Technical article

Introduction to Adaptive Finite Element Analysis for Electromagnetic Simulations

***Abstract*—**The essential components and control structures of modern adaptive finite element methods, for analysis and simulation in electromagnetic applications, are presented and discussed. Whilst the development is intended to survey the full range of techniques, this contribution is focused mainly on fully combined *hp* formulations, designed for parallel and distributed computational environments. In addition to this theoretical presentation and evaluation, a range of implementations and computed results are reported, to demonstrate the advantages and drawbacks of the more promising approaches. These selected overview topics are organized and presented, primarily, to address, inform and educate the non-specialist.

***Index Terms* —** Finite element methods, adaptive systems, error analysis, parallel processing, electromagnetic analysis.

I. Introduction

Adaptive finite element analysis (FEA) for the modeling and simulation of electromagnetic applications has become increasingly important over the years, and is now a mature and well-established research area. The focus of this contribution is to overview the structures, essential components and automatic control systems of modern adaptive finite element methods (AFEMs), and to explore the potential benefits and related costs of employing them in practical and cost-effective electromagnetic computer-aided design environments. Unlike more technically specific contributions in the area, this work is intended to address, inform and educate the non-specialist.

The main difficulty with computational analysis is that a very large number of free modeling parameters are frequently necessary for computing accurate and reliable simulations of realistic systems: sufficient mathematical degrees of freedom (DOF) are required to both resolve the geometric and material features of practical devices, and to represent the fields of the electromagnetic system. As a result, the computational effort required for the electromagnetic simulation can frequently be prohibitive. Currently, one promising approach to overcome this type of computational barrier is to employ adaptive solver technologies which are capable of intelligently evolving and improving an efficient distribution of DOF over the problem domain. Adaptive methods begin with relatively inexpensive initial discretizations for systems, then establish operational solution error distributions over them, and subsequently add DOF to the models to improve them [1], [2].

During the past five to ten years, significant progress has been achieved in the development of advanced AFEMs for electromagnetic analysis and simulation, which can compute sufficiently accurate solutions to problems using many fewer DOF than non-adaptive methods [2]-[7]. In addition, recent research and development progress with advanced strategy *feedback control systems*, which are employed to guide the adaptive process, now hold great promise for overcoming the computational bottleneck [4]-[7]. The main purpose of this contribution is to review the implications of using accurate and reliable AFEMs for efficient computational analysis and design in practical engineering applications.

To date, a number of different types of adaptive systems have been developed for electromagnetic FEA, and some are now in reasonably widespread use, while others represent a relatively new and underdeveloped area of research [1], [2], [4], [5], [8]. In order to accurately and efficiently model the highly non-uniform electromagnetic fields developed within sophisticated systems, AFEMs that can reliably identify and selectively refine the regions of high solution error need to be established. Presently, four basic types of adaption models are applied to refine finite element approximations: *h*-type; *p*-type; combined *h*- and *p*-type (called *hp*-type); and *r*-type. Essentially, these models only differ in the techniques used to update the finite element discretization, within the adaptive feedback loop:

A. Generate initial finite element discretization.
Repeat:
  B. Assemble and solve finite element problem.
  C. Evaluate solution accuracy; if adequate then **STOP**.
  D. Identify regions of inadequate discretization.
  E. Determine required discretization refinements.
  F. Update finite element discretization.
Until STOP

Briefly stated, *h*-type adaption models add new elements into a mesh to improve the discretization; *p*-type adaption models increase the degree of approximation used for the elements in a mesh to improve the discretization; *hp*-type adaption models apply a combination of both of these procedures; and *r*-type adaption models reposition element vertices within the mesh to improve solution accuracy. In addition, a relatively broad range of refinement criteria have been developed during the past decade, that can be used to identify and locate regions of relatively high solution error within the problem domain, due to inadequate discretization [1], [2], [4], [5], [8]. A summary of the essential concepts and some recent research developments in the study of AFEMs for electromagnetic applications is presented in the following section.

II. Adaptive finite element methods

The finite element method (FEM) is a powerful numerical analysis technique which is well-suited to and appropriate for solving a wide variety of electromagnetic applications problems computationally [9]-[17]. Amongst the many methods used in computational electromagnetics [16]-[25], its ability to manage problems with complex geometries, as well as its broad applicability to static, quasi-static, wave and transient systems, and to problems containing material regions that are nonlinear, inhomogeneous and anisotropic, all make the FEM one of the most versatile and powerful computational analysis and simulation schemes available today [13]-[17]. Moreover, the solid theoretical foundations on which the FEM is based, as well as the rigorous mathematical analyses concerning the existence, convergence, and the uniqueness of finite element solutions that have been established, further justify its use in electromagnetics research and design [26]-[37]. Currently, FEA is depended upon frequently in electromagnetic design: typically, FEA tools are applied to numerically simulate and evaluate the performance of a newly proposed device design before building a prototype, or to computationally investigate the electromagnetic characteristics of natural and man-made systems and their interactions with, or their impact on, their surrounding environments [38]-[42].

