int_D E \wedge J^s$ *to the system. Joule losses are* $\int_D E \wedge (J - J^s)$.

Power and losses do not balance. Power not dissipated goes into the field's energy reservoir and is used to perform mechanical work, in proportions which we shall determine in next Section. But we should first satisfy ourselves that the postulate is justified: Indeed $E \wedge J = E \cdot J$, in terms of proxy fields, *if there is no motion,* and we have assessed earlier, by an analysis of Lorentz forces, that $E \cdot J$ is the Joule power density in that case. A similar argument applies to $E \wedge J^s$. Now, for moving parts, one may reckon in the local comoving reference frame, where the electromotive field is $E' = E + v \times B$, and assert that $E' \cdot J$ is Joule power. But this is precisely $E \wedge J$, considering how E has been defined.

III. Energetic balance, forces

First, let us establish the status of $\Psi(u, B)$ as, in actual fact, magnetic energy. Consider a system's evolution $t \to \{u(t), B(t)\}$, and set $W(t) = \Psi(u(t), B(t))$ at time t. The rate of change of this quantity, by the chain rule and the above remark about the derivative $\partial_B \Psi$, is

(6) $\qquad d_t W = \langle \partial_u \Psi, \partial_t u \rangle + \int_D \partial_B \Psi \wedge \partial_t B$
$\qquad\qquad = \langle \partial_u \Psi, \partial_t u \rangle + \int_D H \wedge \partial_t B.$

Here, $\partial_u \Psi$ is a Fréchet derivative again, namely, the linear functional that maps a vector field v to the real number $\langle \partial_u \Psi, v \rangle$, that is to say, the rate of change of W when $\partial_t B = 0$ and $\partial_t u = v$. These brackets are a non-committal notation, which allows one, when v is determined by a finite array of of parameters $\underline{v}$, as discussed earlier, to write the bracket $\langle \partial_u \Psi, \underline{v} \rangle$, where $\partial_u \Psi$ is an array of same dimension as $\underline{u}$ and $\underline{v} = \partial_t \underline{u}$, a point to which we soon return.[6]

Now, let us address equations (1)(2), which we use in their local differential form, $dH = J \equiv (J - J^s) + J^s$ (to distinguish source current and conduction current) and $\partial_t B + dE = 0$. Wedge-multiplying them from the right by $-H$ and E, respectively, adding, and integrating over D, we get

(7) $\qquad -\int_D E \wedge J = \int_D H \wedge \partial_t B + \int_D d(E \wedge H) \equiv \int_D H \wedge \partial_t B,$

because $\int_D d(E \wedge H) = \int_{\partial D} E \wedge H = 0$, thanks to (5). This kind of manipulation, easy to get used to thanks to the above-mentioned "dictionary", gives some of the flavor of calculus in geometrical electromagnetics.

Putting (6) and (7) together, we obtain this:

(8) $\qquad d_t W(t) - \langle \partial_u \Psi, \partial_t u \rangle + \int_D E \wedge (J - J^s) = -\int_D E \wedge J^s.$

Suppose $u(t) = u(0) = u$ at all times (no motion, $\partial_t u = 0$), and $B(0) = 0$. Integrating (8) in time, and remembering that $\Psi(u, 0) = 0$, we get

$$\begin{aligned}\Psi(u, B(t)) \equiv W(t) &= \int_0^t d_s W(s)\, ds \\ &= -\int_0^t [\int_D E(s) \wedge J^s(s)]\, ds - \int_0^t [\int_D E(s) \wedge (J - J^s)(s)]\, ds \\ &= \text{<supplied energy>} - \text{<Joule losses>}\end{aligned}$$

by the energetic postulate. In the absence of any motion, the energy at the right-hand side has had no other outlet than the field's energetic reservoir, hence $\Psi(u, B)$ does qualify as the "magnetic energy" of the induction field B in configuration u. The energy stored by the system is therefore $W(t)$.

