Finite Formulation of Electromagnetic Field

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October 16, 2000

Abstract

This paper shows that the equations of electromagnetism can be directly obtained in a finite (=discrete) form, i.e. without going throught the differential formulation. This finite formulation is a natural extension of the network theory to electromagnetic field and it is suitable for computational electromagnetics.

1. Introduction

Computational electromagnetism requires the transformation of Maxwell's differential equations into algebraic equations. This is obtained using one of the many discretization methods such as Finite Difference Method (FDM); Finite Difference in Time Domain (FDTD); Finite Element Method (FEM); Boundary Element Method (BEM); Edge Element Method (EEM); etc.

Even in Finite Volume Method (FVM) and in Finite Integration Theory (FIT), which both use use an integral formulation, one use integrals of field functions the last being indispensable ingredients of the differential formulation.

A question arises: *is it possible to express the laws of electromagnetism directly by a set of algebraic equations, instead of obtaining them from a discretization process applied to differential equations?*

The answer is: *Yes, it is possible, it is easy and it can be immediately used for computation.* We are introducing an alternative to differential formulation. In order to display field laws in a finite formulation, we must make use of two different classifications of physical quantities.

1.1. Configuration, source and energy variables

A first classification is based on the *role* that every physical variable plays in a theory. This leads to three classes of variables: *configuration*, *source* and *energy* variables as shown in Table 1. In every physical field we can find:

- *Configuration variables* that describe the configuration of the field or of the system. These variables are linked one another by operations of sum, difference, limit, derivative and integral.
- *Source variables* that describe the sources of the field or the forces acting on the system. These variables are linked one another by operations of sum, difference, limit, derivative and integral.
- *Energy variables* that are obtained as the product of a configuration variable by a source one. These variables are linked one another by operations of sum, difference, limit, derivative and integral.

Table 1: The three classes of variables of electromagnetism.

CONFIGURATION VARIABLES:	
gauge function	χ
electric voltage (impulse)	$U,(\mathcal{U})$
e.m.f. (impulse)	$E,(\mathcal{E})$
electric field vector	$\mathbf{E}$
magnetic flux	Φ
electric potential (impulse)	$V,(\mathcal{V})$
magnetic vector potential	$\mathbf{A}$
magnetic induction	$\mathbf{B}$
SOURCE VARIABLES:	
electric charge content	Q^{c}
electric charge flow	Q^{f}
electric current density	$\mathbf{J}$
electric flux	Ψ
electric induction	$\mathbf{D}$
magnetic field strength	$\mathbf{H}$
magnetic voltage (impulse)	$U_m,(\mathcal{U}_{\mathrm{m}})$
m.m.f. (impulse)	$F_m,(\mathcal{F}_{\mathrm{m}})$
magnetic scalar potential	V_{m}
dielectric polarization	$\mathbf{P}$
magnetization	$\mathbf{M}$
ENERGY VARIABLES:	
work, heat	W,Q
electric energy density	w_{e}
magnetic energy density	w_{m}
Poynting vector	$\mathbf{S}$
electromagnetic momentum	$\mathbf{G}$
momentum density	$\mathbf{g}$
electromagnetic action	A

1.2. Global and field variables

A second classification distinguish between *global variables* and *field variables*. Global variables are those commonly called *integral variables*: we avoid the last expression because it refers to an integration process performed on field variables.

We must emphasize that physical measurements deal mainly with em *global variables*: we can directly measure voltages, fluxes, charge content and charge flows, not field vectors. Field variables are needed in a differential formulation because the very notion of derivative refers to point functions. On the contrary a global quantity refers to a system, to a space or time element, such as lines, surfaces, volumes, intervals, i.e. it is a *domain function*. Thus a flow meter measures the electric charge that crosses a given *surface* in a given *time interval*. A magnetic tensiometer measures the magnetic voltage impulse referred to a line and a time interval. The corresponding physical quantities are associated with extensive space and extensive time elements, not only with points and instants

One fundamental advantage of global variables is that they are continuous through the separation surface of two materials while the field variables are discontinuous. This implies that the differential formulation is restricted to regions of material homogeneity. When this is not the case it is necessary to break the domain in subdomains, one for each material, and to introduce jump conditions. If we think about the great number of different materials which we can find in a real device, we can see that the idealization required by differential formulation is too restrictive for the present technology.

