Technical Article

Tree-cotree condensation properties

***Abstract*—Through** topological considerations, some interesting properties of the tree-cotree-gauged discretization of magnetostatic problems are presented. On this basis, several tree-cotree gauging techniques can be interpreted in terms of projections.

I. Introduction

Spanning trees have been used in circuit simulation for a long time. As early as mid-twentieth century, spanning tree techniques were used for generating independent meshes for mesh-current analysis, see e.g. [1].

In electromagnetics, their use is much more recent. In 1988, a tree-cotree technique was proposed by Albanese & Rubinacci [2] for regularizing magnetostatic vector potential discrete formulations.

A way of imposing the Coulomb gauge for the same problem was proposed in [3]. For the calculation of the solenoidal eigenmodes of cavities, tree-cotree techniques were used in [3], [4]. A technique based on tree-cotree techniques for reducing the problem size in mixed formulations is presented in [5].

It was early noticed that, in the case of tree-cotree gauging for the magnetic vector potential, the condition number of the system matrix considerably increased. Quite a number of papers deal with the problem of finding a quasi–optimal tree [6], or try to explain the phenomenon [7].

The goal of the present paper is to highlight some interesting properties of the tree-cotree technique applied to magnetostatic problems. Their algorithmic consequences are yet to be examined.

II. Problem Formulation

Let us consider a domain $\mathcal{D}$ in which the magnetostatic field equations hold:

$$\begin{aligned} \operatorname{curl} \vec{H} &= \vec{J}; \\ \operatorname{div} \vec{B} &= 0; \\ \vec{B} &= \mu \vec{H}. \end{aligned} \tag{1}$$

By defining a vector potential through $\vec{B} = \operatorname{curl} \vec{A}$, the second order equation for $\vec{A}$ is

$$\operatorname{curl}\ \nu \operatorname{curl} \vec{A} = \vec{J}, \tag{2}$$

where $\nu = 1/\mu$. For simplicity, we will consider zero, Dirichlet-type boundary conditions (tangential component of $\vec{A}$ zero on the boundary).

We will assume a discretization of the equation (2) with the Finite Integration Technique (FIT) [8], or with edge finite elements [9]. For both methods, it can be shown that the discretized equation can be brought to the form:

$$\mathbf{C}^T \mathbf{M}_\nu \mathbf{C} \widehat{\mathbf{a}} = \widehat{\mathbf{j}}, \tag{3}$$

where the superscript ^{T} denotes transposition. Here, $\mathbf{C}$ is the discrete curl operator, the edges-to-facets incidence matrix: considering inner-oriented mesh edges and facets, an entry $\mathbf{C}_{fe}$ is equal to 1 (respectively -1) if edge e is a part of the boundary of the facet f and their orientations match (respectively don't match), and is zero otherwise. Note that $\widetilde{\mathbf{C}} = \mathbf{C}^T$ plays the role of the discrete curl operator on a second mesh, the *dual* mesh, defined such that each of its edges/facets are in a one-to-one correspondence with the facets/edges of the primary mesh (except possibly on the boundary).

The matrix of relation (3) is singular. Most of the regularization techniques are based on a gauging for the vector potential, $\operatorname{div} \vec{A} = f$, used explicitly or implicitly in the discrete formulation.

In our case, the discrete divergence operator (defined on the dual mesh) will be denoted by $\mathbf{G}^T$. Its structure is well-known: it is nothing else than the *reduced edge-to-node incidence matrix*, often used in circuit theory. An entry $\mathbf{G}_{en}$ of the matrix $\mathbf{G}$ is 1 (respectively -1) if the edge e contains node n as initial (respectively final) node, and 0 otherwise. One of the meshnodes is considered "grounded" and the corresponding column is eliminated from $\mathbf{G}$.

Therefore, a discretized gauge relation for $\widehat{\mathbf{a}}$ can be written as:

$$\mathbf{G}^T \widehat{\mathbf{a}} = \mathbf{f}, \tag{4}$$

and the discrete "Coulomb gauge" has the form

$$\mathbf{G}^T \widehat{\mathbf{a}} = \mathbf{0}. \tag{5}$$

To make the counting easier, we will consider only simply connected domains. In the primary mesh, let N be the number of meshnodes minus one, E the number of edges, F the number of facets. Thus, the matrix $\mathbf{C}$ has dimension $F \times E$ and the matrix $\mathbf{G}$ has dimension $E \times N$.

