Technical article

Some numerical aspects in electrodynamics of magnetic materials

Abstract

This paper deals with total force computation in magnetic materials. The most popular and widely adopted methods, namely the surface integration of the Maxwell's stress tensor, the energy method and the equivalent source methods are reviewed and their main features and numerical drawbacks are highlighted. In particular, it is shown that equivalent source methods can also be written in terms of the so-called *external field* and evidence is given that these reformulated expressions can provide good accuracy and low computational costs. A comparative study among the discussed methods is finally carried out by means of numerical experiments on axisymmetric and 3D test geometries.

1. Introduction

The evaluation of forces acting upon magnetized continuous media is still a rather critical aspect of computational electromagnetism. This is mainly due to the fact that the most traditional approaches for the resultant force sometimes present numerical drawbacks, which make their results often inaccurate and not much reliable.

At present, the computation of the total force in 2D and 3D electromagnetics is generally carried out by means of the surface integration of Maxwell's stress tensor (MST) or using the magnetic energy (or alternatively co-energy) method, also called *virtual work method* (VWM). The most popular commercial electromagnetic codes for computer-aided design of electromagnetic devices are equipped with tools for force calculation based on MST and VWM.

However, in spite of this large popularity, it is difficult to find in electromagnetic literature any well-established and rigorous proof of the validity of both methods in all the typical configurations in which electromagnetic systems have to operate. By way of example, although MST is successfully used in presence of nonlinear ferromagnetic materials, no demonstration of its applicability in such cases can be found in classical textbooks of electrodynamics (see e.g. [1-3]).

Similarly, VWM is usually justified (see e.g. [4-5]) writing down the energy conservation principle over a magnetic system in which mechanical deformations, exchanges of heat and changes of temperature as well as of thermodynamic internal energy are neglected during the virtual displacement. But these simplifications are, in reality, untenable, as shown in [6-7], since most of the physical quantities involved in the energy balance are mixed thermodynamic-magnetic terms. In addition, VWM can be employed only in absence of hysteretic phenomena, as classical thermodynamics cannot account for such a behavior and this results in the impossibility of defining the concept of magnetic energy.

In recent years, the scientific activity on total force computation in 2D and 3D finite element modeling has been concerned with the research of alternative methods for MST and VWM and with their use within the most popular potential-oriented and field-oriented formulations [8-15]. In this context, the so-called *equivalent source methods* (ESM) have been investigated and their numerical reliability has been verified in connection to the adopted formulation to solve the 2D or 3D field problem and to the level of mesh refinement [14-16].

In this paper all the aforementioned methods are reviewed with the aim to highlight their main features and numerical drawbacks. Particular attention is devoted to the equivalent source methods, since it is possible to reformulate their expressions as functions of the *external field*, namely the difference between the total field (**B** or **H**) and the field produced by the equivalent magnetic sources representing the target body. It will be shown in the next sections that the use of the external field instead of the total one allows circumventing most of the numerical problems arising in force calculations.

The paper is organized as follows. Section II is devoted to recalling all the possible expressions for the total force acting upon a magnetized body. In this context, starting from the concept of Coulombian magnetic dipole, a mathematical derivation of ESM is presented and a general procedure for the numerical evaluation of the external field is proposed. Next, in section III, the numerical drawbacks of all approaches are pointed out and a brief recall on the behavior of the magnetic field on the edges of a generic magnetized body is presented. This enables highlighting the singularity of the field on the corners of the body itself and how this singularity can perturb force calculation. Then, in section IV, some numerical experiments referring to an axisymmetric test case and a 3D one are discussed and a comparison among the methods is carried out.

Finally, in section V, some concluding remarks are drawn.

2. The expressions for the total force acting upon a magnetized body

Basically, the computation of the resultant force upon a magnetized body can be carried out by means of three methods: the equivalent source methods (ESM), the surface integration of Maxwell's stress tensor (MST), and the virtual work method (VWM).

A. The equivalent source methods

Magnetization phenomena can be equally explained by viewing magnetized matter either as a collection of *Amperian dipoles* or of *Coulombian magnetic dipoles* [1], [17]. It is quite obvious that, independently of the point of view, the theory to be developed must be able to lead to the same values for the magnetic force.