While FEMs are presently used extensively for electromagnetics analysis and design, [43], [44], the use of AFEMs has gained considerable attention in recent years from numerical analysts for solving problems more efficiently than standard FEMs permit [45]. The accuracy of a finite element solution is directly dependent on the number of free parameters used to mathematically represent the problem, and how effectively those parameters, or mathematical DOF, are distributed over the problem space. Furthermore, the full computational cost associated with obtaining a finite element solution is related to both the number and the interconnectivity of the DOF used in the problem discretization. Consequently, the most efficient distribution of DOF for a problem is that which provides a sufficiently accurate solution for the lowest number of free parameters. Currently, the only practical way to achieve this objective is by using adaptive solution strategies which are capable of intelligently evolving and improving an efficient distribution of DOF over the problem domain by establishing solution error distributions, and then adjusting or adding DOF to the discretization to correct them [2]-[4]. By increasing the numbers of DOF in the vicinities of higher solution error only, it is possible to make the most significant improvement in the global accuracy of the finite element solution, for the minimum additional computational cost. In contrast, while uniformly increasing the number of free parameters over the problem domain could provide an even greater improvement in the computed solution accuracy, the per capita increase in accuracy for each new DOF may not be as high, since new DOF added to regions which were already sufficiently well discretized would not necessarily contribute to a significant improvement in the overall solution accuracy [1], [3].

One of the primary objectives of AFEMs is to compute the solution to an engineering problem, to within a pre-specified accuracy tolerance, for the smallest possible computational cost. In order to achieve this goal, the fundamental approach underlying the majority of all AFEMs involves the efficient, iterative improvement of an (ideally) convergent sequence of increasingly accurate approximations of the true solution to a given engineering problem. A simple conceptual framework which is meaningful for the study of AFEMs is shown within the context of the general finite element solution scheme, in Fig. 1, where the individual steps of an adaptive method are constituents of one (or the other) of two major, procedural components: namely, the adaption model, and the feedback control system that is used to guide the adaptive finite element process. Simply put, the adaption model includes those steps involved in updating the discretization, while the feedback control system is concerned with those procedures related to resolving *how* to increase the level of discretization over a problem. Consequently, the combined specifications of the adaption model, together with a feedback control strategy, serve to define an adaptive method in this paradigm.

An adaption model is a set of well-defined procedures used within AFEMs to update the finite element discretization. As outlined earlier, there are four basic types of adaption models presently under study in the mainstream literature: (i) *h*-type, (ii) *p*-type, (iii) *combined hp*-type, and (iv) *r*-type. Each of the basic models possesses strong positive attributes, along with distinct disadvantages, which render their use in AFEMs effective under different conditions; all four are considered in this work. These adaption models are described and discussed in greater detail in the following four subsections, to illustrate their importance in developing effective practical AFEMs.

A. The h-type Adaption Model

In *h*-type adaption models, refinement of the finite element discretization is accomplished by modifying the sizes (*h*) of elements within the mesh, while keeping the order (*p*) of the approximating functions of the elements fixed. Consequently, in order to improve the accuracy of a finite element solution using an *h*-type adaption model, the number of free parameters used to compute the solution is increased by increasing

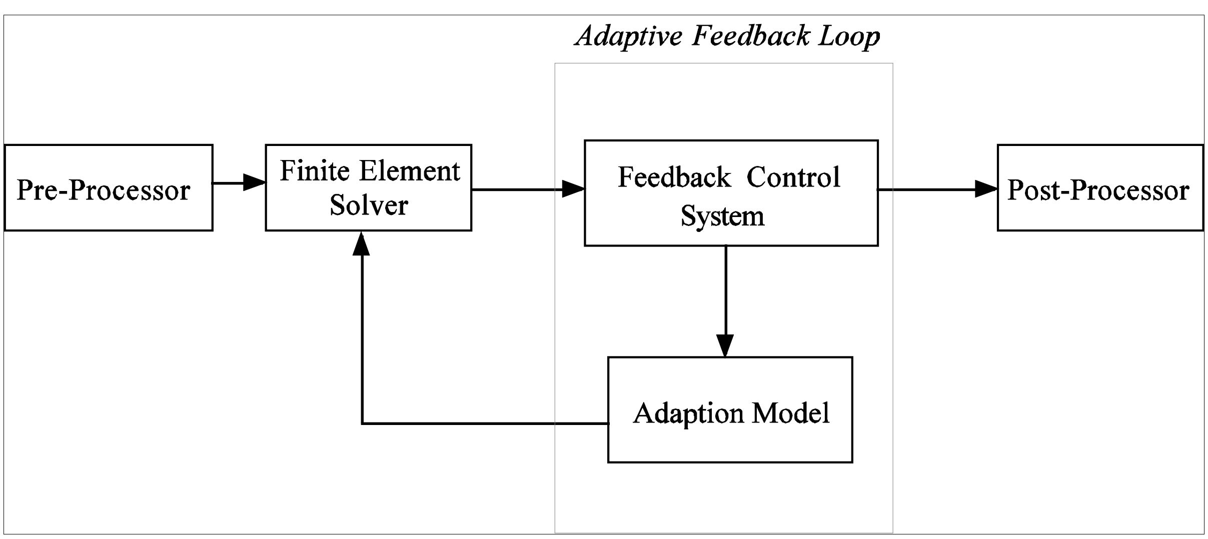

Fig. 1. The adaption model and feedback control system framework for the study of AFEMs, within the context of the general finite element solution scheme. The general finite element solution process, usually, involves: (i) a pre-processing unit for building a computational model of the problem; (ii) a finite element solver for computing solutions to the discretized problems; and (iii) a post-processing unit for analyzing the computed solutions.

the total number of elements in the mesh; thereby, decreasing the overall average size, h_{avg} , of elements in the mesh:

$$h_{avg} = \frac{1}{N} \sum_{i=1}^{N} h_i \tag{1}$$

where h_i is the size of the i^{th} element in a mesh comprised of a total of N elements.