The rank of the $E \times E$ matrix from (3) is $E - N$. The idea of the tree–cotree gauging is to select an independent set of rows, namely those corresponding to the $E - N$ cotree edges.

We will discuss the various techniques to do that in Section IV.

III. Some properties

A. The independent cutset matrix

In order to clarify some properties of the tree-cotree gauging technique, it is useful to write the matrix $\mathbf{G}$ used in (4) under a special form.

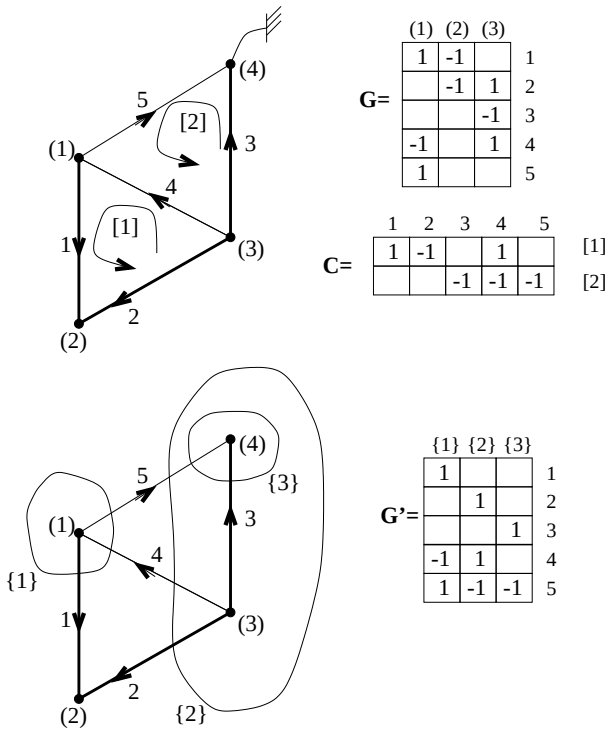

Fig. 1. Matrices $\mathbf{G}$, $\mathbf{G}'$ and $\mathbf{C}$ for a simple circuit.

Let us assume that a tree has been chosen. By numbering the mesh edges such that the cotree edges come first, the gradient matrix $\mathbf{G}$ is partitioned as:

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_c \\ \mathbf{G}_t \end{bmatrix}. \tag{6}$$

The submatrix $\mathbf{G}_t$ is nonsingular, therefore, by multiplying $\mathbf{G}$ to the right by $\mathbf{G}_t^{-1}$ a new matrix is obtained:

$$\mathbf{G}' = \mathbf{G}\,\mathbf{G}_t^{-1} = \begin{bmatrix} \mathbf{F} \\ \mathbf{I} \end{bmatrix}, \tag{7}$$

where the matrix $\mathbf{F}$ is called the essential incidence matrix, of dimension $(E-N)\times N$.

Note that the gauge relation (4) for $\widehat{\mathbf{a}}$ can be written in terms of $\mathbf{G}'$ as:

$$\mathbf{G}'^T \widehat{\mathbf{a}} = \mathbf{G}_t^{-T}\mathbf{f}.$$

The new $\mathbf{G}'$ is an incidence matrix of edges to the independent cutsets[1]. Its generation is equivalent to the following operations on the mesh graph: select a tree; renumber edges such that the cotree edges come first; select the independent cutsets such that each cutset contains only one tree edge and its orientation matches the orientation of the associated tree edge.

The construction of the matrices $\mathbf{G}$, $\mathbf{G}'$ and $\mathbf{C}$ is exemplified in figure 1 for a simple mesh.

[1]We remind that a cutset is a set of edges of a graph chosen such that the elimination of these edges transforms the initially connected graph into a non-connected one. There are N independent cutsets in a graph.

