Scalar and Vector Elements on Overlapping Patches

Abstract – **We consider generalized scalar and vector elements on overlapping patches. No structured FEM-like meshes are required; the patches may have arbitrary shapes and overlap arbitrarily; each patch may have its own system of approximating functions reflecting the local behavior of the solution (e.g. edge or corner singularities). In the vector case, the new elements may be viewed as a generalization of tetrahedral edge elements. Waveguide problems, with or without corner singularities, are presented as illustrative examples.**

I. Introduction: FEM and Its Generalization

The Finite Element Method (FEM) typically guarantees a certain order of piecewise-polynomial approximation on *structured* meshes. It would often be desirable to remove or relax the restrictions on mesh structure *while maintaining the same level of approximation accuracy*. For example, in problems involving mechanical motion it would be natural if finite elements were allowed to overlap. The proper *continuity* of the numerical solution can be maintained easily even if elements do overlap (see [18] and a more detailed explanation below), but the accuracy requirement is critical.

The standard FEM is based on polynomial approximating functions. This is another undesirable limitation: ideally, one would want to tailor the basis functions to a specific local behavior of the field (e.g. spatial oscillations, edges and corner singularities, boundary layers).

Fig. 1 illustrates this very schematically. For a field that behaves differently in different subregions, it would be desirable to use different types of local approximating functions, provided these approximations can be merged seamlessly.

Over the last decade, various generalizations of FEM have been proposed, with a strong emphasis on 'meshlessness' – unfortunately, in some instances at the expense of the most fundamental requirement of approximation accuracy. For example, the Moving Least Squares or Diffuse Element Methods [2] do not generally reproduce even linear polynomials exactly. While the reproducibility of polynomials is not, strictly speaking, necessary for convergence, it is quite desirable both theoretically and practically (few practitioners would accept a numerical method that does not represent a linear solution).

An overview of meshless methods can be found in [2], [7], [11]. In computational electromagnetics, versions of the Diffuse Element Method were applied by Maréchal *et al.* [8]-[10].

A relatively new class of methods that removes the restrictions on mesh structure and on the type of approximation used but still guarantees consistency and accuracy is based on the idea of *partition of unity* (Section II).

A distinctive feature of partition of unity methods that makes them more flexible and accurate than many other 'meshless' methods is the presence of *two* sets of functions. Functions of the first set (denoted with φ_i in this paper) sum up to one everywhere in the domain and form a partition of unity. Each of these functions vanishes outside its respective patch and can be viewed as a weighting factor in the splitting of the global solution into its 'patch components'.

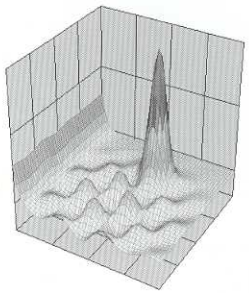

Fig. 1. A schematic illustration of behavior of a field quantity: waves, boundary layers and singularities. Different local approximations of the field can be seamlessly merged using Partition of Unity.

The second set of functions (denoted with g later in this paper) is local approximating functions defined over each patch; they characterize the local behavior of the solution but need not satisfy any additional conditions on patch boundaries.

As a consequence, any relevant approximating functions can be incorporated into the basis. Adaptivity is implemented very easily: since there are few restrictions on the placement of overlapping patches and on the functional bases therein, patches can be added or removed, and / or the approximation accuracy within each patch can be enhanced. The analogs of h- and p-refinement are thus straightforward and do not require 'remeshing'.

In the remainder of the paper, we outline the essence of the Generalized FE – Partition of Unity method [1], [7], [11], [16] its connection to the standard FEM, and remaining problems. In addition, we propose new *vector* elements on overlapping patches of arbitrary shape; these elements can be viewed as a

generalization of Nedelec-Whitney edge elements. The vectorial method gives the same flexibility in the choice of patches and basis functions as its scalar prototype and ensures that tangential, but not necessarily normal, components of the approximate solution are continuous. Numerical examples given in Section IV include applications of both scalar and vector elements on overlapping patches.