Therefore, let us consider a continuous distribution of Coulombian magnetic dipoles placed in the magnetic field $\mathbf{H}_{ext}$ produced by prescribed external sources. The Coulombian magnetic dipole can be modeled as an aggregate of two magnetic charges $+q_m$ and $-q_m$, and can be characterized by its *magnetic dipole moment* **m**:

$$\mathbf{m} = \lim_{d\to 0,\, q_m\to\infty} q_m d\, \hat{\mathbf{u}}_r , \tag{1}$$

where $\hat{\mathbf{u}}_r$ is the unit-vector of the straight line connecting $-q_m$ to $+q_m$.

In strict analogy with the electrostatic Coulomb force, the force acting upon the dipole can be evaluated as:

$$\mathbf{F}_m = q_m \mathbf{H}_{ext}\left(\mathbf{r}_{Q^+}\right) - q_m \mathbf{H}_{ext}\left(\mathbf{r}_{Q^-}\right). \tag{2}$$

On the other hand, it results that:

$$\mathbf{H}_{ext}\left(\mathbf{r}_{Q^+}\right) = \mathbf{H}_{ext}\left(\mathbf{r}_{Q^-}\right) + \left(\mathbf{r}_{Q^+} - \mathbf{r}_{Q^-}\right)\cdot \nabla \mathbf{H}_{ext} . \tag{3}$$

Replacing the expression of $\mathbf{H}_{ext}$ (r_Q^+), arrested at the first order of expansion, into (2), one obtains:

$$\mathbf{F}_m \cong q_m \left(\mathbf{r}_{Q^+} - \mathbf{r}_{Q^-}\right)\cdot \nabla \mathbf{H}_{ext} , \tag{4}$$

and then:

$$\mathbf{F}_m = \lim_{d\to 0,\, q_m\to\infty} \mathbf{F}_m = \mathbf{m}\cdot \nabla \mathbf{H}_{ext} , \tag{5}$$

where the ∇ operator should be considered applied at the point occupied by the dipole.

If we assume now that the continuous distribution of Coulombian dipoles occupies the volume Ω, the expression for the resultant force can be easily obtained as:

$$\mathbf{F}_\Omega = \int_\Omega \mathbf{M}\cdot \nabla \mathbf{H}_{ext} d\Omega , \tag{6}$$

where the magnetization density **M** has been introduced as the dipole moment per unit-volume and the symbol $\int_\Omega (\cdot) d\Omega$ refers to a volume integration.

Let us recall now the following integral identity holding for any vector fields **u** and **v**, which obey to the divergence theorem in a given region Ω [1]

$$\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v} d\Omega + \int_\Omega \mathbf{u}\times(\nabla\times\mathbf{v}) d\Omega - \int_\Omega \mathbf{u}(\nabla\cdot\mathbf{v}) d\Omega = \int_\Omega (\nabla\times\mathbf{u})\times\mathbf{v} d\Omega + \oint_\Sigma (\mathbf{u}\times\mathbf{n})\times\mathbf{v} dS , \tag{7}$$

being Σ the external surface of Ω.

Let us apply this identity to the region Ω occupied by the considered distribution, posing

$$\begin{cases} \mathbf{u} = \mathbf{M} , \\ \mathbf{v} = \mathbf{B}_{ext} , \end{cases} \tag{8}$$

and recalling that the following two equations must hold true:

$$\mathbf{B}_{ext} = \mu_0 \mathbf{H}_{ext} , \tag{9}$$

$$\nabla\times\mathbf{B}_{ext} = 0, \text{ for any } P\in\Omega . \tag{10}$$

One obtains:

$$\int_\Omega (\mathbf{M}\cdot\nabla)\mathbf{B}_{ext} d\Omega = \int_\Omega (\nabla\times\mathbf{M})\times\mathbf{B}_{ext} d\Omega + \oint_\Sigma (\mathbf{M}\times\mathbf{n})\times\mathbf{B}_{ext} dS . \tag{11}$$

Recalling now the well-known expressions for the volume and surface density of *equivalent magnetizing currents* [17]:

$$\mathbf{J}_{vm} = \frac{1}{\mu_0} \nabla \times \mathbf{M} , \quad (12)$$

$$\mathbf{J}_{sm} = \frac{1}{\mu_0} \mathbf{M} \times \mathbf{n} , \quad (13)$$

equation (11) can be rewritten as:

$$\mathbf{F}_\Omega = \int_\Omega (\mathbf{M} \cdot \nabla) \mathbf{H}_{ext} d\Omega = \int_\Omega \mathbf{J}_{vm} \times \mathbf{B}_{ext} d\Omega + \oint_\Sigma \mathbf{J}_{sm} \times \mathbf{B}_{ext} dS . \quad (14)$$

This relationship shows that the resultant force upon the whole distribution may be computed by replacing the distribution itself with suitable equivalent *magnetization currents*.