B. The structure of the discrete curl matrix $\mathbf{C}$

The matrix $\mathbf{C}$ is also partitioned according to the tree-cotree decomposition:

$$\mathbf{C} = \left[\mathbf{C}_c|\mathbf{C}_t\right].$$

The matrices $\mathbf{C}$ and $\mathbf{G}$ satisfy the following fundamental property (see e.g. [8]):

$$\mathbf{C}\,\mathbf{G} = \mathbf{0},$$

the discrete equivalent of curl grad $= 0$. Therefore, since $\mathbf{C}\mathbf{G}\mathbf{G}_t^{-1} = \mathbf{C}\mathbf{G}' = \mathbf{0}$, the following relation holds:

$$\left[\mathbf{C}_c|\mathbf{C}_t\right]\begin{bmatrix} \mathbf{F} \\ \mathbf{I} \end{bmatrix} = \mathbf{C}_c\mathbf{F} + \mathbf{C}_t = \mathbf{0},$$

or

$$\mathbf{C}_t = -\mathbf{C}_c\mathbf{F}. \tag{8}$$

In consequence, the matrix $\mathbf{C}$ has the following special structure:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_c| - \mathbf{C}_c\mathbf{F} \end{bmatrix} = \mathbf{C}_c\begin{bmatrix} \mathbf{I}| - \mathbf{F} \end{bmatrix} = \mathbf{C}_c\mathbf{L}, \tag{9}$$

where

$$\mathbf{L} = \begin{bmatrix} \mathbf{I}| - \mathbf{F} \end{bmatrix} \tag{10}$$

is a matrix of dimension $(E-N)\times E$.

C. The initial problem, rewritten

With the new expression (9) for the curl operator, the matrix relation (3) becomes:

$$\mathbf{L}^T\mathbf{C}_c^T\,\mathbf{M}_\nu\,\mathbf{C}_c\mathbf{L} = \widehat{\widehat{\mathbf{j}}}.$$

By introducing the notation:

$$\mathbf{M}_c = \mathbf{C}_c^T\mathbf{M}_\nu\mathbf{C}_c, \tag{11}$$

the matrix equation becomes

$$\mathbf{L}^T\mathbf{M}_c\mathbf{L}\,\widehat{\mathbf{a}} = \widehat{\widehat{\mathbf{j}}}, \tag{12}$$

with the following block structure:

$$\begin{bmatrix} \mathbf{M}_c & -\mathbf{M}_c\mathbf{F} \\ -\mathbf{F}^T\mathbf{M}_c & \mathbf{F}^T\mathbf{M}_c\mathbf{F} \end{bmatrix}\begin{bmatrix} \widehat{\mathbf{a}}_c \\ \widehat{\mathbf{a}}_t \end{bmatrix} = \begin{bmatrix} \widehat{\widehat{\mathbf{j}}}_c \\ \widehat{\widehat{\mathbf{j}}}_t \end{bmatrix}. \tag{13}$$

Note that the matrix $\mathbf{M}_c$ is full rank, symmetric and positive definite.

IV. Tree-cotree gauging

In magnetostatics, the non-uniqueness of the vector potential is in principle irrelevant: all that matters is that its curl should give the correct magnetic flux density. However, the non-uniqueness is reflected in the singularity of the equation system, an inconvenience for many linear system solvers. That is why gauging, in particular the tree-cotree gauging, has been used to transform the system into a regular one.

Some of the known tree-cotree gauging techniques for (13) are shortly described below.

The technique proposed by **Albanese & Rubinacci** [2] consists in setting to zero the tree-components $\widehat{\mathbf{a}}_t$ of $\widehat{\mathbf{a}}$: $\widehat{\mathbf{a}}_t = \mathbf{0}$. This is equivalent to eliminating those rows and columns in (13) which correspond to the tree edges. Therefore, the reduced-size problem which is solved is:

$$\mathbf{M}_c \widehat{\mathbf{a}}_c = \widehat{\widehat{\mathbf{j}}}_c.$$

Note that this approach does not impose a specified divergence for the vector potential (in particular, it does not impose the Coulomb gauge). It just selects one vector of the right space, whose tree-components are all zero.