II. The Partition of Unity Method

A. Approximation Accuracy is Crucial

In FEM, one considers an infinite sequence of finite-dimensional FE spaces $\{Q_h\}$, where $h > 0$ is a 'small' parameter, usually, but not necessarily, the mesh size (in practice, only one or a few spaces of this conceptually infinite sequence are dealt with). The approximation error ε_a is typically expressed as

$$\varepsilon_a \equiv \min \|u - u_a\| \leq c_1 h^\gamma \qquad (1)$$

where the minimum is taken over all approximation functions $u_a \in Q_h$; c_1 is a constant (the same for all Q_h); parameter $\gamma > 0$ is the order of approximation; $\|\cdot\|$ is a certain norm, usually an energy norm closely related to the boundary value problem under consideration.

Approximation is necessary and in many cases sufficient for *convergence* of the numerical solution $u_h \in Q_h$ obtained usually by Galerkin or other variational methods (see e.g. [17] for more details on the convergence of FEM for low-frequency problems).

We insist that the approximation condition (1) be maintained for all generalizations of FEM that we consider.

B. Objective and the Main Idea

The computational domain Ω is covered by overlapping patches Ω_i, $i = 1, 2, \ldots n_{\text{patches}}$. An almost random 2D setup is shown in Fig. 2. Domain Ω (the large rectangle) is covered by five overlapping patches Ω_i: two rectangles (Ω_1 and Ω_2); an ellipse (Ω_3), and two patches with irregular shapes (Ω_4 and Ω_5). Rectangle Ω_1 overlaps with all other patches except for Ω_5; Ω_2 happens to overlap with all other patches, etc. Of course, this particular arrangement of patches is given here for illustrative purposes only; but in principle it's valid. As noted previously, some of the patches may represent subregions with a special behavior of the solution (singularities, boundary layers, etc.) that need to be approximated accordingly; other patches may represent regions with a generic / unknown behavior where polynomial approximation may be the best choice.

To achieve better approximation and better solution accuracy, one could increase the number of patches (which would be analogous to *h*-refinement in FEM) and / or increase the approximation accuracy within some (or all) patches, which would be analogous to *p*-refinement.

Local approximations over individual patches need to be merged into a global one 'seamlessly,' that is, with little or no loss in the order of approximation. We illustrate the importance of this requirement with an example from one of the authors' experience a decade ago. To model rotor motion in electric machines, overlapping rectangular elements in the air gap were proposed [18] (Fig. 3). Nodes marked by circles are 'active' and have bilinear basis functions associated with them, while the unmarked nodes are 'dummy' (similar to boundary nodes with homogeneous Dirichlet conditions).

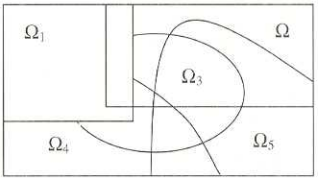

Fig. 2. A schematic 2D example of a domain Ω (the large rectangle) covered by five overlapping patches Ω_i.

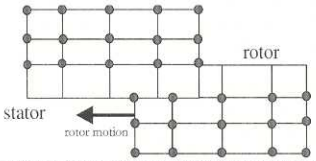

Fig. 3. Rectangular elements with an overlapping layer for problems with rotor motion. For clarity, only fragments of the rotor and stator side meshes in the air gap are sketched and the overlap is only partially shown. Circles indicate 'active' ('non-Dirichlet') nodes.

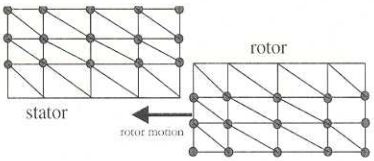

Fig. 4. Overlapping triangular elements may in principle be applied, too, but the order of approximation deteriorates substantially.

The required smoothness of the approximate solution can always be assured, even for overlapping meshes. Indeed, as long as the basis functions are sufficiently smooth (e.g. have first derivatives, not necessarily continuous), their linear combinations are equally smooth, regardless of whether the underlying elements overlap.

However, approximation accuracy remains the key. For an overlapping layer of *rectangular* elements (Fig. 3), the approximation error is of the same order as in the conventional FEM, and quite accurate simulation results were reported in [18]. But for overlapping *triangular* elements (Fig. 4) the accuracy does deteriorate unless the partition of unity proce-

dure is used as described below (in any event, overlapping triangular elements may not be a good practical choice).