Remark. Do not confuse energy as a function (of B) and energy as a number—just as f and $f(x)$ should not be confused (Note 3). The point is important in linear situations where the *values* of energy and coenergy always coincide, in spite of Ψ and Φ being different *functions*. ◊

Next, the force. We may need it at any given instant, for any electromagnetic field $\{E, B\}$, and any mechanical configuration u, but this instant can always be taken as $t = 0$, with $\{E, B\}$ as initial conditions, and the trajectory of placements may start at $u(0) = u$. Equations (1—5) then determine the field's evolution. As a passage to the limit $t = 0$ will be involved, one may assume that $u(t, x) = u(x) + t\, v(x)$, where v is the velocity field at $t = 0$, neglecting higher-order terms. Recall that this is *virtual* motion, the idea being that force will result from an application of the

---

[5] Not quite: The electric field outside conductors is determined up to a gradient only, as usual in eddy-current theory. This is no problem if Coulomb forces are to be ignored, as we assume. Anyhow, electric charge can be computed a posteriori, as $Q = \mathrm{div}(\varepsilon_0 E)$, hence Coulomb forces as a by-product.

[6] In geometric terms: v is a tangent vector, at point u of the configuration manifold U, and $\partial_u \Psi$ is a *co*-vector at this point. Since U is of infinite dimension, some work would be needed to make these notions rigorous, and we gloss over this. When U is reduced to the quotient $\underline{U}$, composed of classes of dynamically equivalent placements, the dimension of $\underline{U}$ is as a rule finite, often as low as 1, and v reduces to $\underline{v}$, tangent to $\underline{U}$.

virtual power principle. So v is any arbitrary continuous vector field. Whether $u(t)$ is dynamically plausible, or even possible, is not an issue.

From (8) with $\partial_t u = v \neq 0$, we see that the power supplied to the mechanical compartment of the system is $-\langle \partial_u \Psi, v \rangle$, by mere bookkeeping: Network-supplied power which doesn't go as heat or into the field's reservoir is mechanical power, which from the point of view of the mechanical compartment of the system comes in addition to whatever it receives from other agencies, and hence is attributed to the electromagnetic field. This enables us, by unplugging the virtual field v, to *define* force *as* the mapping $v \to -\langle \partial_u \Psi, v \rangle$, that is to say (cf. Note 6) as the covector $-\partial_u \Psi$ at point u of U.

Remark. The scrupulous reader may object to the assumption $\Psi(u, 0) = 0$. But remark that, should any term $g(u)$ be added to $\Psi(u, B)$, which as we have noticed doesn't change the B–H law if the same term is subtracted from $\Phi(u, H)$, force as predicted by the previous theory would be $v \to \langle \partial_u [g(u) - \partial_u(\Psi(u, B) + g(u))], v \rangle$ — the same result. With our convention, $\Psi(u, 0) = 0$ for an "electrically dead" system, that is to say $B = 0$ and[7] $E = 0$, from which no electromagnetic forces come. ◊

Since Φ and Ψ play symmetric roles, it should be possible to express the force in terms of variations of the co-energy, $\Phi(u, H)$, in a similar way. Indeed,

Proposition 1. *If* B *and* H *satisfy* $\Psi(u, B) + \Phi(u, H) = (B, H)$, *then*

$$\partial_u \Phi(u, H) = -\partial_u \Psi(u, B). \tag{9}$$

Proof. Given a velocity field v, consider a trajectory $t \to u(t)$ in U such that $v = \partial_t u$ at $t = 0$, and let $t \to \{B(t), H(t)\}$ be a correlative evolution of B and H such that $\Psi(u, B) + \Phi(u, H) = (B, H)$. [Such evolutions exist: for instance, keep B fixed, and take $H = \partial_u \Psi(u, B)$. The Maxwell equations need not be satisfied for our present purpose.] Differentiate in t:

$$\langle \partial_u \Psi, v \rangle + \int H \wedge \partial_t B + \langle \partial_u \Phi, v \rangle + \int B \wedge \partial_t H = \int B \wedge \partial_t H + \int H \wedge \partial_t B,$$

hence $\langle \partial_u \Psi, v \rangle + \langle \partial_u \Phi, v \rangle = 0$ for all v, hence (9). ◊

So force is *the partial derivative of coenergy with respect to the configuration parameter*, which amounts to saying it is obtained by varying the coenergy while keeping fixed the material magnetic strength H, i.e., the material mmf's, which implies "keeping the material currents constant". We have found back the classical rule (a weakened form of it, actually, which is for the better).

