Technical article

APPLICATION OF DIFFERENTIAL FORMS IN THE FINITE ELEMENT FORMULATION OF ELECTROMAGNETIC PROBLEMS

INTRODUCTION

In many physical problems, we have to study the integral of a quantity over a p-dimensional manifold in an n-dimensional Euclidean space. In the study of these integrals, it is important to know in what manner the integral over the manifold depends on the position of the manifold in the Euclidean space. The purpose of differential forms is to study these integrals in a broad setting of geometry, topology, algebra and analysis. In the calculus of differential forms, the local field quantities are associated with the geometric and topological property of the manifold. One works with the integral quantities instead of local scalar or vector fields. The integral quantities have usually physically measurable units and are more interesting in the engineering applications. In addition, the fact that differential forms carry the geometric information of the manifold makes clear the difference between vectors related to the line integral such as the electric or magnetic field intensity and vectors related to the surface integral such as the flux or current density, which are ambiguously defined in vector algebra. Association of field quantities with the topological property of the manifold makes more understandable the modelling of multiply connected domains. In one word, the calculus of differential forms presents numerous advantages compared to the conventional vector or tensor algebra and turn out to be more efficient in the description of physical problems [1-4].

In electromagnetics, application of differential forms enables a simple and clear representation of Maxwell's equations. With the help of the exterior differential operator and Hodge operator founded in the calculus of differential forms, Maxwell equations can be presented in a formal flow diagram known as Tonti's diagram [5]. This diagram shows clearly the duality between the two systems of Maxwell equations. It is helpful for the derivation of dual formulations in the computation of electromagnetic fields and allows a better understanding of dual approximation schemas.

Application of differential forms in the numerical computation of electromagnetic fields was notably marked by the appearance of Whitney elements [6]. Whitney elements consider the differential forms as degrees of freedom. Their advantages are principally their capacity of allowing natural discretization of the systems with appropriate continuity of scalar and vector variables. With the help of differential forms, the high order curl-conformal and div-conformal elements proposed in [7] can be put in a more clear circumstance.

One of the important notions in the calculus of differential forms is De Rham's complex [1-2]. De Rham's complex shows the relation between the spaces of differential forms of different degrees. Application of De Rham's complex in electromagnetism shows the mathematical structure involved in electromagnetic theory. It allows a deep understanding in the study of differential operators grad, curl, div with their kernels and their co-domains. De Rham's cohomology groups reveal the global topological property of the study domain. In three dimensions, they are related to the loops and the cavities in the study domain and have particular interest in the study of problems in multiply connected regions. De Rham's complex can also be used to describe the relation between the functional spaces of differential form based elements of different degrees [8]. It illustrates not only the inclusion property of these elements, but also is helpful for the determination of the rank of the matrix system of an electromagnetic field problem and enables an easy understanding of the gauge condition in the case of working with potential variables.

This paper gives a short presentation of the application of differential forms in the finite element computation of electromagnetic field. We give first a brief introduction of differential forms, in particular the notion of exact and closed forms, and De Rham's complex. Maxwell equations in electromagnetics are then written in terms of differential forms and represented by Tonti's diagram. The very useful De Rham's complex is used to describe the relation between the functional spaces (domains of differential operators) in electromagnetics. The discrete spaces, i.e. elements based on differential forms of different degrees, as well as the discrete spaces of De Rham's cohomology groups for the modelling of cuts and links in the case of multiply connected regions, are presented. Their relation is clarified by using De Rham's complex. As an application, dual formulations of eddy current problems in the case of non-trivial study domain are presented. The rank of the matrix system is determined and the gauge condition in the case of working with vector potentials is discussed with the help of De Rham's complex.

We intentionally use alternately the vector notation and the differential form notation, whenever there is no confusion, in order to have a comparison of two calculus.