C. Using Partition of Unity to Merge Local Approximations

Suppose we have a system of functions $\{\varphi_i\}$, $1 \le i \le n_{\text{patches}}$ that form *a partition of unity*; that is,

$$\sum_{i=1}^{n_{\text{patches}}} \varphi_i \equiv 1 \quad \text{in } \Omega, \qquad \text{supp}(\varphi_i) = \Omega_i \tag{2}$$

where each φ_i is associated (in the manner described below) with the respective patch Ω_i and vanishes outside that patch. If this is the case, the φ's act essentially as weighs that let one decompose any function (e.g. the exact field and its numerical approximation) into its 'patch components' as

$$u = \sum_{i=1}^{n_{\text{patches}}} u\varphi_i \tag{3}$$

If the solution u is approximated *locally*, i.e. over each patch Ω_i, by a certain function u_{hi}, these local approximations can be merged as

$$u_h = \sum_{i=1}^{n_{\text{patches}}} u_{hi}\varphi_i \tag{4}$$

Then the field error over the whole domain Ω is

$$\varepsilon_a \equiv \left\|\nabla(u_h - u)\right\|_{L_2(\Omega)} = \left\| \sum_{i=1}^{n_{\text{patches}}} \nabla\left[(u_{hi} - u)\varphi_i\right] \right\|_{L_2(\Omega)}$$

$$\le \sum_{i=1}^{n_{\text{patches}}} \left\| \nabla\left[u_{hi} - u\right] \right\|_{L_2(\Omega_i)} \left\|\varphi_i\right\|_{\infty} +$$

$$\sum_{i=1}^{n_{\text{patches}}} \left\| u_{hi} - u \right\|_{L_2(\Omega_i)} \left\| \nabla\varphi_i \right\|_{\infty} .$$

Thus, under the assumption that the above norms of φ_i are bounded, the global approximation essentially reduces, due to the partition of unity, to local approximations over each patch. For the original and more rigorous analysis, see [11], [1].

D. Generating a Partition of Unity

Within each patch, one defines a single function (that we call a *patch shape function*, denoted $\tilde{\varphi}_k$) that vanishes on the boundary $\partial\Omega_k$ and outside Ω_k. Exception: $\tilde{\varphi}_k$ can be *non-zero* on parts of $\partial\Omega_k$ lying on the domain boundary ∂D.

One way to define the shape function for a rectangular patch $[x_k^{\min}, x_k^{\max}] \times [y_k^{\min}, y_k^{\max}]$ *strictly inside* the domain is

$$\tilde{\varphi}_k = (x - x_k^{\min})(x_k^{\max} - x)(y - y_k^{\min})(y_k^{\max} - y)$$

(Fig. 5). The shape function in a patch adjacent to the domain boundary (say, $x_k^{\min} = x^{\min}$ and $y_k^{\min} = y^{\min}$, where $x^{\min}$ and $y^{\min}$ correspond to the edges of the computational domain as a whole, see Fig. 6) could be

$$\tilde{\varphi}_k = (x_k^{\max} - x)(y_k^{\max} - y)$$

Obviously, there are many other possibilities of defining the shape functions.

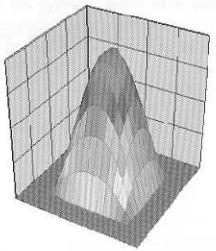

Fig. 5. A typical 'patch shape function' inside a 2D domain.

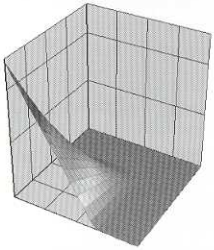

Fig. 6. A 'patch shape function' at the domain boundary.

To avoid any misunderstanding, let us compare the terminology used for overlapping patches with the notions of the conventional FEM. In the standard FEM, the 'shape functions' double as basis functions. The numerical solution is a linear combination of these functions. Hence their primary role is to represent the numerical solution, and their dependence on shape is to some extent superfluous.

Not so for the partition of unity method. We are dealing with *two* sets of functions. The first one is the patch shape functions $\tilde{\varphi}_k$ and their normalized versions φ_k, below. These functions have the effect of *localizing* the approximation: any function multiplied by $\tilde{\varphi}_k$ will obviously vanish outside the respective patch. The second set consists of local approximating functions within each patch, as described below.