The simplest example, already evoked, is when $B = \mu_0 H$. Then $\Psi(u, B) = 1/2 \int \mu_0^{-1} |B_u|^2$, where B_u is B's proxy. Differentiating with respect to u boils down to computing the rate of change of B_u under the condition $\partial_t B = 0$, the placement being $u(t, x) = u(x) + tv(x)$. An auxiliary result, first:

Lemma 1. *Let* B (not necessarily divergence-free) *be a steady vector field. The time derivative of the material flux* $\int_{u(t, S)} n \cdot B$ *of* B, *at* $t = 0$, *is*

$$\int_{u(t, S)} n \cdot [v \operatorname{div} B - \operatorname{rot}(v \times B)]. \tag{10}$$

Proof. For convenience, let us define the *extrusion* $\operatorname{extr}(x, v, t)$ of point x by the velocity flow v between time 0 and time t as the segment $[x, x + tv]$. The extrusion of a set of points is defined as the union of their extrusions, hence the extrusion of a line [resp. a surface] is (generically) a surface [resp. a volume]. By Ostrogradskii,

$$\int_{\operatorname{extr}(S, v, t)} \operatorname{div} B = \int_{u(t, S)} n \cdot B - \int_S n \cdot B + \int_{\operatorname{extr}(\partial S, v, t)} n \cdot B,$$

with all the normals n directed as shown on Fig. 2. When t tends to 0, one has

$$\lim_{t \to 0} t^{-1} \int_{\operatorname{extr}(S, v, t)} \operatorname{div} B = \int_S n \cdot (v \operatorname{div} B),$$

$$\lim_{t \to 0} t^{-1} \int_{\operatorname{extr}(\partial S, v, t)} n \cdot B = \int_{\partial S} (\tau \times v) \cdot B = \int_{\partial S} \tau \cdot (v \times B) = \int_S n \cdot \operatorname{rot}(v \times B),$$

hence (10). ◊

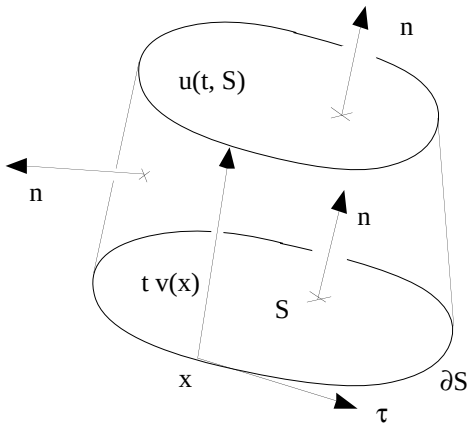

Fig. 2. Notations for Lemma 1. (The associated computation is a standard in differential geometry, in connexion with the so-called "Lie derivative".)

Now, if $\partial_t B = 0$, the rate of change of the proxy B_u must compensate for (10), hence the following corollary

$$\partial_t B_u = -v \operatorname{div} B + \operatorname{rot}(v \times B), \tag{11}$$

where B stands for B_u at time 0. The rate of change of $1/2 \int_D \mu_0^{-1} |B_u|^2$ is thus (since $\operatorname{div} B = 0$),

$$\int_D \mu_0^{-1} B \cdot \operatorname{rot}(v \times B) = \int_D H \cdot \operatorname{rot}(v \times B) = \int_D \operatorname{rot} H \cdot (v \times B)$$

(no boundary term, because $v = 0$ on ∂D). The force field is therefore $J \times B$, as expected.

A second corollary: Suppose two velocity fields v^1 and v^2 such that $v = v^2 - v^1$ is supported by a region Ω entirely contained in the air, or in an "electrically passive" material, i.e., non-conductor with permeability μ_0. The variations of $\Psi(u, B)$, for these two velocities, differ by that of $1/2 \int_\Omega \mu_0^{-1} |B_u|^2$, which is zero by what precedes, since $\operatorname{rot} H = 0$ in Ω. Two dynamically equivalent placements, therefore, generate the same forces, which justifies this terminology, and more importantly, allows one to reduce U to a much smaller configuration manifold.