This shows that the *differential formulation imposes derivability conditions on field functions that are restrictive from the physical point of view*. On the opposite a direct finite formulation based on global variables can hold material discontinuities, i.e. does not add regularity conditions to those requested by the physical nature of the problem.

To help the reader accustomed to think in terms of traditional field variables, we first examine the global variables corresponding to traditional field functions:

$$\begin{array}{cccccc} \rho & \mathbf{J} & \mathbf{B} & \mathbf{D} & \mathbf{E} & \mathbf{H} \\ Q^{\mathrm{c}} & Q^{\mathrm{f}} & \Phi & \Psi & U & U_{\mathrm{m}} \end{array} \tag{1}$$

These are collected in Table 2. This table shows that integral variables arise by integration of field functions on space and time domains i.e. lines, surfaces, volumes and time intervals. The time integral of a physical variable, say U, will be called its *impulse* and will be denoted by the corresponding calligraphic letter, say $\mathcal{U}$.

It is remarkable that all *global* configuration variables have the dimension of a magnetic flux and that all *global* source variables have the dimension of a charge. The product of a global configuration variable and a global source variable has the dimension of an action (energy x time).

1.3. Cell complexes

It is well known that there is a strict link between physics and geometry. In spite of this, it is not usually stressed that global physical variables are naturally associated with space and time elements, i.e. points, lines, surfaces, volumes, instants and intervals.

In the differential formulation points play a fundamental role: field functions are point functions. In order to associate points with numbers we then introduce *coordinate systems*.

In finite formulation we need to consider not only points ($\mathbf{P}$) but also lines ($\mathbf{L}$), surfaces ($\mathbf{S}$) and volumes ($\mathbf{V}$). We shall call these entities *space elements*.

Table 2: Integral (=global) physical variables of electromagnetism and corresponding field functions.

configuration variables (unit: weber)	*source variables (unit: coulomb)*
$\mathcal{V} = \int_{\mathbf{T}} V \,\mathrm{d}t$	$Q^{\mathrm{c}} = \int_{\tilde{\mathbf{V}}} \rho \,\mathrm{d}V$
$p = \int_{\mathbf{L}} \mathbf{A} \cdot \mathrm{d}\mathbf{L}$	$Q^{\mathrm{f}} = \int_{\tilde{\mathbf{T}}} \int_{\tilde{\mathbf{S}}} \mathbf{J} \cdot \mathrm{d}\mathbf{S} \,\mathrm{d}t$
$\mathcal{U} = \int_{\mathbf{T}} \int_{\mathbf{L}} \mathbf{E} \cdot \mathrm{d}\mathbf{L} \,\mathrm{d}t$	$\Psi = \int_{\tilde{\mathbf{S}}} \mathbf{D} \cdot \mathrm{d}\mathbf{S}$
$\Phi = \int_{\mathbf{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S}$	$\mathcal{U}_{\mathrm{m}} = \int_{\tilde{\mathbf{T}}} \int_{\tilde{\mathbf{L}}} \mathbf{H} \cdot \mathrm{d}\mathbf{L} \,\mathrm{d}t$

The natural substitute of coordinate systems are *cell complexes*. They exhibit vertices, edges, faces and cells that are representative of the four spatial elements.

Once we have introduced a cell complex we can consider a *dual* complex. If we make use of a simplicial complex as a primal complex, then the commonst choices are either the barycentres of every simplex or the circumcentres (in 2D) and the circumspheres (in 3D). *In this paper we consider only circumcentres and circumspheres*. Since the straight line connecting the circumcentres of two adjacent simplexes in 2D is orthogonal to the common edge the dual polygon thus obtained has its sides orthogonal to the common edge. This is called *Voronoi polygon* in 2D and *Voronoi polyhedron* in 3D. The circumcentres have the disadvantage that for triangles with obtuse angles they lie outside the triangle. This is inconvenient when the circumcentre of one obtuse triangle goes beyond the one of the adjacent triangle with the common sides. This is avoided when the primal triangulation satisfies the *Delaunay condition*. This leads us to consider only Delaunay-Voronoi complexes, as we shall do in this paper. While in coordinate systems it is preferable to deal with orthogonal coordinate systems, in a simplicial complex it is preferable to deal with a Delaunay complex and its associated Voronoi complex as dual, as shown in Figure 1.