A variety of AFEMs which are solely based on *h*-adaption models have been applied quite successfully in a broad range of electrical engineering applications [4], [8], [46], [47]-[58]. In particular, for problems where mathematical singularities in the field solutions exist, such as those which occur at sharp material edges and corners [59], *h*-type adaption models have proven to be quite effective, where a large number of smaller elements are required close to these singularities, but fewer, larger elements of the same order suffice farther away [1], [2], [50], [60], [61]. Given that the approximate error in a finite element solution is $O(h^{min\,[p+1,\,\varsigma]})$, where $\varsigma < 1$ in the vicinity of most singularities, it is evident that reducing the element sizes (*h*) near a singularity may be more advantageous than raising the degree of approximation (*p*). Numerical studies have also shown that *h*-type refinement near singularities in finite element electromagnetics can yield near optimal rates of convergence, for certain levels of discretization [2].

B. The p-type Adaption Model

In *p*-type adaption models, refinement of the finite element discretization is accomplished by adapting the orders (*p*) of approximating functions on elements, while holding the sizes (*h*) of the elements in the mesh constant. Standard Lagrangian elements require the same order approximating functions be used over the full mesh, to ensure a continuous finite element solution [13]; however, hierarchal elements permit increasing the order of individually selected elements in the mesh, while still ensuring C^0 continuity of the computed solution. Hence, it is possible to evolve efficient distributions of DOF by only raising the polynomial order of the elements in the inaccurate regions of the mesh. Based on the interpolation theory error model described in the previous section, the point-wise error in a finite element solution is approximately $O(h^{p+1})$ within sub-regions of a problem domain where no singularities are present. Therefore, if the finite element mesh is such that the element sizes are sufficiently small in regions far away from any singularity, then the improvement in the accuracy of the computed solution should be greater with an increase in the polynomial orders (*p*) of the approximating functions, when compared with a decrease in element sizes (*h*), according to the interpolation theory error model. Numerical studies have shown that, under certain conditions, *p*-type refinement can result in better rates of solution error convergence than that which can be achieved using *h*-adaption models [2].

In finite element electromagnetics, *p*-type adaption models incorporating hierarchal elements have been shown especially useful in high frequency problems, where the fields possess a wave-like variation, and are better modeled in certain parts of the mesh by higher order elements, and lower order elements provide a sufficiently accurate approximation in other regions of the mesh [62]. Further, with high frequency analyses, the wave-like fields located far away from material boundaries are particularly well represented by higher order polynomial approximating functions, and *p*-type refinement can be a very attractive alternative to *h*-type refinement since it avoids the cost of re-meshing [1], [62]-[64]. Finally, it is worth noting that the use of hierarchal finite elements in *p*-type adaption models has also been shown effective for low-frequency finite element electromagnetics [65].

The practical implementation of *p*-type adaption models involves key issues which must be addressed. The choice of basis functions used to form hierarchal elements can play a major role in the effectiveness of a practical *p*-type adaption model. Attention must be paid to the linear independence of the basis functions. If the basis functions that are used to form the approximating functions over an element are not linearly independent (even if they are nearly linearly dependent), then the resulting finite element matrices required to compute the numerical solution for the discretized problem might be ill-conditioned. Depending on whether a direct or an iterative method is used to solve the matrix problem that results, ill-conditioned matrices can lead to inaccurate solutions and slow convergence rates, respectively [66], therefore, research into hierarchal basis functions has constituted an important component of the literature related to *p*-type adaption models over recent years [62], [67], [68]. One successful approach that has been adopted in order to develop hierarchal elements which preserve a manageable degree of linear independence between their basis functions, is to employ orthogonal polynomials in the formulation of basis functions [67], [68].

AFEMs which employ *p*-type adaption models have been especially valuable in the computational analysis and design of three-dimensional systems [3]. In particular, the formation of a well-structured mesh of tetrahedral elements, based on Delaunay or other types of algorithms, is a complicated and relatively expensive computational task [69]-[72]. Therefore, *p*-type adaptive refinement for hierarchal tetrahedra is often considered more favorable than *h*-type adaption, for three-dimensional problems [3]. However, a sufficiently *h*-refined mesh is often an important prerequisite for *p*-type adaption models to be effective [1], [73].

C. The hp-type Adaption Model

For *hp*-type adaption models, finite element discretization refinement is achieved by adapting both the sizes (*h*) and the orders (*p*) of elements in a mesh. In general, combined *hp*-adaptive approaches use both *h*-type and *p*-type refinements in order to exploit the valuable advantages of each of these adaption models. Theoretical studies and numerical results indicate that the ability to independently vary the two basic discretization parameters, *h* and *p*, should provide adaptive methods which employ combined *hp*-type adaption models with the possibility and potential of realizing superior rates of solution error convergence compared against those methods that utilize only pure *h*-type or *p*-type adaption models [61], [74]-[77]. The putative enhanced performance of combined *hp*-type adaption-based approaches arises from the fact that the solution accuracies may be more efficiently improved by reducing the element sizes in certain regions of the problem domain, whereas, increasing the order of the approximating functions over other regions of the solution realm may yield the most significant impact on the overall solution accuracy. Therefore, a hybrid adaption model capable of both types of refinements should, at least in theory, yield optimal rates of solution error convergence.