In reference [3], **Manges & al.** use the first block line of the matrix in (13):

$$\mathbf{H} = \left[\begin{array}{c|c} \mathbf{M}_c & -\mathbf{M}_c\mathbf{F} \end{array} \right] = \mathbf{M}_c\mathbf{L}.$$

Through the change of variable $\widehat{\mathbf{a}} = \mathbf{H}^T\mathbf{y}$, the reduced-size system to solve becomes $\mathbf{H}\mathbf{H}^T\mathbf{y} = \widehat{\widehat{\mathbf{j}}}_c$, i.e.

$$\mathbf{M}_c\mathbf{L}\mathbf{L}^T\mathbf{M}_c\mathbf{y} = \widehat{\widehat{\mathbf{j}}}_c.$$

In [10], **Tičar & al.** use the following change of variable:

$$\widehat{\mathbf{a}} = \mathbf{C}^T\mathbf{y},$$

so that, after multiplication of (3) to the left by $\mathbf{C}$, the reduced system (which is still irregular, but has smaller dimension) becomes:

$$\mathbf{C}\ \mathbf{C}^T\mathbf{M}_\nu\mathbf{C}\ \mathbf{C}^T\mathbf{y} = \mathbf{C}\widehat{\widehat{\mathbf{j}}},$$

or, after using (9) and (12),

$$(\mathbf{C}_c\mathbf{L})\ \mathbf{L}^T\mathbf{M}_c\mathbf{L}\ (\mathbf{C}_c\mathbf{L})^T\ \mathbf{y} = \mathbf{C}\widehat{\widehat{\mathbf{j}}}.$$

Finally, a non-symmetric version can be obtained, following [4], based on the
Coulomb gauge" (5):

$$\mathbf{G}^T\widehat{\mathbf{a}} = 0 \Rightarrow \mathbf{G}'^T\widehat{\mathbf{a}} = 0 \Rightarrow \widehat{\mathbf{a}}_t = -\mathbf{F}^T\widehat{\mathbf{a}}_c, \qquad (14)$$

where the form (7) of the matrix $\mathbf{G}'$ was used. By using this relation and (8), the matrix equation (3) becomes:

$$\mathbf{C}^T\mathbf{M}_\nu(\mathbf{C}_c\widehat{\mathbf{a}}_c + \mathbf{C}_t\widehat{\mathbf{a}}_t) = \widehat{\widehat{\mathbf{j}}},$$

$$\mathbf{C}^T\mathbf{M}_\nu(\mathbf{C}_c\widehat{\mathbf{a}}_c + \mathbf{C}_c\mathbf{F}\ \mathbf{F}^T\widehat{\mathbf{a}}_t) = \widehat{\widehat{\mathbf{j}}},$$

and by preserving only the cotree block of $\mathbf{C}^T$

$$\mathbf{C}_c^T\mathbf{M}_\nu\mathbf{C}_c(\mathbf{I} + \mathbf{F}\ \mathbf{F}^T)\widehat{\mathbf{a}}_c = \widehat{\widehat{\mathbf{j}}}_c.$$

Observe that $\mathbf{I} + \mathbf{F}\,\mathbf{F}^T = \mathbf{L}\,\mathbf{L}^T$. Therefore, the reduced system has the form:

$$\mathbf{M}_c\mathbf{L}\,\mathbf{L}^T\widehat{\mathbf{a}}_c = \widehat{\widehat{\mathbf{j}}}_c. \qquad (15)$$

Note that, since the current density is divergence-free, a relation similar to (14) holds for the tree components of $\widehat{\widehat{\mathbf{j}}}$: $\widehat{\widehat{\mathbf{j}}}_t = -\mathbf{F}^T\widehat{\widehat{\mathbf{j}}}_c$.