Let us now show that the same result for the resultant force is also obtained by replacing the magnetization currents with equivalent *magnetic charges*.

Recalling another integral identity, holding for any vector fields **u** and **v** obeying the divergence theorem in the region Ω

$$\int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{v}\, d\Omega = \int_\Omega [\nabla \cdot (\mathbf{uv}) - (\nabla \cdot \mathbf{u}) \mathbf{v}] d\Omega = \int_\Omega [-(\nabla \cdot \mathbf{u}) \mathbf{v}] d\Omega + \oint_\Sigma \mathbf{n} \cdot (\mathbf{uv}) dS , \quad (15)$$

one has (setting $\mathbf{u} = \mathbf{M}$ and $\mathbf{v} = \mathbf{H}_{ext}$):

$$\mathbf{F}_\Omega = \int_\Omega (\mathbf{M} \cdot \nabla) \mathbf{H}_{ext} d\Omega = \int_\Omega (-\nabla \cdot \mathbf{M}) \mathbf{H}_{ext} d\Omega + \oint_\Sigma \mathbf{n} \cdot (\mathbf{M} \mathbf{H}_{ext}) dS . \quad (16)$$

Let us focus the attention on the last integral of (16). The term $\mathbf{MH}_{ext}$ is a second-order tensor representing the *dyadic product* of the two vectors **M** and $\mathbf{H}_{ext}$. The operation $\mathbf{n}\cdot(\mathbf{MH}_{ext})$ is hence a *dot product at left* between the vector **n** and the tensor $\mathbf{MH}_{ext}$. Therefore it results:

$$\oint_\Sigma \mathbf{n} \cdot (\mathbf{M} \mathbf{H}_{ext}) dS = \oint_\Sigma (\mathbf{n} \cdot \mathbf{M}) \mathbf{H}_{ext} dS . \quad (17)$$

In the light of (17), (16) becomes:

$$\mathbf{F}_\Omega = \int_\Omega (\mathbf{M} \cdot \nabla) \mathbf{H}_{ext} d\Omega = \int_\Omega (-\nabla \cdot \mathbf{M}) \mathbf{H}_{ext} d\Omega + \oint_\Sigma (\mathbf{n} \cdot \mathbf{M}) \mathbf{H}_{ext} dS . \quad (18)$$

It can easily be observed that equation (18) can be further simplified, noting that [17]:

$$\rho_m = -\nabla \cdot \mathbf{M} , \quad (19)$$

$$\sigma_m = \mathbf{M} \cdot \mathbf{n} , \quad (20)$$

being ρ_m and σ_m, respectively, the volume and surface density of the *equivalent magnetic charges*.

We then conclude that:

$$\mathbf{F}_\Omega = \int_\Omega (\mathbf{M} \cdot \nabla) \mathbf{H}_{ext} d\Omega = \int_\Omega \rho_m \mathbf{H}_{ext} d\Omega + \oint_\Sigma \sigma_m \mathbf{H}_{ext} dS , \quad (21)$$

as previously anticipated.

Equations (6), (14) and (21) show that the force acting upon a magnetized body placed in an "external" field ($\mathbf{B}_{ext}$ or, the same, $\mathbf{H}_{ext}$) can be evaluated in three different ways, according to the representation used for simulating the field produced by the magnetization of the body itself. These three formulae are generally referred to as *equivalent source methods* for force calculation, since they are based on the definition of suitable distributions of magnetic dipoles or currents or charges.

They are expressed as a function of $\mathbf{B}_{ext}$ or $\mathbf{H}_{ext}$, but can also be rewritten in terms of the total field $\mathbf{B}_{tot}$, sum between the former and the field due to the magnetization of the body, namely $\mathbf{B}_M$ (Fig. 1).

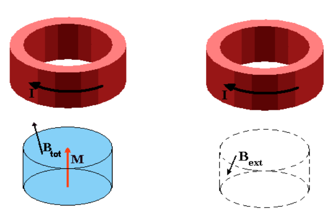

Fig. 1 - Definition of $\mathbf{B}_{tot}$ and $\mathbf{B}_{ext}$ for an observation point inside a uniformly magnetized body near a circular coil carrying a stationary current.