The implementation and control of a fully hybrid *hp*-type system can be inherently more complex and subtle than that of its simpler *h*-type or *p*-type counterparts. In addition to the key issues that are relevant to the design and implementation of individual *h*-type and *p*-type adaption models, a number of further concerns arise, related to the coupling of the *h*-type and *p*-type refinement procedures. Although many of these issues have been addressed (at least to certain extents) in the literature, the focus has been primarily on structured meshes, for which the relationships between the discretization parameters, associated with consecutive iterations, remain rather well-defined [75], [76], [78]-[80].

One major research problem that has emerged associated with the implementation of true combined *hp*-type adaption models, is the development of systematic approaches for generating discretizations with optimized relative distributions of *h*-type and *p*-type DOF [2],[81]-[84]. For example, in fully-coupled *hp*-type adaption models, for which *h* and *p* can be adapted simultaneously in any given iteration of the adaptive process, one of the primary difficulties is to determine which parts of the discretization to enhance with *h*-refinement, and which to update through *p*-refinement, such that the greatest improvement in solution accuracy is gained for a prescribed increase in the previous number of DOF used to compute the approximate solution [2], [81]. Similarly, in decoupled *hp*-adaption models, for which only one or the other of the two basic refinements are exploited during each iteration of the adaptive feedback loop, the dilemma of which discretization parameter, *h* or *p*, to adapt at a given iteration to achieve the maximal decrease in solution error per unit new DOF exists [74], [84]-[86]. Resolving these difficulties is equivalent to determining the optimal trajectory through an abstract space of admissible *hp*-distributions, beginning from a fixed initial discretization, and given a final required solution accuracy; where, the space of permissible trajectories is dependent upon the constraints of the specific adaptive method under consideration, i.e., the specific combination of the chosen adaption model, and associated feedback control system. In this case, the optimal trajectory is interpreted as that which yields the lowest cumulative computational cost [87].

Although some theoretical approaches have been suggested for determining optimal *hp*-trajectories, the resulting discrete optimization problems are not readily solvable in a rigorous, analytical manner, if at all, for systems of realistic complexity [81], [83], [88]; therefore, numerical experiments have also been relied upon to gain insight into these problems [2], [82], [89]. Based in part on such theoretical and numerical investigations, practicable approaches have been developed which can, although not necessarily optimally, evolve distributions of the discretization parameters in such a way that hybrid *hp*-based adaptive methods outperform pure *h*-type and *p*-type systems. One intuitive approach which has been developed recently, and used successfully, for electromagnetic adaptive FEA (AFEA) is reported in [5]; it is based on using parallel processing to evaluate two or more competing discretization strategies at each *hp*-refinement step to help guide the overall evolution of the adaption. However, it should be stated that numerical studies have shown, that, despite the advantages of *hp*-type adaption models, sometimes the simpler non-hybrid models can yield superior results under certain conditions.

D. The r-type Adaption Model

In *r*-type adaption models, the finite element discretization is refined by adapting the positions (*r*) of element vertices in the mesh, in order to improve the accuracy of the computed solution [90]. *r*-type adaption models can evolve efficient finite element discretizations through repositioning element vertices such that there is a sharper focus of DOF in regions where the solution variation is most significant. The *r*-type adaption model is most frequently employed when maximal solution accuracy is required from discretizations of a given fixed number of DOF [2], [75], [78], [91], [92]. As a result, *r*-adaption has been primarily investigated in the context of adaptive systems which are based on evolving optimal finite element discretizations [2], [4], [90], [93]-[110].

III. ADVANCED STRATEGY AFEMs

Today, the research and development of *optimized* AFEMs which are effective, reliable and versatile enough for general application in electromagnetics analysis and design, represent a critical component of the state-of-the-art in FEA research. The purpose of this section is to review recent advances in this area, which have been achieved through the development of practical adaptive refinement procedures, that are capable of effectively reproducing the main modeling characteristics and performance attributes of optimal discretizations, without the expense of solving the optimal discretization problem.

One relatively recent and promising research direction for *hp*-adaptive FEA has targeted the development of enhanced adaptive refinement strategies which exploit the strengths of practical workstation-level parallel and distributed processing environments [5]. The remainder of this survey presentation will focus on highlighting a representative sampling of these strategies and techniques.

To date, conclusive contributions towards an efficient and effective coupling of parallel processing and *hp*-adaption for electromagnetics remain rare. Unlike standard approaches, the most interesting avenues of this research are focused on developing and tuning *hp*-adaptive subsystems and strategies to exploit the inherent strengths of a parallel environment, as opposed to constructing parallelizations of existing algorithms which were not developed with parallel processing in mind. The thrust of these new research directions is founded on the strong belief that only fundamental approaches, which take the nature of parallel and distributed computing environments into consideration from the ground up, can yield *hp*-adaptive systems that will be capable of exploiting the full potential of such environments.

The *hp*-adaptive process for FEA may be interpreted as a simple system, consisting of complex subsystems. While the individual subsystems can be sophisticated, and rather large, they are also well-understood, and essentially non-adaptive. On the other hand, while the main feedback system (outlined in section I) consists of only a handful of steps, they can be quite sensitive and fairly subtle in their interaction.