[7] Notice again (cf. Note 5) how we drag E along as extra weight. If displacement currents were not ignored, there would be an electric energy $\Psi_{el}(u, D)$ and an associated coenergy $\Phi_{el}(u, E)$ which together would prescribe the D–E law, both defined up to some arbitrary function of u. The natural convention is then $\Psi_{el}(u, D) = 0$ when $\partial_D \Psi_{el}(u, D) = 0$, i.e., when $E = 0$.

IV. One-DoF systems

To test our understanding, consider two parallel plates with opposite parallel currents (Fig. 3). Neglecting side effects, we have a uniform field H in the airgap. Imagine a kind of finite-difference net in the airgap, each edge of which bears a definite mmf which we keep constant, according to the force computation rule, as the airgap width is varied. For definiteness, let d stand for this width in the reference configuration, the modified one then being $\underline{u}$. Only the horizontal edges bear a nonzero mmf, and since their length doesn't change with u, the intensity of the proxy H_u doesn't change either. The coenergy in the airgap is thus proportional to $\underline{u}$, from which we conclude that $\partial_{\underline{u}}\Phi > 0$ —a repulsive force.

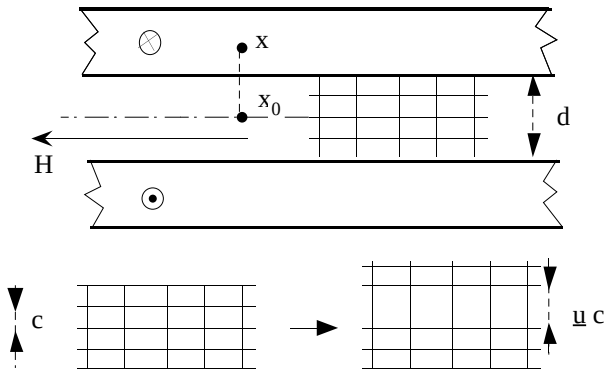

Fig. 3. Virtual deformation of an imaginary material grid in the airgap to compute the force between two parallel conducting plates.

To better quantify this information, compute the coenergy, $\mu_0/2 \int |H_u|^2$, for one square-meter of plate surface, which is $\underline{u}\ \mu_0\ |H|^2/2$ in the u-modified configuration. The derivative $\mu_0\ |H|^2/2$ of $\Phi(u, H)$, as taken at the reference value $\underline{u} = d$, is the *magnetic pressure* exerted on the plates.

Reasoning in terms of the energy should give the same result, which we check. Now, it's the horizontal fluxes (through the vertical faces of the imaginary net that helps visualize the material manifold) that are kept constant. Suppose the placement u associated with $\underline{u}$ is a uniform widening of the gap. Then areas of these faces behave like $\underline{u}$, hence the vector proxy is a horizontal uniform B_u with $\underline{u}\ B_u = d\,B$. Energy per square meter is $\underline{u}^{-1}\, d^2\ |B|^2/(2\mu_0)$, and *minus* the derivative of that, at $\underline{u} = d$, yields $|B|^2/(2\mu_0)$ —the same pressure, indeed.

Implicitly, the placement we have been using in the latter computation was $u(x) = x + \underline{u}(x - x_0)/d$ in the airgap, where x_0 is the point of the symmetry plane above or below x, and $u(x) = x \pm d/2$ for other points (sign + above the symmetry plane, – below). This shows how cumbersome it may be to specify a placement u that corrresponds to the parameter $\underline{u}$ one is really interested about. Fortunately, such definiteness is rarely required (to wit our reasoning about coenergy), and the possibility to choose between a whole family of dynamically equivalent placements may help a lot.