The external field can therefore be rewritten as [17-18]:

$$\begin{cases} \mathbf{B}_{\mathrm{ext}} = \mathbf{B} - \mathbf{B}_{\mathrm{M}}, \\ \mathbf{H}_{\mathrm{ext}} = \mathbf{H} - \mathbf{H}_{\mathrm{M}}, \end{cases} \tag{22}$$

having posed, for shortness, $\mathbf{B} = \mathbf{B}_{\mathrm{tot}}$ and $\mathbf{H} = \mathbf{H}_{\mathrm{tot}}$.

It is possible to show that the following three equalities must hold true [17]:

$$\mathbf{F}_{\Omega} = \int_{\Omega} \mathbf{M} \cdot \nabla \mathbf{H}_{\mathrm{ext}} d\Omega = \int_{\Omega} \mathbf{M} \cdot \nabla \mathbf{H} d\Omega + \oint_{\Sigma} \frac{1}{2} \frac{\mathrm{M}_{\mathrm{n}}^2}{\mu_0} \mathbf{n}\, dS . \tag{23}$$

$$\mathbf{F}_{\Omega} = \int_{\Omega} \rho_{\mathrm{m}} \mathbf{H}_{\mathrm{ext}} d\Omega + \oint_{\Sigma} \sigma_{\mathrm{m}} \mathbf{H}_{\mathrm{ext}} dS = \int_{\Omega} \rho_{\mathrm{m}} \mathbf{H} d\Omega + \oint_{\Sigma} \sigma_{\mathrm{m}} \frac{\mathbf{H}_{+} + \mathbf{H}_{-}}{2} dS \tag{24}$$

$$\mathbf{F}_{\Omega} = \int_{\Omega} \mathbf{J}_{\mathrm{vm}} \times \mathbf{B}_{\mathrm{ext}} d\Omega + \oint_{\Sigma} \mathbf{J}_{\mathrm{sm}} \times \mathbf{B}_{\mathrm{ext}} dS = \int_{\Omega} \mathbf{J}_{\mathrm{vm}} \times \mathbf{B} d\Omega + \oint_{\Sigma} \mathbf{J}_{\mathrm{sm}} \times \frac{\mathbf{B}_{+} + \mathbf{B}_{-}}{2} dS \tag{25}$$

Equation (23) shows that equivalent magnetic dipole method can be also formulated in terms of the total field $\mathbf{H}$ and this implies the addition of a surface term, depending on the normal component of magnetization. The two integrands at the r.h.s. of (23) are usually referred to as *Kelvin's formulae* and they provide the correct expressions for the volume and the surface force densities acting within a magnetized body [17].
In equations (24)-(25), representing the expressions as functions of the total field for the equivalent magnetic charge method and the equivalent magnetizing current method respectively, both fields $\mathbf{H}$ and $\mathbf{B}$ are averaged on the external surface Σ of the body, since, differently from $\mathbf{H}_{\mathrm{ext}}$ and $\mathbf{B}_{\mathrm{ext}}$, they experience a jump in their normal components on Σ.

It should be highlighted that, both from a theoretical and a numerical standpoint, formulae involving the *external field* should be considered the most appropriate way to compute force, due to the smooth behavior of this latter field on the surface of the target body (the external field does not suffer any discontinuity on Σ).

As far as the numerical computation of $\mathbf{H}_{\mathrm{ext}}$ and $\mathbf{B}_{\mathrm{ext}}$ is concerned, this can be accomplished in different ways. We propose to evaluate firstly the contribution to $\mathbf{B}$ provided in empty space by the equivalent magnetizing currents flowing inside the target body and on its surface. Once this field, namely $\mathbf{B}_{\mathrm{M}}$, is obtained in each node of the discretized domain, the external field $\mathbf{B}_{\mathrm{ext}}$ can be derived by simply subtracting it to the total field $\mathbf{B}$ provided by the FEM solution of the complete case.

The problem can be consequently approached in a two-step procedure: first, calculating the distribution of magnetizing currents from (12) and (13); then, imposing them as field sources in a second FEM solution with all the materials replaced by air [14].

B. Maxwell's stress tensor

The expression of Maxwell's stress tensor for the magnetic field in free space is [17]

$$\overline{\mathbf{T}}^{(\mathrm{M})} = \frac{1}{\mu_0} \mathbf{BB} - \frac{\mathrm{B}^2}{2\mu_0} \overline{\mathbf{I}} , \tag{26}$$

where $\mathbf{BB}$ is the dyadic product of the magnetic field $\mathbf{B}$ by itself and $\overline{\mathbf{I}}$ is the identity-tensor.