DIFFERENTIAL FORMS AND DE RHAM'S COMPLEX

Differential forms are expressions on which integration operates [1]. A differential form of degree p, or a p-form, is an expression where the integral is performed over a manifold of dimension p in a space of dimension n, i.e. the integrand of a p-fold integral in an n-dimensional space. The differential forms can be introduced according to the dimension of the manifold on which the variable is integrated. In electromagnetics, a scalar potential is a 0-form; the circulation of a vector potential or a (electric or magnetic) field intensity along a small segment is a 1-form, a flux (or current) across a small area is a 2-form and charges contained in a small volume are a 3-form. For example, the electric field is identified with a 1-form, its expression is $\mathbf{e}\cdot d\mathbf{l}$, representing the electromotive force along a short line. Another example is the current across a small surface, $\mathbf{j}\cdot d\mathbf{s}$, which defines a 2-form. An example of 3-form is the charges containing in a small volume, given by ρdv. We can note in particular that, the vectors related to line integral such as electric and magnetic fields, vector potentials, and the vectors related to surface integral such as current and flux densities correspond, respectively, to differential forms of degree 1 and 2. These vectors, different on their nature, cannot be distinguished with the vector presentation.

Differential forms operate in exterior algebra. Exterior (wedge) product of a p-form ω and a q-form υ produces a (p+q)-form with the skew symmetry property: $\omega\wedge\upsilon = (-1)^{pq}\upsilon\wedge\omega$, (where p+q<n, n denotes the dimension of space). Two other operators permit transformation of a differential form of one degree to the other. One is the exterior derivation 'd'. Application of this operator to a differential form leads to a form of higher degree. In three dimensions, it generalizes and unifies the familiar 'grad', 'curl' and 'div' operators of vector algebra. The other operator is the star (Hodge) operator '*'. It transforms a p-form to an (n-p)-form. We will see later that this operator corresponds to the constitutive laws in physic problems.

Let $D^p(M)$ be the set of p-forms defined on an n-dimensional differentiable manifold M. The inclusion property $dD^p(M) \subset D^{p+1}(M)$ holds. This property is represented by a sequence called De Rham's complex [2].

$$D^0 \xrightarrow{d} \dots D^p \xrightarrow{d} D^{p+1} \dots \xrightarrow{d} D^n$$

A form ω is said to be closed if $d\omega = 0$. A form ω is said to be exact if there exists a form υ (of one degree lower) such that $\omega = d\upsilon$. Since $d(d\upsilon) \equiv 0$, every exact form is closed. Can we also say 'every closed form is exact'? The answer is positive for a manifold not too complex (topologically trivial domains). But in general, the answer is negative. Let $Z^p(M)$ be the set of closed p-forms, $B^p(M)$ be the set of exact p-forms. We have in general $B^p(M) \subset Z^p(M)$. The complement of $B^p(M)$ in $Z^p(M)$, $\mathsf{H}^p(M) = Z^p(M) \setminus B^p(M)$ is called De Rham's p^{th} cohomology group. The property of $\mathsf{H}^p(M)$ depends on the topology of the manifold [2]. The dimension of $\mathsf{H}^p(M)$ is finite and called p^{th} Betti number of M. In particular, H^0 is equal to the number of connected components of M. H^p (for p>0) vanishes if M is topologically trivial. The spaces of exact forms and closed forms are related by the Hodge decomposition: $Z^p(M) = B^p(M) \oplus \mathsf{H}^p(M)$. Taking the Hodge decomposition into account, De Rham's complex can be shown in the form of Fig.1.

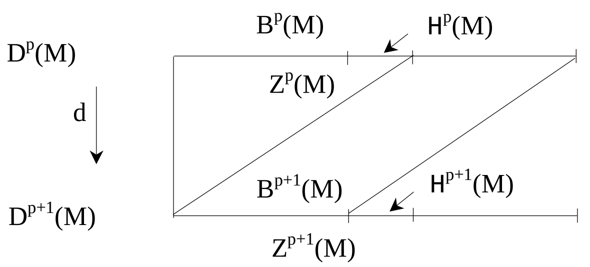

Fig.1. De Rham's complex showing relation of p^{th} cohomology groups.