Figure 1: Three kinds of dual entities: those of the first and second columns satisfy the orthogonality condition.

1.4. Inner and outer orientation

The notions of inner and outer orientation of a space element play a pivotal role in electromagnetism as well as in all physical theories.

Inner orientation. Points can be oriented to be ``sources" or ``sinks". The notion of source and sink, borrowed from fluid dynamics, can be used to define an inner orientation of points because it allows to maintain the notion of incidence number from lines and points. A line is endowed with inner orientation when we choose a direction on the line. A surface is endowed with inner orientation when its boundary has an inner orientation. A volume is endowed with inner orientation when its boundary is so. The four space elements endowed with inner orientation will be denoted $\mathbf{P}, \mathbf{L}, \mathbf{S}, \mathbf{V}$.

Outer orientation. In order to write a balance we need a notion of exterior of a volume, because we speak of charge *contained* inside the volume. This is usually done by fixing outwards or inwards normals to the volume boundary. A surface is endowed with outer orientation when one of its faces has been chosen as positive and the other as negative: this is equivalent to fixing the direction of an arrow crossing the surface from the negative to the positive face. We need the outer orientation of a surface when we consider a flow crossing the surface. A line is endowed with outer orientation when a direction of rotation *around* the line has been defined: for example let us think to the rotation of the polarization plane of a ligth beam. A point is endowed with outer orientation when all line segments with their origin in the point have an outer orientation. Let us think, for example, of the sign of the scalar magnetic potential of a coil at a point: it depends on the direction of the current in the coil.

The four space elements endowed with outer orientation will be denoted $\tilde{\mathbf{P}}, \tilde{\mathbf{L}}, \tilde{\mathbf{S}}, \tilde{\mathbf{V}}$

A cell complex and its dual enjoy a peculiar property: once the vertices, edges, faces and cells of the primal complex have been endowed with inner orientation, then an outer orientation on the cells, faces, edges and vertices of its dual is induced. It follows that a pair formed by a cell complex and its dual is the natural frame to exhibit all space elements and their orientations.

1.5. Cell complex in time

Let us consider a given interval on the time axis and divide it into small time intervals, as shown in Figure 2. A primal instant $\mathbf{I}$ is oriented as sink, such as space points. A primal interval $\mathbf{T}$ will be

Figure 2: Cell complex on time axis and its dual.

endowed with an *inner* orientation, i.e. it is oriented towards increasing time. If we choose an instant inside every time interval we obtain a dual instant $\tilde{\mathbf{I}}$ that is authomatically endowed with an *outer* orientation. The interval $\tilde{\mathbf{T}}$ between two dual instants is a dual interval and it is automatically endowed with an *outer* orientation.

In this fashion, every instant of the primal complex there corresponds an interval of the dual and every interval of the primal corresponds an instant of the dual. Thus we have the correspondence $\mathbf{I} \leftrightarrow \tilde{\mathbf{T}}$ and $\tilde{\mathbf{I}} \leftrightarrow \mathbf{T}$ and this is a duality map.

1.6. Global variables and space-time elements

From the analysis of a great number of physical variables of classical fields we can infer [3], [4] that

> FIRST PRINCIPLE: *Global configuration variables are associated with space and time elements which are endowed with inner orientation while global source variables are associated with space and time elements which are endowed with outer orientation.*

This principle gives the reason for which differential forms are used in electromagnetism [5].

The reason for associating source variables with outer orientation is that source variables are used in balance equations and a balance requires a volume with an outer orientation (outwards or inwards normals). In short:

configuration variables	source variables
inner orientation	outer orientation

This principle offers a rational criterion for which

Table 3: Global variables of electromagnetism.