It is easy to show that:

(i) at any *regular point*, where a finite volume current density exists, the result is:

$$\mathbf{f}_{\Omega} = \mathbf{J} \times \mathbf{B} = \nabla \cdot \overline{\mathbf{T}}^{(\mathrm{M})}, \tag{27}$$

(ii) at any point where a finite surface current density $\mathbf{J}_{\mathrm{S}}$ exists, the result is:

$$\mathbf{f}_{\mathrm{S}} = \mathbf{J}_{\mathrm{S}} \times \frac{\mathbf{B}_{+} + \mathbf{B}_{-}}{2} = \mathbf{n} \cdot \left(\overline{\mathbf{T}}_{+}^{(\mathrm{M})} - \overline{\mathbf{T}}_{-}^{(\mathrm{M})} \right) , \tag{28}$$

with obvious meaning of the symbols.

The use of Maxwell's stress tensor is always limited to situations where the resultant force exerted by the magnetic field upon a magnetized body *surrounded by vacuum* is to be calculated.

In such cases, one has:

$$\mathbf{F}_{\Omega} = \oint_{\Sigma} \left(\mathbf{n} \cdot \overline{\mathbf{T}}^{(\mathrm{M})} \right) dS , \tag{29}$$

having indicated with Σ any closed surface (lying entirely in vacuum) bounding the region Ω where the current flows.

C. Virtual work method

According to this method, the force is derived from the variation of the magnetic energy (or, alternatively, co-energy) caused by a suitable infinitesimal rigid displacement of the considered body. The use of the energy rather than the co-energy is determined by the nature of the transformation performed: if the magnetic fluxes are kept constant during the displacement of the body, the magnetic energy must be employed, whereas the magnetic co-energy should be used if, on the contrary, the magnetic fluxes can vary and the currents remain unchanged.

In this latter case, it can be shown [7] that the resultant force $\mathbf{F}_\Omega$ acting upon the target body can be expressed as:

$$\mathbf{F}_\Omega \cdot d\mathbf{r} = dW_m^{'}, \tag{30}$$

where $d\mathbf{r}$ is an elementary rigid displacement of the body and W_m' is the *magnetic co-energy*, defined as:

$$W_m^{'} = \int_{\Omega_\infty} \left[\int_0^{\mathbf{H}} \mathbf{B}(\mathbf{H}) \cdot d\mathbf{H} \right] d\Omega, \tag{31}$$

being $\mathbf{\Omega}_\infty$ the whole space.

Similarly, in the case of constant magnetic fluxes, it can be proven [7] that the following result must hold true:

$$\mathbf{F}_\Omega \cdot d\mathbf{r} = -dW_m, \tag{32}$$

where W_m is the *magnetic energy*, defined as:

$$W_m = \int_{\Omega_\infty} \left[\int_0^{\mathbf{B}} \mathbf{H}(\mathbf{B}) \cdot d\mathbf{B} \right] d\Omega, \tag{33}$$

being the meaning of the symbols now obvious.

3. Critical analysis of the methods: the problem of the corner singularity of the self-field of the target body

From a numerical standpoint, the methods described in the previous section are based on volume integration (VWM), surface integration (MST) and both a volume and a surface integration (ESM).

Typically, the surface integration of Maxwell's stress tensor has to be performed in very thin air regions like the air-gap between the fixed part of an electromagnetic device and the moving one. In such situations, the following problems can arise: if the surface is chosen very close to the body, the numerical results are very sensitive to the field discontinuity occurring at the boundary of the magnetized body; if, on the contrary, the surface is far from the body, the numerical errors affecting the evaluation of the surface integral may grow unavoidably. That is why it is usually common practice to perform several simulations varying the distance between the closed surface in air and the target body and to consider as reference force value for MST the arithmetic average of the results obtained. This procedure permits gaining accuracy but increases the post-processing time to acquire force value.

Also VWM is characterized by high computational costs, since it needs the evaluation of the magnetic energy W_m (or co-energy W_m') on the whole domain for several small rigid displacements of the target body around its initial position and along its movement spatial coordinate. This requires the execution of several field simulations and then a suitable interpolation of the obtained values of W_m (W_m') in order to compute finally the force as the derivative of W_m (W_m') with respect to the displacement.

As far as ESM are concerned, they main source of numerical errors affecting their results is the surface integration along the boundary of the target body, where a field discontinuity occurs. It should be noticed however that this drawback affects only ESM written in terms of the total field.

The numerical integration process is undoubtedly a source of errors in the force value, but accuracy can be lost also because of numerical errors affecting the fields, due to the adopted numerical formulation and to the singular behavior of the magnetic field on the edges of the target body.