The matrix in equation (15) is, unlike the previous ones, not symmetric. A symmetric form can be obtained by noting that (14) implies

$$\widehat{\mathbf{a}} = \left[\begin{array}{c} \widehat{\mathbf{a}}_c \\ \widehat{\mathbf{a}}_t \end{array} \right] = \left[\begin{array}{c} \mathbf{I} \\ -\mathbf{F}^T \end{array} \right] \widehat{\mathbf{a}}_c = \mathbf{L}^T\widehat{\mathbf{a}}_c.$$

Using this relation in (12), and premultiplying with $\mathbf{L}$, the following symmetric reduced-size system is obtained:

$$\mathbf{L}\,\mathbf{L}^T\,\mathbf{M}_c\,\mathbf{L}\,\mathbf{L}^T\,\widehat{\mathbf{a}}_c = \mathbf{L}\widehat{\widehat{\mathbf{j}}}.$$

The matrix of this system is much less sparse than that of the unsymmetric version.

To summarize, the following versions for reducing the size of the problem

$$\mathbf{L}^T\mathbf{M}_c\mathbf{L}\,\widehat{\mathbf{a}} = \widehat{\widehat{\mathbf{j}}}$$

have been used:

A. Albanese & Rubinacci [2]

$$\mathbf{M}_c\widehat{\mathbf{a}}_c = \widehat{\widehat{\mathbf{j}}}_c, \quad \widehat{\mathbf{a}} = \left[\begin{array}{c} \widehat{\mathbf{a}}_c \\ \mathbf{0} \end{array} \right] \qquad (16)$$

B. Manges & al. [3]

$$\mathbf{M}_c\mathbf{L}\mathbf{L}^T\mathbf{M}_c\ \mathbf{y} = \widehat{\widehat{\mathbf{j}}}_c, \quad \widehat{\mathbf{a}} = \mathbf{L}^T\mathbf{M}_c\ \mathbf{y} \qquad (17)$$

C. Tičar [10]

$$\mathbf{C}_c\ \mathbf{L}\mathbf{L}^T\mathbf{M}_c\mathbf{L}\mathbf{L}^T\ \mathbf{C}_c^T\mathbf{y} = \mathbf{C}\widehat{\widehat{\mathbf{j}}}, \quad \widehat{\mathbf{a}} = \mathbf{C}^T\mathbf{y} \qquad (18)$$

D. Munteanu (present paper), unsymmetric version

$$\mathbf{M}_c\mathbf{L}\mathbf{L}^T\widehat{\mathbf{a}}_c = \widehat{\widehat{\mathbf{j}}}_c, \quad \widehat{\mathbf{a}} = \mathbf{L}^T\widehat{\mathbf{a}}_c \qquad (19)$$

E. Munteanu (present paper), symmetric version

$$\mathbf{L}\mathbf{L}^T\mathbf{M}_c\mathbf{L}\mathbf{L}^T\widehat{\mathbf{a}}_c = \mathbf{L}\widehat{\widehat{\mathbf{j}}}, \quad \widehat{\mathbf{a}} = \mathbf{L}^T\widehat{\mathbf{a}}_c \qquad (20)$$

In the above relations, $\mathbf{L}$ is the topological matrix given by (10).

The above relations are a good starting point for analyzing the effective condition number $\kappa = \lambda_{\max}/\lambda_{\min}$ of the various tree-cotree formulations ($\lambda_{\min}$ is the minimum nonzero eigenvalue). Without dealing with this issue here, we will just mention a few points.

For the technique **A**, it has been noticed by many authors that the condition number is larger than the one of the initial matrix. An explanation of the phenomenon is given in [11]. It is expected that the condition number will increase for all the formulations except **D**, where, due to the fact that the nonzero eigenvalues of $\mathbf{MN}$ are equal to the nonzero eigenvalues of $\mathbf{NM}$, we have $\kappa(\mathbf{L}^T\mathbf{M}_c\mathbf{L}) = \kappa(\mathbf{M}_c\mathbf{L}\mathbf{L}^T)$.

Our numerical tests indicate that

$$\kappa_{\mathbf{D}} < \kappa_{\mathbf{C}} < \kappa_{\mathbf{A}} < \kappa_{\mathbf{E}} < \kappa_{\mathbf{B}}.$$

Also note that only the formulation **A** maintains the sparsity of the initial matrix: all the others produce matrices which are (sometimes much) less sparse than the matrix in (3).

V. Tree-cotree gauging as projections

It is interesting to find an interpretation of the tree-cotree-gauged formulations in terms of projections.