According to current theory and practice, it is important to keep the cumulative computational cost (measured as elapsed runtime) of the adaptive control and discretization refinement processes small relative to that of the finite element solver, to gain the full potential of an *hp*-adaptive approach [1], [111]. Based on typical *h*-type and *p*-type adaptive implementations, elapsed runtime cost ratios of 10:1 (or better) are commonly realized. This distribution of computational effort indicates there is an empirical balance between adaptive refinement and solution calculations that seems to be effective for most practical adaptive systems running on sequential machines. However, for parallel and distributed implementations, this balance can be significantly less efficient, since the full-scale speedup potential of local error estimator evaluation and *hp* discretization refinement can be far superior to that of finite element solver execution [5], [66], [112], [113]. Therefore, relative to serial implementations, the use of more complex and computationally intensive adaption strategies is indicated. Furthermore, a parallel environment permits the comparative assessment of competing discretization schemes, at each *hp*-refinement step, to help guide the evolution of the adaption.

A sampling of investigative studies and illustrative results taken from the research project described in [5] is presented in the following section.

IV. Investigative studies and illustrative results

At each refinement step, the main concerns of *hp*-adaption are: *where* should extra DOF be inserted; *what type* of DOF should be used; and *how many* DOF should be added. The following investigations have been designed to explore the potential advantages of addressing these points in a parallel processing environment. The first study examines the value of using pairs of complementary error estimators to determine where to insert new DOF into a discretization. In this case, an unbiased average of two complementary errors was used to assess each element, at each adaptive step, within a practical "*h*- followed by *p*-refinement" *hp*-adaption formulation. The second case study was designed to investigate the potential benefits of constructing (and solving) both *h*- and *p*-refined discretizations, at each adaptive step, to determine what type of DOF should be added into the discretization at each step. In this case, both a single-step depth search (two refinement scenarios: pure *h* and pure *p*) and a double-step depth search (four scenarios: pure *h* followed by pure *h*; pure *h* followed by pure *p*; pure *p* followed by pure *p*; and pure *p* followed by pure *h*) were examined. The third study also addresses the "what type of DOF" question. In this case, the potential value of adding a mixture of *h*- and *p*-type DOF at each adaptive step was examined: 50% of the prescribed DOF update was inserted as *p*-type, guided by a *p*-type error estimator; and the remaining 50% was added as *h*-type, according to an *h*-type error estimator. The fourth study examines the advantages of constructing and solving a range of discretizations that differ only in the number of new DOF added, at each adaptive step, in order to determine how many DOF should be added to the discretization at each step. In this case, four different %DOF refinement levels, ranging from a 25% increase in DOF, to refining each element in the discretization, were investigated and compared within the practical *h*-followed by *p*-adaption system. The final study also addresses the "how many DOF" question. In this case, two straightforward schemes that are based on the distribution and relative strengths of the errors throughout the discretization were utilized to determine how many DOF should be added at each adaptive step. The first scheme simply directs that all elements with above average error levels should be refined; the second scheme scans the error level list, sorted by descending magnitude, for the first statistically significant abrupt jump in error level, and selects all elements with errors higher than that level for refinement.

These five investigative studies were carried out using two basic FEA test systems: the standard "L" benchmark setup, prescribed in [4]; and a high-frequency variation defined on the same 2D geometry and initial mesh. These investigations form a subset of the larger body of work described in [5], and the illustrative results reported below have been reproduced from these earlier studies. Briefly, Fig.2 represents ¼ of the

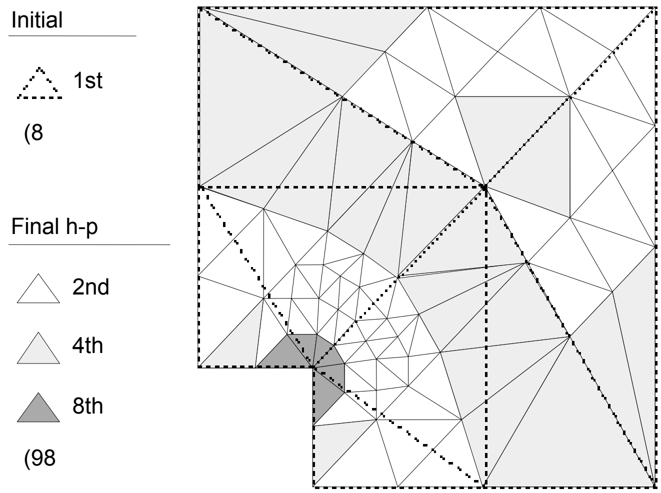

Fig. 2. Two-dimensional geometry and the initial mesh for the two investigative study test problems; and the final *hp* discretization for the first investigative study, based on the averaged error estimators.

cross-section of an infinitely long, translationally symmetric, air-filled, coaxial line – for test system 1. The objective is to solve for the electrostatic scalar potential, in the air between the conductors, when a unit voltage difference is maintained across the conductors. For test problem 2, Fig. 2 represents a sharply truncated 90^{o} corner in a planar microstrip circuit. In this case, the objective is to resolve the variation of E_N in the substrate between the strip and the ground plane, given that: one port is prescribed to unit excitation; the second port is a short-circuit; the exterior region boundaries are modeled as perfect magnetic walls; and, the system is set to operate at a normalized frequency equal to 1/3 of the width of the ports. The results for the first three investigative studies are based on test problem 1 computations, using an initial discretization of eight first-order triangles. The results for studies four and five are based on test problem 2, and an initial discretization of eight second-order triangles. For each case, the adaption performance results are represented in terms of normalized FEA functional error, versus cumulative computational cost. Further, each result is derived from and representative of test data spanning a minimum 1000-fold reduction in functional error. Finally, simple 50% DOF updates (per adaptive step) were employed for the first three studies; and *h*- followed by *p*-adaption was applied in studies one, four and five.