Let us examine in detail this latter problem, considering a two-dimensional magnetized body with arbitrary polygonal shape. The field due to its magnetization can be computed by means of the equivalent current method. Considering, for the sake of simplicity, linear materials, the equivalent current results in a surface distribution expressed by (13).

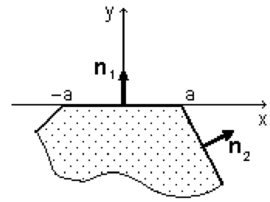

Fig. 2 - Generic corner of a polygonal magnetized body.

Focusing now the attention on a generic corner of the body (Fig. 2), we are interested in analyzing the properties of the magnetic induction field at the points (x = ± a, y = 0). The z-component of the magnetic vector potential due to the magnetizing current sheet on the x axes, between x = -a and x = a and with surface density $J_{sm1}(x)=(\mathbf{M}\times\mathbf{n}_1\cdot\mathbf{e}_z)/\mu_0 = M_x(x)/\mu_0$ can be expressed as:

$$A_z(x,y)=\frac{1}{2\pi}\left\{\int_{-a}^{a} M_x(x')\ln\sqrt{\left[(x-x')^2+y^2\right]}dx'\right\}, \tag{34}$$

and the magnetic induction is consequently given by:

$$B_x(x,y)=\frac{1}{2\pi}\left\{\int_{-a}^{a} M_x(x')\frac{y}{(x-x')^2+y^2}dx'\right\}, \tag{35}$$

$$B_y(x,y)=-\frac{1}{2\pi}\left\{\int_{-a}^{a} M_x(x')\frac{x-x'}{(x-x')^2+y^2}dx'\right\}. \tag{36}$$

With the assumption that $\exists$ M > 0 such that $M_x \geq M$ for $x\in[-a,a]$, it can be shown that B_y is singular at (x = ±a, y = 0), while B_x not only exhibits a discontinuity for $x\in(-a,a)$ and y=0 but also cannot be defined at the ends, even for continuity. To this aim, let us consider the limit of $B_y(x,0)$ when x tends to a^+. It results that:

$$B_y\Big|_{\substack{y=0\\x\geq a}}=-\frac{1}{2\pi}\left\{\int_{-a}^{a} M_x(x')\frac{1}{(x-x')}dx'\right\}\leq-\frac{1}{2\pi}\left\{\int_{-a}^{a} M\frac{1}{(x-x')}dx'\right\}=\frac{M}{2\pi}\ln\frac{x-a}{x+a}\to-\infty, \tag{37}$$

for $x\to a^+$.

For B_x, instead, one obtains:

$$B_x\Big|_{\substack{x=a\\y\geq 0}}=\frac{1}{2\pi}\left\{\int_{-a}^{a} M_x(x')\frac{y}{(a-x')^2+y^2}dx'\right\}\geq\frac{1}{2\pi}\left\{\int_{-a}^{a} M\frac{y}{(a-x')^2+y^2}dx'\right\}=\frac{M}{2\pi}\tan^{-1}\frac{2a}{y}\to\frac{M}{4}, \tag{38}$$

for $y\to 0^+$;

$$B_x\Big|_{\substack{x=a\\y\leq 0}}=\frac{1}{2\pi}\left\{\int_{-a}^{a} M_x(x')\frac{y}{(a-x')^2+y^2}dx'\right\}\leq\frac{1}{2\pi}\left\{\int_{-a}^{a} M\frac{y}{(a-x')^2+y^2}dx'\right\}=\frac{M}{2\pi}\tan^{-1}\frac{2a}{y}\to-\frac{M}{4}, \tag{39}$$

for $y\to 0^-$.

Further, $B_x \equiv 0$ for y=0 and x < -a or x > a.

As previously mentioned, in FEM modeling the singularity of the self-field of the target body gives rise to a numerical error affecting the total field, sum of the former and that created by other sources.

For a given mesh size, the relative magnitude of this error and its influence on the evaluation of the total force acting on the target body depend on the ratio between the magnetization field $\mathbf{B}_M$, and the external field $\mathbf{B}_{ext}$. In ferromagnetic materials, both $\mathbf{B}_M$ and $\mathbf{B}_{ext}$ are linked to external sources and when the contribution of these is prevalent, the error is high only in close proximity to the edge. On the contrary, in permanent magnets, magnetization is (almost) independent on the external sources. As it will be shown in the next section, in these situations, especially when low currents are involved and the total field is practically coincident with that of the magnet, the error spreads over a more extended area surrounding the singularity point. This leads to the necessity of very dense discretizations to obtain reliable values of force with the expressions involving the total field. On the other side, it should be stressed again that ESM using the external field do not suffer of this drawback, since the field of the target body is not included in these relations.