For instance (Fig. 3, bottom), one might imagine all horizontal layers of our imaginary net unchanged, except one of them, the width of which is $\underline{u}c$, with now $\underline{u} = 1$ in the reference configuration. This amounts to tailor u in such a way that all the gap-widening is due to the swelling of this layer. The proxy is B all over except within this thin layer, where it's $B/\underline{u}$. A factor c/d thus affects the derivative of the energy, but the factor d/c must be thrown in to achieve the same airgap width variation, hence the same result for the pressure.

This example has general value: Each time one has to compute forces, torques, etc., involving the relative motion of rigid bodies, the placement may be imagined as supported by the airgap, or, to use a previous metaphor, only the putty that fills up the voids between conductors, magnets, etc., is virtually stretched, and this stretching can be designed as most convenient. A classical method to compute forces consists in selecting a surface that separates bodies in relative motion, and in designing u to confine the stretching to a thin layer of air on one side of this surface. I won't spoil the pleasure readers may find in checking that what is recovered, from this procedure, is the standard integration of the Maxwell stress tensor.

It may be more instructive to wonder how this will be transposed in real-life finite element computations. A case such as displayed on Fig. 1 would be treated as follows: (1) A finite element mesh of the whole computation box is designed, element boundaries fitting material interfaces, as usual. (2) Arrange elements so that a small subset of them, for instance those forming a separating layer between the magnet and the bottom plate, in the airgap, have variable geometry under the placement. (3) At the assembly phase, when computing the magnetic stiffness matrix (which corresponds to the discretization of energy or coenergy, depending on which kind of field-DoF is used), precompute the *derivatives* of the elementary stiffness matrices with respect to $\underline{u}$. (4) Compute the field, hence either edge mmf's or edge circulations of the vector potential. (5) Using these values, sum up the variation of coenergy, or energy, in the deformable elements, hence the virtual (co-)energy variation, hence the force.

Step (3) of this recipe can be detailed as follows: (a) For all "stretchable" elements, take the relative positions of one or more nodes as "local" configuration parameters—say u_n for node n (a translation vector). Precompute the elementary stiffness matrix derivatives with respect to u_n. (b) Throw in the derivative of the position of node n with respect to $\underline{u}$, and sum up the matrices thus obtained. Left- and right-multiplying the matrix sum by the DoF arrays, at step (5), will give the force. But one may as well, equivalently, postpone step (3b) until edge DoF's are known, then left- and right-multiply the elementary matrix derivatives w.r.t. u_n by the DoF arrays, and sum up. At this stage, "nodal generalized forces" f_n, vector valued, will have been obtained, the variation of the coenergy (let's say) being $\delta\Phi = \sum_n f_n \cdot \delta u_n$. It remains to substitute $\partial u_n/\partial \underline{u}\ \delta\underline{u}$, there, hence $\delta\Phi = \underline{f}\ \delta\underline{u}$, to have the force.

Obviously, once such a computational chain is in place, one could use it to compute nodal forces, independently, and hence the force *field*. One should just be aware that, in such a process, the generalized nodal force f_n is *not* the value at node n of the force field f: For the latter is the field such that $\int_D f(x) \cdot \delta u(x) = \sum_n f_n \cdot \delta u_n$, hence the necessity to ponder by the elementary volumes to relate f_n with $f(x_n)$. Apart from this, it would seem that we have solved, at least practically, the problem of computing the force field inside a deformable body, by this procedure.

Yet we are not there, because the elementary matrix differentiation requires knowing the B–H law inside the element *as a function of the deformation*, which is no problem in what precedes, since all stretchable elements are in the air, where the law suffers no ambiguity (it's $B_u = \mu_0 H_u$ in terms of the proxies), but may be a problem inside, say, a magnet. *Do we know, really, about the B–H law inside a* permanent *magnet?*

V. What is a "permanent" magnet, exactly?

There is no such thing as a "permanent" magnet, obviously—if magnetization was an immovable feature of matter, no magnet could be made in the first place. What counts as a permanent magnet, in practice, is a chunk of magnetized material placed in circumstances where its B–H curve is approximately linear:[8] $B = \mu_0(H + H^c)$, or equivalently $B = \mu_0 H + B^r$, where the "coercive field" H^c or the "remanent induction" $B^r = \mu_0 H^c$ (thus called by analogy) do not depend on the ambient field. They are vector fields, borne by a bounded region M, the magnet, which they characterize.