global physical variable	*symbol*
electric potential impulse	$V[\mathbf{T},\mathbf{P}]$
electric voltage impulse	$U[\mathbf{T},\mathbf{L}]$
e.m.f. impulse	$E[\mathbf{T},\mathbf{L}]$
magnetic flux	$\Phi[\mathbf{I},\mathbf{S}]$
electrokinetic momentum	$p[\mathbf{I},\mathbf{L}]$
magnetic potential impulse	$V_{\mathrm{m}}[\tilde{\mathbf{T}},\tilde{\mathbf{P}}]$
magnetic voltage impulse	$U_{\mathrm{m}}[\tilde{\mathbf{T}},\tilde{\mathbf{L}}]$
m.m.f. impulse	$F_{\mathrm{m}}[\tilde{\mathbf{T}},\tilde{\mathbf{L}}]$
electric flux	$\Psi[\tilde{\mathbf{I}},\tilde{\mathbf{S}}]$
electric charge content	$Q^{\mathrm{c}}[\tilde{\mathbf{I}},\tilde{\mathbf{V}}]$
electric charge flow	$Q^{\mathrm{f}}[\tilde{\mathbf{T}},\tilde{\mathbf{S}}]$

global variables of every physical theory can be associated to space and time elements and is useful in computational electromagnetism. Figure 3 shows this association for physical variables of electromagnetism.

It is important to note that each one of the six variables of Eq.(1) admits an operational definition [7]. We can say that *the role of the dual complex is to form*

a reference structure to which source variables can be referred.

The space and time association of global electromagnetic variables is summarized in Table 3.

Figure 3: Global physical variables are associated with elements of a cell complex and its dual.

1.7. Physical laws and space-time elements.

The first principle states that all global physical variables refer to oriented space and time elements. From the analysis of a great number of physical variables of classical fields one can infer [3], [4] the

> SECOND PRINCIPLE: *In every physical theory there are physical laws that link global variables referred to an oriented space-time element, say* $\mathbf{\Omega}$*, with others referred to its oriented boundary, say* $\partial\mathbf{\Omega}$

The field laws of electromagnetism satisfy this principle.This principle motivates the role of exterior differential on differential forms [5].

1.8. Field laws in global form

Experiments lead us to infer the following four laws of electromagnetism (refer to Figure 4):

Figure 4: Field laws are associated with space and time elements.

- The *electromotive force impulse* E referred to the *boundary* $\partial\mathbf{S}$ of a surface endowed with inner orientation during a time interval $\mathbf{T}$ is opposite to the *magnetic flux* Φ variation across the surface $\mathbf{S}$ in the same interval (Faraday).

- The *magnetic flux* Φ referred to the *boundary* $\partial\mathbf{V}$ of a volume $\mathbf{V}$ endowed with inner orientation vanishes at any instant $\mathbf{I}$ (Gauss).

- The *magnetomotive force impulse* F_{m} referred to the *boundary* $\partial\tilde{\mathbf{S}}$ of a surface endowed with outer orientation in a time interval $\tilde{\mathbf{T}}$ is equal to the sum of the *electric charge flow* Q^{f} across the surface $\tilde{\mathbf{S}}$ in that time interval and the *electric flux* Ψ variation across the surface $\tilde{\mathbf{S}}$ in such interval (Ampère-Maxwell).

- The *electric flux* Ψ across the *boundary* $\partial\tilde{\mathbf{V}}$ of a volume endowed with outer orientation at any instant $\tilde{\mathbf{I}}$ is equal to the *electric charge* Q^{c} contained inside the volume $\tilde{\mathbf{V}}$ at such instant (Gauss).

In formulae:

$$
\begin{aligned}
&\mathsf{E}[\mathbf{T},\partial\mathbf{S}]=\Phi[\mathbf{I}^{-},\mathbf{S}]-\Phi[\mathbf{I}^{+},\mathbf{S}]\\
&\Phi\left[\mathbf{I},\partial\mathbf{V}\right]=0\\
&F_{\mathrm{m}}[\tilde{\mathbf{T}},\partial\tilde{\mathbf{S}}]=\Psi[\tilde{\mathbf{I}}^{+},\tilde{\mathbf{S}}]-\Psi[\tilde{\mathbf{I}}^{-},\tilde{\mathbf{S}}]+Q^{\mathrm{f}}[\tilde{\mathbf{T}},\tilde{\mathbf{S}}]\\
&\Psi[\tilde{\mathbf{I}},\partial\tilde{\mathbf{V}}]=Q^{\mathrm{c}}[\tilde{\mathbf{I}},\tilde{\mathbf{V}}]
\end{aligned}
\qquad (2)
$$

Equations (2) are the four laws of electromagnetism in the finite formulation we were searching for [7]. These algebraic equations enjoy the following properties:

- they link physical variables of the same kind, i.e. configuration variables with configuration variables and source variables with source variables;

- they are valid in *whatever medium* and then they are free from any material parameter;

- they are valid for *whatever surface* and *whatever volume* and then they are valid both in the large and in the small;

- they *do not involve metrical notions*, i.e. lengths, areas, measures of volumes and durations are not required.