The standard FEM – say, with triangular or tetrahedral elements – can be viewed as a particular case of the partition of unity method. In nodal FEM, 'patches' are clusters of elements sharing a common node, and the barycentric coordinates λ_k defined on these clusters form a partition of unity. An example of such function λ_k supported on a cluster of six triangular elements is shown in Fig. 7.

A partition of unity is created by *normalizing* the $\tilde{\varphi}_k$'s (Fig.8):

$$\varphi_k = \tilde{\varphi}_k \,/\, \sum_{i=1}^{n_{\text{patches}}} \tilde{\varphi}_i \tag{5}$$

(clearly, the sum of φ_k is identically equal to one).

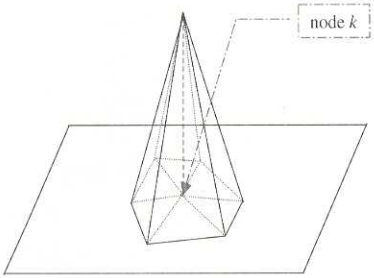

Fig. 7. Barycentric coordinate functions λ_k sum up to unity everywhere in the domain and thus form a partition of unity.

Once the partition of unity is available, the numerical solution is sought as a linear combination of n approximating functions ψ_i:

$$u_h(x) = \sum_{i=1}^{n} a_i \psi_i \tag{6}$$

and the standard Galerkin or Ritz procedure can be applied to find the unknown coefficients.

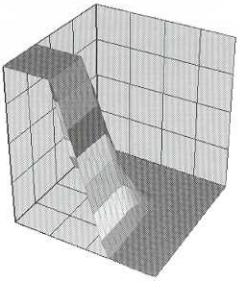

Fig. 8. A normalized patch shape function φ_k.

The ψ functions are constructed as follows. First, each patch k is endowed with a *local* system of basis functions $\{g_{kj}\}$. There is no restriction on the behavior of these functions at the patch boundary. They can be chosen as polynomials, sinusoids, exponentials, or any other functions providing a good local approximation to the solution over the patch.

Functions ψ_i of (6) are obtained by multiplying each local basis function g_{kj} on Ω_k by the respective normalized patch shape function φ_k:

$$\psi_i = g_{kj}\varphi_k \tag{7}$$

Local approximations over the patches are merged into a global one by the partition of unity $\{\varphi_k\}$ (see Section II-C).

III. Vector Elements by Partition of Unity

New vector elements constructed in this section generalize tetrahedral edge elements to arbitrary overlapping patches. We show that using a set of scalar functions with reasonable approximation properties in the Sobolev space $H^1(\Omega)$ as a starting point, one can generate an 'edge-element'-type space with equally reasonable approximation properties in $H(\mathrm{curl}, \Omega)$. The underlying scalar functions can, in particular, be obtained by partition of unity on overlapping patches.

Thus we start with a set of *scalar* functions $\{\psi_i\}$ $(i = 1,2, \ldots n)$ in $H^1(\Omega)$ and let

$$\Psi = \mathrm{span}\,\{\psi_i\}$$

We shall make a reasonable assumption that any linear function of coordinates can be represented exactly by the ψ's, that is

$$ax + by + cz + d \in \Psi \tag{8}$$

for any numbers a, b, c, d.

Two vector spaces can now be introduced by analogy with Nedelec-Whitney elements on tetrahedra [13], [14], [4], [23]:

$$V = \mathrm{span}\,\{v_{ij}\} \text{ with } v_{ij} = \psi_i \nabla\psi_j\,,\ 1 \le i, j \le n \tag{9}$$

$$W = \mathrm{span}\,\{w_{ij}\} \text{ with } w_{ij} = \psi_i\nabla\psi_j - \psi_j\nabla\psi_i,\ 1 \le i < j \le n \tag{10}$$

Remarkably, several critical properties of spaces V and W can be formulated and proven with almost no additional assumptions. These properties are formally summarized as theorems below and given without proofs; see [20], [21] for details.