The results of the first investigative study are reported in Fig. 3. The standard $\boldsymbol{D_N}$ field discontinuity and the recently developed functional-gradient error estimators [4] were used as a complementary pair with the *h*-adaption; while the PDE residual and hierarchal coefficient estimators [1] were used for the *p*-adaption. In addition to the performance obtained by applying the averages of the two sets of estimators, three related curves are plotted to gauge adaption efficiency: the two conventional adaption results, corresponding to the best (most efficient) and worst of the four possible pairings of the two *h*-type and the two *p*-type estimators, together with the optimal uniform refinement *h*- followed by *p*-adaption result, are provided for direct performance comparisons.

The results of the second study are reported in Fig. 4. To fix the focus of the investigation, only the PDE residual error estimator was used for both *h*- and *p*-adaption. In this study, optimal *h*- followed by *p*-adaption results for both 50% DOF (Ideal h-p) and uniform (Uniform h-p) updates are also plotted for convenience of comparison. The single-step depth search

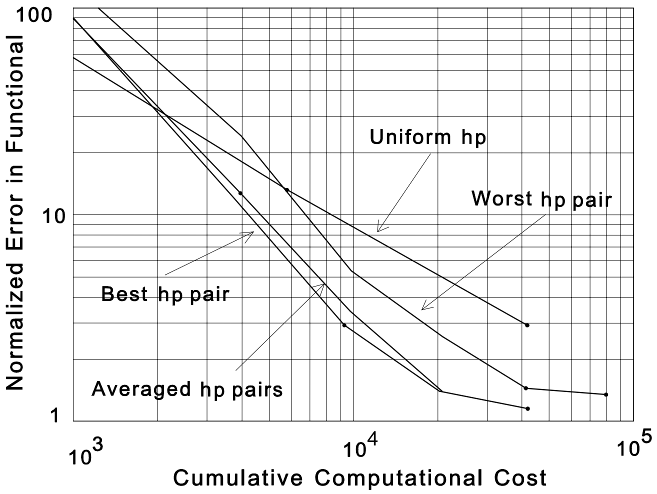

Fig. 3. Comparative *hp*-adaption performance results for the first investigative study, addressing the potential benefits of applying the average of complementary pairs of standard AFEA error estimators.

performance curve is denoted "Level 1 h-p"; the double-step depth search results plot is labeled "Level 2 h-p".

The results of investigative study three are reported in Fig. 5. As in the previous study, the PDE residual error estimator was used in each case. In this study, performance results for pure *h*-adaption, pure *p*-adaption, and optimal uniform refinement *h*- followed by *p*-adaption are plotted for comparison.

The results of the fourth study are reported in Fig. 6. In this case, the PDE residual estimator was used for *h*-adaption, and the hierarchal coefficient estimator for *p*-adaption. The four DOF refinement levels examined were: 25% DOF; 50% DOF; 100% DOF; and uniform mesh refinement. These four refinement updates were constructed, solved and compared at each adaptive step, to determine how many DOF to add to the discretization at each step. For comparison, the performance of this search-based result (labeled "Mixed") is also plotted, together with the four constant-level %DOF update curves.

The results of the fifth investigative study are reported in Fig. 7. As done with the fourth study, only the PDE residual error estimator was used for the *h*-adaption and the hierarchal

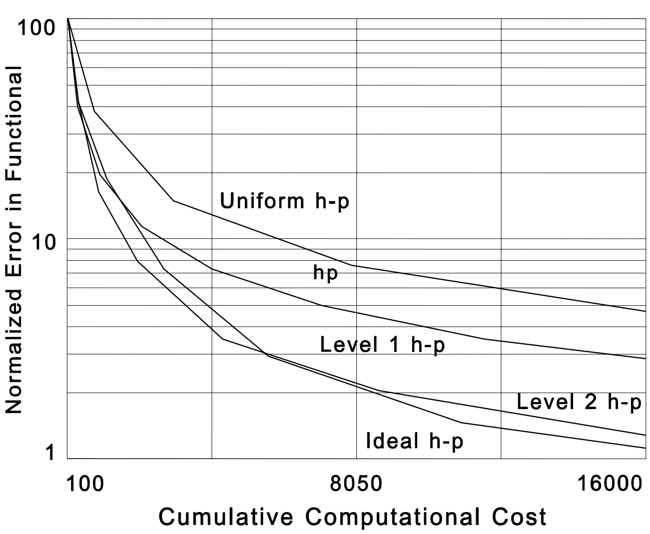

Fig. 4. Comparative *hp*-adaption performance results for the second investigative study, addressing the potential benefits of monitoring both *h*-type and *p*-type FEA discretization updates simultaneously.