These properties show that field equations do not require infinitesimal space elements and then they are not responsible for differential formulation.

These four laws can be expressed *without* recourse to cell complexes. They are the finite formulation of field laws corresponding to Maxwell's equations.

Local formulation. In order to obtain a set of algebraic equations we must introduce a cell complex and its dual. All elements must be labelled. Let $\mathbf{l}_{\alpha}$, $\mathbf{s}_{\beta}$ be the edges and faces of the primal complex respectively; $\tilde{\mathbf{l}}_{\beta}$ and $\tilde{\mathbf{s}}_{\alpha}$ the same for the dual complex; $c_{\beta\alpha}, d_{\kappa\beta}, \tilde{c}_{\alpha\beta}, \tilde{d}_{h\alpha}$, the incidence numbers.

When equations (2) are applied to the corresponding cells of the two complexes, we obtain a *local* form of the field equations of the electromagnetic field in a finite setting. We can write

$$
\begin{aligned}
&\sum_{\alpha} c_{\beta\alpha} E\left[\boldsymbol{\tau}_{n+1},\mathbf{l}_{\alpha}\right]+\Phi\left[\mathbf{t}_{n+1},\mathbf{s}_{\beta}\right]-\Phi\left[\mathbf{t}_{n},\mathbf{s}_{\beta}\right]=0\\
&\sum_{\beta} d_{\kappa\beta}\,\Phi\left[\mathbf{t}_{n},\mathbf{s}_{\beta}\right]=0\\
&\sum_{\beta} \tilde{c}_{\alpha\beta} F_{\mathrm{m}}\left[\tilde{\boldsymbol{\tau}}_{n},\tilde{\mathbf{l}}_{\beta}\right]-\Psi\left[\tilde{\mathbf{t}}_{n+1},\tilde{\mathbf{s}}_{\alpha}\right]+\Psi\left[\tilde{\mathbf{t}}_{n},\tilde{\mathbf{s}}_{\alpha}\right]=Q^{\mathrm{f}}\left[\tilde{\boldsymbol{\tau}}_{n},\tilde{\mathbf{s}}_{\alpha}\right]\\
&\sum_{\alpha} \tilde{d}_{h\alpha}\,\Psi\left[\tilde{\mathbf{t}}_{n},\tilde{\mathbf{s}}_{\alpha}\right]=Q^{\mathrm{c}}\left[\tilde{\mathbf{t}}_{n},\tilde{\mathbf{v}}_{h}\right]
\end{aligned}
\qquad (3)
$$

For computational purposes it is useful to perform the following changes of symbols: $\Phi\left[\mathbf{t}_n,\mathbf{s}_\beta\right]\rightarrow\Phi_\beta^n$; $\Psi\left[\tilde{\mathbf{t}}_n,\tilde{\mathbf{s}}_\alpha\right]\rightarrow\Psi_\alpha^{n+1/2}$ etc. In particular the two evolution equations can be written as ($\tilde{c}_{\alpha\beta}=c_{\beta\alpha}$, see [7])

$$\Phi_\beta^{n+1}=\Phi_\beta^n-\sum_\alpha c_{\beta\alpha}E_\alpha^{n+1/2}$$
$$\Psi_\alpha^{n+1/2}=\Psi_\alpha^{n-1/2}+\sum_\beta c_{\beta\alpha}\left(F_{\mathrm{m}}\right)_\beta^n-\left(Q^{\mathrm{f}}\right)_\alpha^n \tag{4}$$

This gives rise to the *leapfrog algorithm*.