Theorem 1.

$$\sum_{i=1}^{n} \vec{c}_i\,\psi_i \in V \tag{11a}$$

$$\sum_{i=1}^{n} a_i\,\nabla\psi_i \in W \subset V \tag{11b}$$

$$\nabla \times \sum_{i=1}^{n} \vec{c}_i\,\psi \in \nabla\times W = \nabla\times V \tag{11c}$$

for any *vector* coefficients $\vec{c}_i$ and scalar coefficients a_i.

Corollary 1. If a scalar function f: $\Omega\to R$ is a linear combination of ψ's, i.e. $f \in \Psi$, then $\nabla f \in W$. In short,

$$\nabla\Psi \subset W \subset V \tag{12}$$

Corollary 2. Let $\vec{F}$ be a *vector* function $\Omega\to R^3$ in $H(\mathrm{curl}, \Omega)$ that can be approximated with a given degree of accuracy by functions in Ψ; that is,

$$\left\| \vec{F} - \sum_{i=1}^{n} \vec{c}_i\,\psi_i \right\|_{H(\mathrm{curl},\,\Omega)} < \varepsilon \tag{13}$$

$\vec{c}_i$ again being some *vector* coefficients. Then, if (8) holds, $\nabla\times\vec{F}$ can be approximated with the same, or better, accuracy ε by a function in W:

$$\min_{w_i \in W} \left\| \nabla \times \left(\vec{F} - \sum_{i=1}^{n} a_i w_i \right) \right\|_{L_2(\Omega)^3} < \varepsilon \qquad (14a)$$

$$\min_{v_i \in V} \left\| \vec{F} - \sum_{i=1}^{n} a_i v_i \right\|_{H(\mathrm{curl},\,\Omega)} < \varepsilon \qquad (14b)$$

(The last inequality implies approximation of both $\vec{F}$ and its curl by functions in V.)

Theorem 2. If the set $\{\psi_i\}$ is constructed on (overlapping) patches as in Section II, then w_{ij} and v_{ij} have continuous tangential components (but not necessarily continuous normal components) across all patch boundaries.

IV. Numerical Examples

A. Linearly Polarized EM Wave in a Γ-shaped Waveguide with Imperfectly Conducting Walls

The electric field experiences a singularity in the vicinity of an edge of a good conductor. If the character of this singularity is known *a priori*, the appropriate (singular) approximating functions can be incorporated in the partition of unity method.

Let us consider propagation of a linearly polarized wave in a Γ-shaped waveguide. We assume that the only component of the electrical field is directed along the z-axis (Fig.9); Γ_1 and Γ_2 are surfaces of imperfect and perfect conductors, respectively, and Γ_0 is the waveguide port.

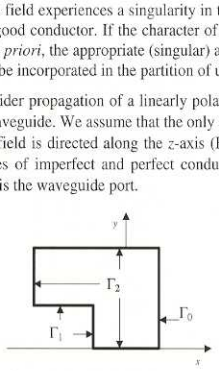

Fig. 9. Waveguide with imperfectly conducting walls

The problem can be formulated in terms of E_z as follows:

$$\nabla^2 E_z + k^2 E_z = 0$$
$$E_z\big|_{\Gamma_0} = E_0 \sin(k_y y + \varphi_0) \qquad (15)$$
$$\left(E_z + Z \frac{\partial E_z}{\partial \vec{n}} \right)\Big|_{\Gamma_1} = 0, \quad E_z\big|_{\Gamma_2} = 0$$

In (15) Z is the electric impedance whose behavior in the vicinity of the edge can be approximated as [15]:

$$Z_x \equiv \frac{E_z}{H_x}\Big|_{\Gamma_1} \cong -\frac{\tan(\pi/6)}{\sigma\rho}, \quad Z_y \equiv \frac{E_z}{H_y}\Big|_{\Gamma_1} \cong \frac{\tan(\pi/6)}{\sigma\rho} \qquad (16)$$

We represent the electric field as the sum

$$E_z = E_0 + u \qquad (17)$$

where E_0 is any smooth function satisfying the inhomogeneous boundary conditions at the port:

$$E_0\big|_{\Gamma_0} = E_0 \sin(k_y y + \varphi_0), \quad E_0\big|_{\Gamma_1 \cup \Gamma_2} = 0. \qquad (18)$$