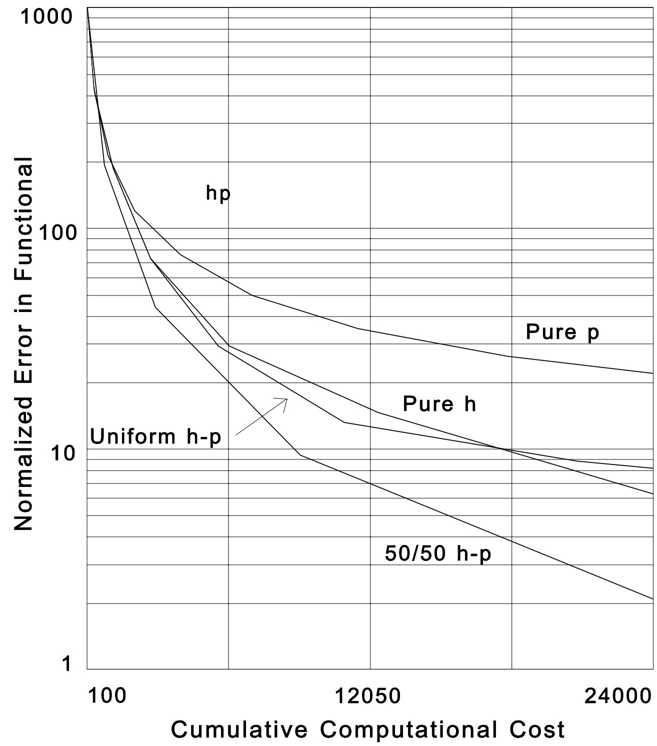

Fig. 5. Comparative *h*-, *p*- and *hp*-adaption performance results for the third investigative study, addressing the potential advantages of applying both *h*- and *p*-type refinements in the same adaptive step.

coefficient error estimator was used for the *p*-adaption. The "above average" DOF result curve is labeled "Average"; the "abrupt jump" DOF curve is labeled "Variable". The constant level %DOF update result curves (given earlier in Fig. 6) are also plotted in Fig. 7, to facilitate a direct comparison.

V. Analysis of Test Results

All five of the sample investigative studies outlined above were designed to explore a selection of possibilities, and their associated potential advantages, of incorporating parallel and

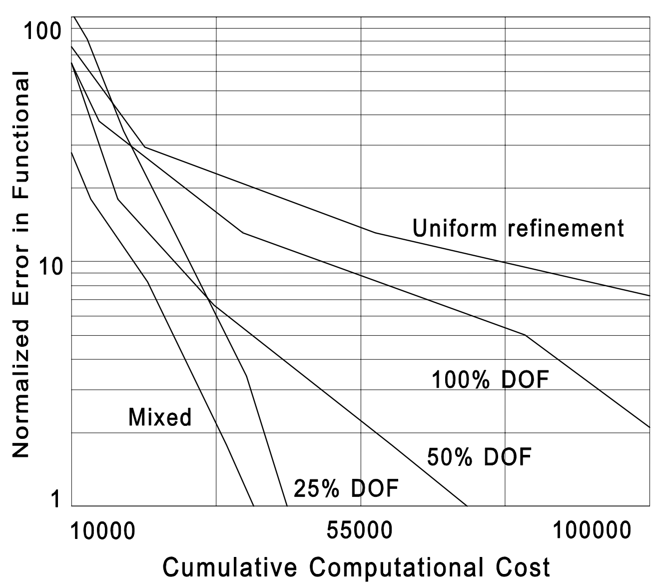

Fig. 6. Comparative *hp*-adaption performance results for the fourth investigative study, addressing the potential benefits of monitoring multiple %DOF mesh updates simultaneously in each adaptive step.

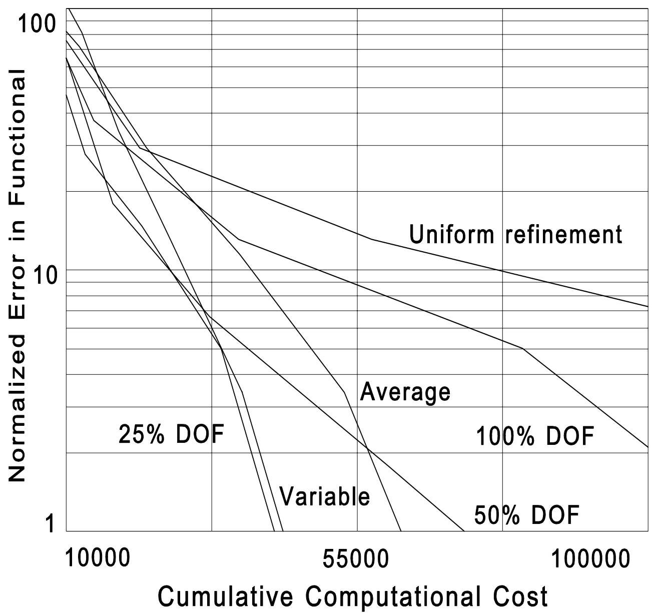

Fig. 7. Comparative *hp*-adaption performance results for the fifth investigative study, addressing the potential benefits of using local error distributions to determine DOF updates at each adaptive step.

distributed processing concepts "from the ground up" within *hp*-adaptive FEA, for applications in electromagnetics. The results of the computational experiments provide supportive evidence for a range of basic hypotheses on the considerable benefits associated with a parallel processing environment. The most significant of these findings are summarized below.