1.9. Constitutive laws

The equations that link the source variables with configuration ones are *constitutive* or *material* equations. In a region of *uniform* field the three constitutive equations of electromagnetism in finite form *and for orthogonal duals* $\tilde{\mathbf{s}}_\alpha\perp\mathbf{l}_\alpha$ are

$$\frac{\Psi\left[\tilde{\mathbf{t}}_n,\tilde{\mathbf{s}}_\alpha\right]}{\tilde{s}_\alpha}=\in\frac{E\left[\boldsymbol{\tau}_n,\mathbf{l}_\alpha\right]}{\tau_n\,l_\alpha}$$
$$\frac{\Phi\left[\mathbf{t}_n,\mathbf{s}_\beta\right]}{s_\beta}=\mu\frac{F_{\mathrm{m}}\left[\tilde{\boldsymbol{\tau}}_n,\tilde{\mathbf{l}}_\beta\right]}{\tilde{\tau}_n\,\tilde{l}_\alpha}$$
$$\frac{Q^{\mathrm{f}}\left[\tilde{\boldsymbol{\tau}}_n,\tilde{\mathbf{s}}_\alpha\right]}{\tilde{\tau}_n\,\tilde{s}_\alpha}=\sigma\frac{1}{2}\left(\frac{E\left[\boldsymbol{\tau}_n,\mathbf{l}_\alpha\right]}{\tau_{n+1}\,l_\alpha}+\frac{E\left[\boldsymbol{\tau}_{n+1},\mathbf{l}_\alpha\right]}{\tau_{n+1}\,l_\alpha}\right) \tag{5}$$

in which $\tau_n,\tilde{\tau}_n,l_\alpha,\tilde{l}_\beta,s_\beta,\tilde{s}_\alpha$, are the extensions of the corresponding elements. To explain the particular form of Ohm's law let us remark that while the electric charge Q^{f} is referred to dual intervals, the electric tension impulse E is referred to primal ones. These equations are valid if cells are cubes or they belong to a Delaunay-Voronoi complex, as shown in Figure 1. In these cases 1-cells of the dual are orthogonal to the primal 2-cells and viceversa. The orthogonality condition is not necessary, and also barycenters may be used to define the dual complexes [2].

The main properties of constitutive laws are:

- They are valid in regions in which the field is *uniform* because they are tested under such conditions.
- They link a variable referred to a $p-$cell of a complex with the dual $(n-p)-$cell of the dual complex. This geometrical property is not apparent in differential formulation.
- They contain *material parameters.*
- They require *metrical notions* such as length, areas, volumes and orthogonality.

While field equations in finite form describe the corresponding physical laws *exactly*, the constitutive ones in finite form describe the corresponding physical laws *approximately* because they are validated in regions of uniform field.

Combining the two set of equations, the field and the constitutive ones, we obtain a set of algebraic equations.

At this point we have two alternatives:

1. To perform the limit process on the dimensions of the cells in order to obtain a uniform field at the limit. This is the traditional way that leads to *differential* formulation.
2. To make the approximation that the field is uniform inside every cell. In this way we obtain a *finite* formulation.

The last choice is the one pursued in this paper.

1.10. Comparison with other methods.

Since the distinction between the two classes of physical variables is not commonly done and since it is not recognized that two kind of orientations are needed, it follows that the differential formulation uses only one kind of *infinitesimal* cells. It follows that FEM, arising from a discretization of differential equations, ignores the need of two dual complexes and therefore uses only one complex formed by the *elements*. In 1966 Yee [10], using a cartesian complex and an appropriate choice of points at which the various field components are to be evaluated, opened the way to the introduction of a pair of dual complexes called by Weiland [8], the *electric* and *magnetic* grids. Nevertheless the two complexes were not justified by physical considerations but only by computational advantages. In the realm of differential forms the two kinds of orientation give rise to *normal* and *twisted* forms [1, p.183].

1.11. Conclusion

It is possible to express both field and constitutive laws of electromagnetism in a finite (i.e. algebraic) form starting directly from experimental facts, i.e. avoiding the passage through the differential formulation. This provide a set of algebraic equations that can be directly used in computational electromagnetism. The unique approximation lies in the hypotesis that inside every simplex the field is uniform. If we use an adaptive simplicial complex we can reduce the size of the simplexes in the regions where global variables undergo large variations. Doing so, the hypotesis of uniformity inside every simplex can be under control.

The finite formulation can easily handle inhomogeneous and anisotropic materials (this is of particular interest when dealing with absorbing boundary conditions) and can easily handle concentrated sources because infinities do not appear.

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