The boundary value problem for u is obtained by substituting representation (17) into problem (15) and taking into account boundary conditions (18):

$$\nabla^2 u + k^2 u = -\nabla^2 E_0 + k^2 E_0$$
$$u\big|_{\Gamma_0} = 0, \quad \left(u + Z \frac{\partial (u + E_0)}{\partial \vec{n}} \right)\Big|_{\Gamma_1} = 0, \quad u\big|_{\Gamma_2} = 0 \qquad (19)$$

The weak formulation of this problem is well known:

$$\int_{\Omega} (\nabla u \cdot \nabla u' - k^2 u u')\, d\Omega - \int_{\Gamma_1} \frac{1}{Z} u u'\, dS = - \int_{\Omega} (\nabla E_0 \cdot \nabla u' - k^2 E_0 u')\, d\Omega \qquad (20)$$

Here u' is a 'test' function from $H^1(\Omega)$ satisfying the essential boundary conditions on $\partial\Omega$.

Discretization of (20) can be obtained from the weak formulation in the standard way, by restricting both u and u' to the finite-dimensional space constructed by partition of unity (see Section II) and then applying the Galerkin method:

$$u' = \psi_i \equiv g_{km}\, \varphi_m \qquad (21)$$

In (21) subscript "i" is used for simplicity in lieu of a pair of subscripts "m" (for the patch number) and "k" (the local index for an approximating function within a patch m). Thus the numerical solution is sought as a linear combination of approximating functions ψ that act as trial functions u' in (20).

To generate a partition of unity, we start with the patch shape functions $\tilde{\varphi}$ chosen in the form:

$$\tilde{\varphi} = \begin{cases} (x - x_0)(x - x_1)(y - y_0)(y - y_1), \\ (x, y) \in [x_0, x_1] \times [y_0, y_1] \\ \hline 0, \quad \text{otherwise} \end{cases} \qquad (22)$$

where x_i and y_i are defined in Fig. 10. The function family $\{\hat{\varphi}_i\}$ with its elements:

$$\hat{\varphi}_i = \tilde{\varphi}_i \Big/ \sum_n \tilde{\varphi}_n \qquad (23)$$

defines a partition of unity over an *extended* domain $\tilde{\Omega}$ (ABCDEF) that includes the waveguide: $\tilde{\Omega} \supset \Omega$ (Fig. 10).

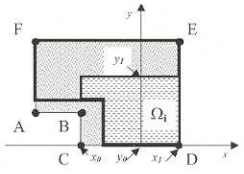

Fig.10. The geometric setup.

Partition of unity over Ω (as opposed to $\tilde{\Omega}$) can be obtained by simple truncation:

$$\varphi_i(x,y) = \begin{cases} \hat{\varphi}_i(x,y), & (x,y) \in \Omega \\ 0, & \text{otherwise} \end{cases} \tag{24}$$

The local (patch-wise) approximating functions g_k were taken as polynomials of order up to six:

$$g_k \equiv g_{nm} = (x - x_0)^n (y - y_0)^m \tag{25}$$

(in the patches adjacent to the boundaries only polynomials satisfying the homogeneous boundary conditions were considered). To approximate singularity effectively, function $v(x, y) = \rho^{-1/3}$ (ρ being the distance from the edge) was added to the set of approximating functions in the patches adjacent to the edge. Figs. 11, 12 represent numerical solutions obtained for 12 patches with 258 degrees of freedom altogether.

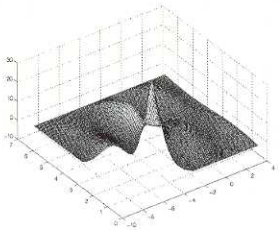

Fig.11 Electric field in a waveguide with imperfectly conducting walls.

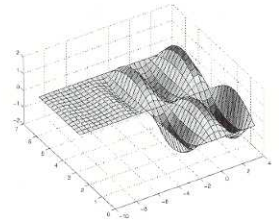

Fig. 12. Electric field in the waveguide with perfectly conducting walls.