We will start by a short introduction in projections. Let us consider an n-dimensional matrix equation $\mathbf{Ax} = \mathbf{f}$ and two subspaces S and T of $\mathbb{R}^n$. If we want that the solution $\mathbf{x} \in S$ and impose the condition that the residual $\mathbf{Ax} - \mathbf{f}$ be orthogonal to T, what results is a projected equation [12].

Now if $\mathbf{V}$ and $\mathbf{W}$ are bases for S and T respectively, then $\mathbf{x} \in S$ is written as $\mathbf{x} = \mathbf{Vy}$. If T is a subspace of dimension m, imposing the condition $\mathbf{Ax} - \mathbf{f}$ orthogonal to T is equivalent to requiring that the residual be orthogonal to m independent vectors of T, in particular to the vectors of the basis $\mathbf{W}$:

$$\mathbf{W}^T(\mathbf{Ax} - \mathbf{f}) = \mathbf{0}.$$

The combination of these two conditions yields the system

$$\mathbf{W}^T\mathbf{AVy} = \mathbf{W}^T\mathbf{f},$$

called an oblique projection on S, orthogonally to T.

If S and T are the same subspace and we choose $\mathbf{W} = \mathbf{V}$, what results is an orthogonal projection on S:

$$\mathbf{V}^T\mathbf{AVy} = \mathbf{V}^T\mathbf{f}.$$

For our problem, let us consider a chosen tree, and write any vector in $\mathbb{R}^n$ as

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_c \\ \mathbf{x}_t \end{bmatrix},$$

in which the cotree and the tree components of the vector intervene.

Let us denote by S_c the subspace of vectors which have zero tree components, i.e. $\mathbf{x}_t = 0$, and by S_s the subspace of "solenoidal" vectors, i.e. ones that satisfy $\mathbf{G}^T\mathbf{x} = \mathbf{0}$, or $\mathbf{x}_t = -\mathbf{F}^T\mathbf{x}_c$. An orthonormal basis for S_c is

$$\mathbf{B}_c = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix}.$$

A (non-orthogonal) basis for S_s is

$$\mathbf{B}_s = \mathbf{L}^T.$$

It is easy to verify that S_c and S_s are indeed subspaces.

A. Albanese

The matrix of (16) is obtained by performing a projection with

$$\mathbf{V} = \mathbf{W} = \mathbf{B}_c.$$

The Albanese method is therefore an orthogonal projection on the subspace S_c of vectors with zero tree components. It is trivial to show that the solution $\mathbf{x} = \mathbf{Vy}$ is in S_c, and that the residual is orthogonal to S_c. Figure 2 a shows schematically the geometric interpretation of this approach. The system matrix, unknown vector and the right-hand side are denoted generically by $\mathbf{A}$, $\mathbf{x}$ and $\mathbf{f}$, respectively.

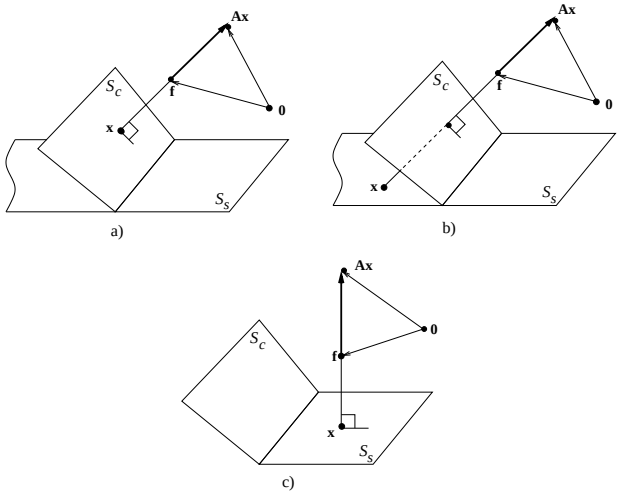

Fig. 2. a) Orthogonal projection on S_c; b) Oblique projection on S_s, orthogonally to S_c; c) Orthogonal projection on S_s. S_c is the subspace of vectors with zero tree components, and S_s is the subspace of "solenoidal" vectors, satisfying $\mathbf{G}^T\mathbf{x} = \mathbf{0}$.