4. Numerical computations

The discussed methods have been numerically tested on axisymmetric and 3D problems. The examined test cases have been selected in order to highlight critical aspects in force computation.

A. Axisymmetric test case

Team Workshop problem 23 [20-21] was considered as axisymmetric case study. It consists in determining the repulsive force between a coil and a magnet (Fig. 3).

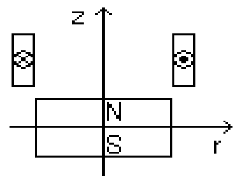

Fig. 3 - Team problem 23.

The problem definition provides dimensions for four different configurations but we analyzed only the one involving the larger coil and the larger Samarium-Cobalt magnet, when both are completely aligned. In particular, we calculated the repulsive force for a fixed value of the distance δ between the coil and the magnet (δ = 0.381 mm) and for the lowest current I = 10 mA. This configuration, critical from the force computation viewpoint both for the reduced dimensions of the air gap and for the extremely low value of the force itself, is not simply a theoretical example but it can represent, in our experience, also an acoustic transducer for biomedical applications.

Since it has been highlighted in [21] that the reported material properties of the permanent magnet differ from the real values, a disagreement between measured and computed values of force was observed. So we decided to adopt as reference value the force calculated on the coil with the volume integration of $\mathbf{J}\times\mathbf{B}$, when **B** comes from an adaptive solution with tolerance set to 10^{-4}.

We used five different meshes (mesh 1→ coarse; mesh 5→ very dense) to solve this problem. In Table I the results obtained with MST, VWM and with the application of the equivalent charge method (ESM_Q), the equivalent dipole method (ESM_D) and the equivalent current method (ESM_I) are reported. ESM results refer to the formulae expressed as functions of the external field.

TABLE I
FORCE ON THE MAGNET [N]

	Mesh 1	Mesh 2	Mesh 3	Mesh 4	Mesh 5
F_{ref}	-0.0020677	-0.0020677	-0.0020677	-0.0020677	-0.0020677
MST	-	-	-0.0021582	-0.0020406	-0.0020734
VWM	-	-	-	-0.0016001	-0.0025012
ESM_Q	-0.0020520	-0.0020628	-0.0020664	-0.0020674	-0.0020675
ESM_D	-0.0020520	-0.0020628	-0.0020664	-0.0020674	-0.0020675
ESM_I	-0.0020667	-0.0020688	-0.0020706	-0.0020705	-0.0020703

As far as MST is concerned, we performed several simulations varying the distance *d* between the closed surface in air and the magnet (Fig. 4). Results obtained with mesh 1 and 2 exhibited so high oscillations to result wholly unfounded.

When the level of mesh refinement is increased, MST becomes more stable for surfaces not very close to the magnet, although oscillations can still be observed.

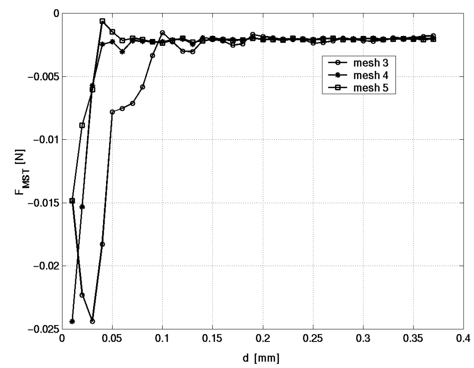

Fig. 4 - Results obtained with MST on several surfaces enclosing the magnet.

In order to provide a reference force value for MST we therefore considered the arithmetic average of the results obtained only in this more stabilized region.

Force computations carried out with VWM seem to be more sensitive to the accuracy of the mesh than those carried out with MST. For the considered problem, W_m' showed extremely small variations, since the field provided by the coil is negligible with respect to the field of the magnet. Thus, only with the most refined meshes (mesh 4 and 5) we succeeded in interpolating W_m' and in computing the force. It is worth noting that the obtained results are affected anyway by large inaccuracies (differences higher than 20 %).

As far as ESM are concerned, they exhibited high numerical robustness, since they provided reliable results even when poor meshes were used (Fig. 5).