B. Linearly Polarized EM Wave in a Γ-shaped Waveguide with Perfectly Conducting Walls

Consider a linearly polarized monochromatic EM wave in a Γ-shaped waveguide (Fig. 13). Walls Γ_1 are now perfectly conducting. We solve the problem using new generalized vector elements. The total magnetic field can be decomposed into the incident field $\vec{H}_{inc}$ (satisfying inhomogeneous conditions at the port) and the remaining two-component field $\vec{H}$, the boundary value problem for $\vec{H}$ being

$$\nabla\times\nabla\times\vec{H} - k^2\vec{H} = -(\nabla\times\nabla\times\vec{H}_{inc} - k^2\vec{H}_{inc}) \tag{26}$$
$$\hat{n}\cdot\vec{H}\,|_{\partial\Omega} = 0$$

($\hat{n}$ is the unit normal vector).

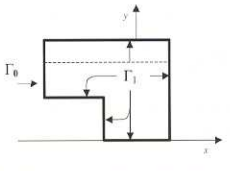

Fig.13. Waveguide with perfectly conducting walls

The weak formulation of problem (26) is

$$\int_\Omega \left[(\nabla\times\vec{H})\cdot(\nabla\times\vec{H}') - k^2\vec{H}\cdot\vec{H}'\right]dV = -\int_\Omega \left[(\nabla\times\vec{H}_0)\cdot(\nabla\times\vec{H}') - k^2\vec{H}_0\cdot\vec{H}'\right]dV \tag{27}$$

In this example, the domain is covered by three overlapping patches (Fig.14).

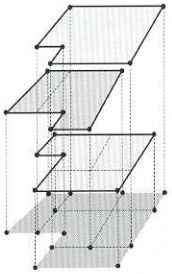

Fig. 14. Three overlapping patches (lifted for clarity)

We restrict both $\vec{H}$ and $\vec{H}'$ to the finite-dimensional space V formed by functions $v_{mn\,pq} = (g_m\varphi_n)\nabla(g_p\varphi_q)$. The boundary condition is imposed on the normal components of the basis functions:

$$\hat{n}\cdot v_{mn\,pq}\big|_{\partial\Omega} = 0 \tag{28}$$

This condition is fulfilled automatically if

$$\hat{n}\cdot(g_i\nabla g_j)\big|_{\partial\Omega} = 0 \tag{29}$$

Local functions g_i on each patch are polynomials of order up to 8. These polynomials satisfy the homogeneous condition (29) on the domain boundaries. There are 2,025 degrees of freedom altogether. The field distribution (Fig.15, 16) is in a good agreement with the PUM modeling on the same patches (Fig. 17).

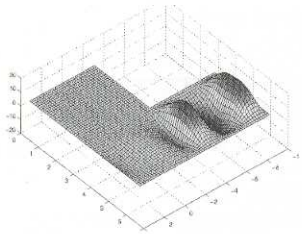

Fig.15. The x-component of the magnetic field.

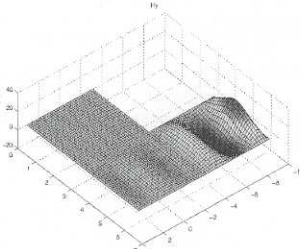

Fig.16. The y-component of the magnetic field.

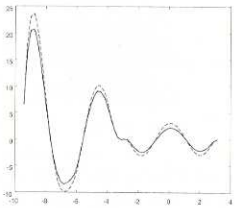

Fig.17. The y-component of the magnetic field along the dashed line in Fig. 12. Dashed line – scalar elements by partition of unity; solid line – vector elements.

V. The Price

The price to be paid for the high generality and flexibility of generalized FE-partition of unity methods is related to (adaptive) numerical integration over the intersections of arbitrary patches in the Galerkin method, to possible ill-conditioning of the algebraic system, and consequently to the overall numerical efficiency. In practice, we expect that reasonable compromises will be adopted, whereby the structure of the overlapping patches and the systems of approximating functions will remain manageable.

Conclusion

The generalized FE-partition of unity method is highly flexible and allows one to approximate different types of local behavior of the solution (including singularities and boundary layers) and seamlessly merge these local approximations. The proposed novel *vector* spaces generalize tetrahedral Whitney-Nedelec edge elements to a system of arbitrary overlapping patches. This class of methods (in both scalar and vector cases) may prove useful for various problems with singularities, boundary layers, and motion.

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Igor A. Tsukerman and Leonid B. Proekt are with the Electrical Engineering Department, The University of Akron, OH 44325-3904, USA
E-mail: itsukerman@uakron.edu