1. Facts: The placement of added DOF in AFEA can be very important to the adaption performance; and, two (or more) error estimators can be evaluated nearly as inexpensively as one in a parallel or distributed computing environment.

   Hypothesis: Complementary error estimators (pairs/sets) should be able to identify more detailed and decisive local error distributions, with less refinement-model dependent bias and distortion, and therefore yield increased adaption efficiency and stability at minimal added cost in a parallel or distributed computing environment.

   Support: (study one) Using combined, or averaged, error estimator techniques can yield substantially better performance results than methods that apply the same estimators singly; in this test, the averaged approach was comparable to the most effective individual strategy available.

2. Facts: The types of DOF added to an evolving discretization can be very important to *hp*-adaption performance; two (or more) refinement scenarios can be constructed and evaluated in almost the same elapsed runtime as one, in an efficient parallel or distributed processing environment.

   Hypothesis: Comparative evaluations of potential *h*- and *p*-refinement update scenarios at each step should be able to deliver more appropriately focused discretizations, and thereby lead to better optimized *hp*-adaption trajectories.

   Support: (study two) Locally optimized refinement type selections can yield substantial improvements in adaption efficiency over standard *h* followed by *p* strategies; and, the benefit seems to increase with the depth of the search, and number of refinement update scenarios considered.

3. Facts: The amount of DOF added to an evolving AFEA discretization can be important to overall adaption performance; two (or more) mesh refinements can be constructed and evaluated nearly as inexpensively as one in a parallel or distributed computing environment.

   Hypothesis: Comparative evaluations of various sizes of DOF refinements should be able to identify more effective and efficient discretization updates, and thereby facilitate increasingly optimized *hp*-adaption trajectories.

   Support: (study four) Stepwise-optimized AFEA %DOF updates can produce remarkable improvements in overall adaption efficiency, compared to conventional, fixed-size %DOF updates; in this test, the optimized strategy strongly outperformed all of the standard updates which were used to determine this final mixed %DOF result.

4. Facts: The performance of *hp*-adaptive formulations for sequential environments can be improved when the types and amount of DOF can be "tuned" to each adaptive step.

   Hypothesis: The parallel processing strategies discussed above should be able to be adapted to sequential computing AFEA systems, and thereby yield comparable related benefits within serial adaption environments.

   Support: (study three) The enhanced performance of the 50/50 mixture of *h*- and *p*-refinements (per adaptive step) indicates that non-trivial *hp*-adaption efficiency improvements can be realized at relatively small additional cost in a serial environment. Further, (study five) the remarkable efficiency achieved by the "abrupt jump" (Variable) DOF updates shows that excellent serial adaption performance improvements are possible at a very reasonable cost.

VI. Conclusions

The primary benefit of AFEMs is that they provide for the efficient, accurate and reliable computational analysis of very large continuum problems, for only a relatively small fraction of the cost associated with non-adaptive FEA. The objective of this contribution has been twofold: first, to present a user-friendly introduction to modern AFEA for electromagnetics applications to the non-specialist, through an overview of the main structures, essential components and feedback control systems of AFEMs. The second goal has been to provide a portal onto certain interesting aspects of the *state-of-the-art* research currently under investigation, on the development of practicable AFEMs for parallel and distributed processing workstation environments. Regarding the first objective, the structure and underlying importance of the adaptive feedback loop, and its essential control systems, have been introduced and explained. Further, the full range of adaptive refinement models has been defined, compared and critically evaluated. Finally, the primary use, and critical importance, of efficient and effective local error estimation procedures, appropriate for the individual refinement models, have been summarized.

In regards to the second objective of this contribution, it is noteworthy that the reported investigative studies have served to demonstrate that substantial advantages are available with the development and application of *hp*-adaptive FEMs, for electromagnetic analysis and design, in practical parallel and distributed computing environments. Furthermore, this work has served to illustrate some of the possibilities and potential benefits of designing parallel processing AFEA strategies and modules "from the ground up", as opposed to implementing direct parallelizations of existing sequential algorithms, that were never intended for parallel and distributed computing in the first place. Finally, new *hp*-adaption error estimation and local refinement strategies for practicable parallel processing workstation environments have been developed and tested.

In closing, it should be remarked that a number of issues associated with adaption have been purposefully excluded in this contribution, to balance the scope, specificity and clarity of the presentation. These subjects should not necessarily be considered as secondary, or unimportant: the authors believe that many of these interesting, and potentially critical, topics justify the focus of future research endeavors, e.g.

- Different parallel and distributed computing environments possess different intrinsic strengths and weaknesses, e.g., numbers of available processors, processing power, data communications costs and overhead. *Are some adaptive strategies, formulations and constructs more appropriate for (or poorly suited to!) certain parallel and distributed processing environments, and if so, which?*

- There have been increasingly clear indications that a few of the other standard AFEA modules could also benefit from a "ground up" redesign for parallel and distributed computing, e.g., *hp* mesh generation and refinement [73]. *What are the potential gains and the associated costs of implementing these algorithm and software updates?*

- The parallel processing AFEA strategies and techniques studied in this introductory overview have been treated as essentially independent components, e.g., determining the best type of adaption model was considered separate from the determination of the most effective size (%DOF) for the refinement update. *Can these associated components be integrated and applied effectively in combination, to yield even stronger performance characteristics; and if so, how — and what are the attendant implications?*

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