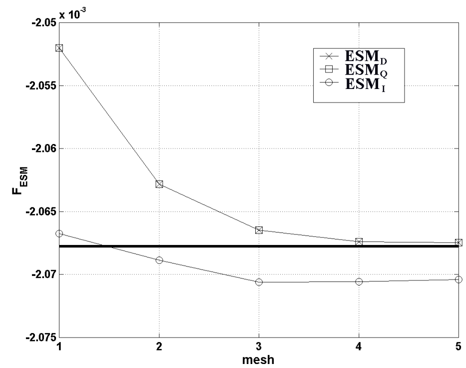

Fig. 5 - ESM vs. discretization. The bold line represents the correct value of force.

The poor accuracy and efficiency of classical methods is not only due to the singularity on the edges of the self-field of the magnet, but also to the dominant contribution that it gives to the total field [16].

B. 3D test cases

The presented 3D test case concerns with the evaluation of the attracting force between two cubic rigid magnets (side equal to 10 mm, B_r = 1 T along the same direction for both magnets) placed face to face at a distance of 5 mm. This problem, solved analytically in [22], was discretized using three different meshes with 48, 500 and 4000 elements for each magnet respectively. The obtained results are listed in Table II.

TABLE II
FORCE ON THE CUBIC MAGNET [N]

	Mesh 1	Mesh 2	Mesh 3
Analytical	6.568	6.568	6.568
MST	7.09754	7.47062	6.73695
VWM	7.20314	6.56755	6.57234
ESM_Q	6.33122	6.68592	6.64007
ESM_D	6.42301	6.60710	6.61891
ESM_I	6.19460	6.41129	6.65433

In this case, MST did not reach stability with respect to the distance between the integration surface and the first magnet (Fig. 6). This must be attributed to the presence of the second magnet. Therefore, the values obtained with surfaces in proximity of the two magnets were neglected for the evaluation of the averaged MST.

It is apparent from Table II that VWM provides a good accuracy in the results. In this case, indeed, significant variations of the co-energy W_m' can be appreciated when the rigid displacements are performed. This is due to the fact that the two fields produced by magnets have the same intensity. Thus, even small displacements can give rise to appreciable variations of the field configuration.

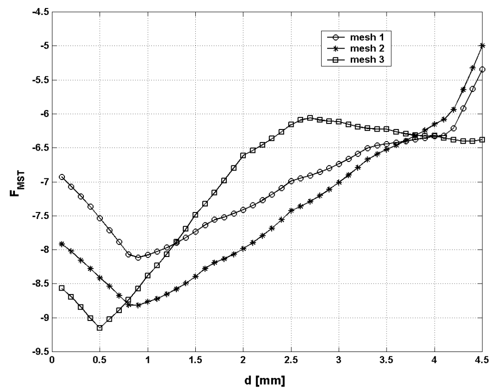

Fig. 6 - MST vs. *d* for the two magnets face to face.

This problem, solved with the most refined mesh, was also used to compare the CPU time of the tested methods. In particular, we compared VWM, MST and ESM_Q using 5 solutions for VWM and 45 values of *d* for MST. The surface integration, required both by MST and ESM_Q, was performed using respectively 20×20 and 200×200 gaussian points (min and max values in Fig. 7). As can be seen in Fig. 7, ESM_Q show the best performances.

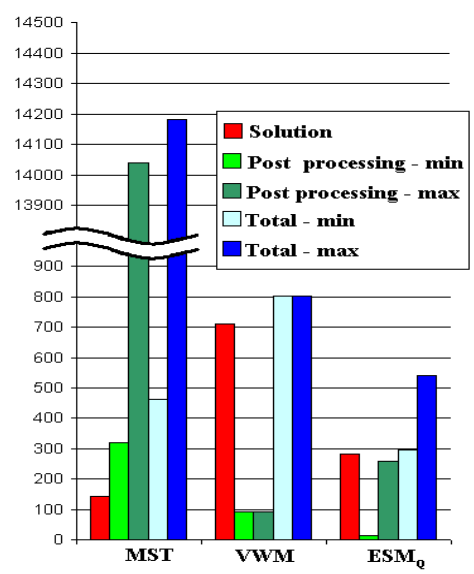

Fig. 7 – CPU time (in seconds) on a Digital Personal Workstation 500 AU.

5. Conclusions

The most popular methods for total magnetic force calculation have been reviewed in this paper. It has been shown that the equivalent source methods can be reformulated in terms of the so-called external field and it has been pointed out that these new expressions allow circumventing numerical errors and instabilities due to the singularity of the self-field of the magnetized target body on its corners. All methods have been finally compared on axisymmetric and three-dimensional problems, and their accuracy has been analyzed in connection to the level of refinement of the used FEM grid.

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