Or do they? That H^c and B^r do not change with time is granted, but what of their dependence on the deformation of the material, that is, on u? If H^c and B^r do not depend on u, the associated *material coercive field* h^c and *material induction field* b^r for which they stand as proxies *have to*, and we *need* to know how in order to estimate $\Psi(u, b)$ and $\Phi(u, h)$. One may, trying another tack, choose to define as "permanent" a magnet for which the material B–H law contains an invariable material coercive field,

(12) $\quad h = \nu_u b - h^c,$

which derives from the magnetic energy

$$\Psi(u, b) = 1/2 \int_D \nu_u b \wedge b - \int_D b \wedge h^c.$$

This is an attractive proposition, because the additional term term $-\int_D b \wedge h^c$ does not depend on u, so we may recycle the previous computation of the rate of change of $1/2 \int_D \mu_0^{-1} |B_u|^2 \equiv 1/2 \int_D \nu_u b \wedge b$, which we found was $\int_D \mu_0^{-1} B \cdot \mathrm{rot}(v \times B)$. Proceeding from this point on, we find now

$$\begin{aligned} -\langle \partial_u \Psi, v\rangle &= \int_D \mu_0^{-1} B \cdot \mathrm{rot}(B \times v) \\ &= \int_D (H + H^c) \cdot \mathrm{rot}(B \times v) \\ &= \int_D \mathrm{rot}(H + H^c) \cdot (B \times v) = \int_D (J + J^c) \times B \cdot v, \end{aligned}$$

if we denote by $J^c \equiv \mathrm{rot}\, H^c$ the Amperian currents. The field force inside the magnet is

(12') $\quad f = \mathrm{rot}(H + H^c) \times B,$

exactly what the $J \times B$ rule would give if applied not only to the conduction currents (if any) but to the Amperian currents as well.

Reassuring as this result may seem, we should take it with much suspicion, because the following alternative to (12):

(13) $\quad h = \nu_u (b - b^r),$

with an invariable material remanent induction, has just as good a claim to describe a "permanent" magnet, as the following thought experiments suggest. First, sprinkle some plastic substrate with a large number of small hard magnets, evenly distributed, all roughly oriented the same way. The "micro" scale, for such a composite, has the small magnet's size as characteristic length. Consider a material surface S whose dimensions correspond to an appropriately larger "meso" scale. It cuts a large number of magnets. The fluxes of their fixed magnetizations sum up to some number, an additive function of S, independent of both time and the deformation of the composite, which is therefore properly represented by a fixed 2-form b^r. So at such a mesoscale, the composite behaves like a homogeneous material, whose constitutive law is (13). It is not too difficult to imagine how (12) could be realized: Instead of the magnets, immerse a large number of small solenoids, arranged more or less like mattress springs, each bearing a constant intensity provided by some unobtrusive power source. This time, any mesoscale material circuit encloses a definite intensity, whatever the overall deformation, and (12) is the appropriate "homogenized" mesoscale law.

Thus (13), too, is a plausible constitutive law for a permanent magnet. But the force field it predicts *differs* from (12'), for the magnetic energy is now

$$\begin{aligned} \Psi(u, b) &= 1/2 \int_D \nu_u b \wedge b - \int_D \nu_u b \wedge b^r, \\ &\equiv 1/2 \int_D \mu_0^{-1} |B_u|^2 - \int_D \mu_0^{-1} B_u \cdot B^r_u \end{aligned}$$

in terms of proxy fields. By (11), $\partial_t B^r_u = -\,v\ \mathrm{div}\, B^r + \mathrm{rot}(v \times B^r)$, hence an easy calculation of the rate of change of $\int_D \mu_0^{-1} B^r_u \cdot B_u$. Collecting all such rates of change, we have

$$\begin{aligned} -\langle \partial_u \Psi, v\rangle &= \int_D \mu_0^{-1} (B - B^r) \cdot \mathrm{rot}(B \times v) \\ &\quad + \int_D \mu_0^{-1} B \cdot [-\,v\ \mathrm{div}\, B^r + \mathrm{rot}(v \times B^r)] \\ &= \int_D \mathrm{rot}\, H \cdot (B \times v) - \int_D \mathrm{div}\, B^r\, (H + H^c) \cdot v \\ &\quad + \int_D \mathrm{rot}(H + H^c) \cdot v \times B^r. \end{aligned}$$