B. Manges

The method given by (17) uses

$$\mathbf{V} = \mathbf{L}^T\mathbf{M}_c = \mathbf{B}_s\mathbf{M}_c, \quad \mathbf{W} = \mathbf{B}_c.$$

Since $\mathbf{B}_s$ is a basis for S_s and since $\mathbf{M}_c$ is nonsingular, $\mathbf{V}$ is also a basis for the subspace S_s of solenoidal vectors. The Manges & al. method is therefore an oblique projection on the subspace S_s of solenoidal vectors, orthogonally to the subspace S_c of vectors with zero tree components (figure 2 b).

C. Tičar

It can be considered that this method uses

$$\mathbf{V} = \mathbf{W} = \mathbf{C}^T.$$

The method is special in that, in 3D, it cannot be interpreted as projection, since $\mathbf{VW}^T$ is singular. However, the obtained solution is solenoidal. Indeed, $\mathbf{x} = \mathbf{C}^T\mathbf{y}$ and $\mathbf{G}^T\mathbf{x} = \mathbf{G}^T\mathbf{C}^T\mathbf{y} = \mathbf{0}$ due to $\mathbf{CG} = \mathbf{0}$. The resulting matrix is not regular.

In the 2D case, $\mathbf{C}^T$ is a basis, therefore the method represents an orthogonal projection on the subspace of solenoidal vectors (figure 2 c).

D. Munteanu nonsymmetric

The nonsymmetric version (19) uses

$$\mathbf{V} = \mathbf{B}_s = \mathbf{L}^T, \mathbf{W} = \mathbf{B}_c.$$

It is therefore obtained by an oblique projection on the subspace of solenoidal vectors, orthogonally to the subspace of vectors with zero tree components (figure 2 b).

Note that although the subspaces are the same as in Manges, another basis is used for the solenoidal vectors subspace. As shown before, this choice does not increase the matrix condition number, unlike the Manges version.

E. Munteanu symmetric

In this case,

$$\mathbf{V} = \mathbf{W} = \mathbf{L}^T.$$

Therefore, the methods is an orthogonal projection on the subspace S_s of solenoidal vectors (figure 2 c). The matrix is less sparse than the initial one.

Note that, while the orthogonal *projector* is unique, therefore basis- (and tree)-independent, the projection applied to the system of equations does depend on the basis.

If an orthonormal basis $\mathbf{L}_1^T$ for S_s is constructed from $\mathbf{L}^T$, and by performing the projection with

$$\mathbf{V} = \mathbf{W} = \mathbf{L}_1^T,$$

then the resulting system is tree-independent. However, $\mathbf{L}_1^T$ will be almost full, so that the initial sparsity is lost in the resulting system of equations.

VI. Conclusions

As the last decade's tendencies in computational electromagnetics have already shown, the separation of topology and metric in the discrete formulations proves to be very fruitful not only in understanding the numerical properties of the algorithms, but also in generating new, more efficient and robust techniques.

On the same line, the present paper has revealed new structural properties of the tree-cotree formulations, and provided an interpretation thereof in terms of projections.

Apparently, each of the tree-cotree techniques has its own characteristics, advantages and disadvantages.

The technique **A** of Albanese & al is very simple, preserves the sparsity of the matrix, but increases the condition number.

The technique **B** (Manges) is also quite simple, but it typically has a very large condition number and matrix density increases.

The formulation **C** (Tičar & al.) produces only a slight increase of the condition number and of the matrix density, however, the matrix is still singular.

The technique **D** is relatively simple to implement and preserves the condition number of the initial matrix, but the resulting reduced matrix is not symmetric.

Finally, the symmetric version **E** also produces an increase of the condition number (much less than in method **B**, but it can be higher than in the Albanese version). The matrix density increases (typically somewhat less than in method **B**). If an orthonormal basis is used in this method, then the condition number is not influenced, but the matrix becomes full.

Note that the techniques **B**, **D**, **E** are based on the imposition, at the discrete level, of the "Coulomb gauge" given by (5).

VII. Acknowledgment

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