The force field is therefore

(13') $\quad f = J \times B - (H + H^c)\ \mathrm{div}\, B^r - (J + J^c) \times B^r.$

Now which is right, of (12') and (13')? Which of the laws (12) or (13) does describe a "permanent" deformable magnet?

Wrong questions, obviously. *You don't speculate about constitutive laws, you measure them.* Being handed a sample of the material our permanent magnet is made of, it's up to us to determine what the B–H law is for various deformation states (likely, neither (12) nor (13), but something in-between), and to encode this information in functionals such as Ψ and Φ. In practice, this is done by a complex process, which allies experiments with numerical simulations, as suggested by Fig. 4. One may take a sample rod, plunge it in some known magnetic field created by a current I, pull on it with force P, record the length L and the flux F, for a series of values of I and P. The "macroscopic" relations thus observed between F, I, L, and P do not immediately translate into "mesoscopic" constitutive laws between B, H, ε (strain), and σ (stress). One must posit an appropriate mesoscopic law, with a number of adjustable parameters, set up the functionals $\Psi(u, b)$ and $\Phi(u, h)$ that apply in the particular experimental situation, predict the F, I, L, P charts by computations (simple computations, if at all possible—but one well imagines that finite element simulations become necessary in some cases), compare that with the experimental results, and adjust the parameters

---

[8] One rather has $B = \mu(H + H^c)$, with $\mu \neq \mu_0$ and independent of the ambient field, for actual magnets ($\mu \sim 4\,\mu_0$ is a typical value). We take $\mu = \mu_0$ here to simplify the discussion.

accordingly, in an iterative and hopefully converging identification process.

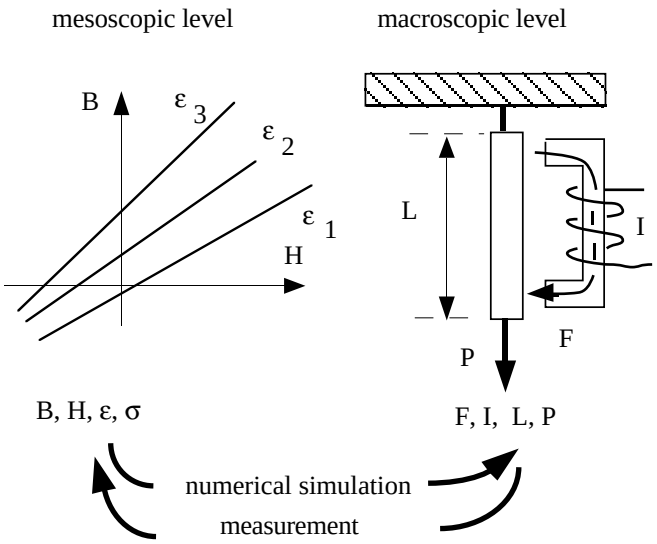

Fig. 4. From mesoscopic constitutive laws to macroscopic observations and back.

This is the first application for a theory of forces: *to experimentally determine the* B–H *laws as parameterized by the local deformation of the material*, and also, for the two things cannot be separated as Fig. 4 suggests, *determine the* σ–ε *mechanical behavior laws as parameterized by the local magnetic field.* Once this is done, that is to say, once materials are characterized at the mesoscopic level, the road is open to the numerical simulation of magnetostriction phenomena, static and dynamic, in order to predict the macroscopic quantities our sponsors and clients may be interested in.

AUTHOR'S NAME AND AFFILIATION

Alain Bossavit, Laboratoire de Génie Électrique de Paris, CNRS, 11 Rue Joliot-Curie, 91192 Gif-sur-Yvette, France
bossavit@lgep.supelec.fr