De Rham's p^{th} cohomology group has the particular interest because they are related to the topology property of the manifold. In three dimensions, H^1 and H^2 are related to the loops and cavities in the manifold M [9-10]. The first and second Betti numbers, i.e. $\dim(\mathsf{H}^1)$ and $\dim(\mathsf{H}^2)$, correspond respectively to the number of loops and the number of cavities in M.

MAXWELL EQUATIONS IN TERMS OF DIFFRENTIAL FORMS

The fundamental equations of many physical problems can be put into a formal mathematical structure, they are classified by definition, balance and constitutive equations [5]. In electromagnetics, the Maxwell equations are put into two dual systems: Ampere's system and Faraday's system. In terms of differential forms, they are written as:

Faraday's system: $\quad \mathrm{d}e = -\partial_t b, \quad \mathrm{d}b = 0,$

Ampere's system: $\quad \mathrm{d}h = j + \partial_t d, \quad \mathrm{d}d = \rho$ (or $\mathrm{d}j = -\partial_t \rho$),

where e and h are 1-form electric and magnetic fields, b, d and j are 2-form magnetic, electric flux and current, ρ is a 3-form charge. The two dual systems are related by the Hodge (star) transformation, they are constitutive laws of the media:

$$d = \varepsilon{*}e, \qquad b = \mu{*}h, \qquad j = \sigma{*}e,$$

where ε, μ and σ are, respectively the permittivity, the permeability and the conductivity.

The 1-form electric field and magnetic field can also be expressed in terms of 1-form vector potentials a, t (or u) and the exterior derivative of 0-form scalar potentials v, ϕ :

$$e = -\partial_t a - \mathrm{d}v, \qquad h = t - \mathrm{d}\phi \; (h = \partial_t u - \mathrm{d}\phi).$$

Different scalar and vector variables written in terms of differential forms in the two dual systems of electromagnetic problems are shown in Table 1. It can be noted that differential forms are measurable quantities. Their units are also given in this table.

Table 1. Differential forms and their units in electromagnetic systems

differential forms	Faraday system		Ampere system	
0-form	v = v		$\phi = \phi$	
1-form	$e = \mathbf{e}\cdot\mathrm{d}\mathbf{l}$	$a = \mathbf{a}\cdot\mathrm{d}\mathbf{l}$	$h = \mathbf{h}\cdot\mathrm{d}\mathbf{l}$ $t = \mathbf{t}\cdot\mathrm{d}\mathbf{l}$	$u = \mathbf{u}\cdot\mathrm{d}\mathbf{l}$
2-form		$b = \mathbf{b}\cdot\mathrm{d}\mathbf{s}$	$j = \mathbf{j}\cdot\mathrm{d}\mathbf{s}$	$d = \mathbf{d}\cdot\mathrm{d}\mathbf{s}$
3-form				$\rho = \rho \, \mathrm{d}v$
Units	Volt	Weber	Ampere	Coulomb

With the help of the exterior differential and the Hodge (star) operators, Maxwell's equations can be represented in a formal flow diagram known as Tonti diagram [5]. The case of the eddy current problem ($\partial_t d = 0$, $\rho = 0$) is shown in Fig.2. The left hand side represents Faraday's equations and the unit of the forms is Weber or Volt. On the right hand side, we have the Ampere's equations and the forms take the unit Ampere. The two dual systems are related by the constitutive laws (the Hodge transformation). This diagram illustrates clearly the sequence and the duality of Maxwell equations. It is very helpful for the derivation of dual finite element formulations. They are obtained by performing a conformal approach of one system using appropriate elements (elements based on differential forms as described later) and by solving the other system using the weak variational principle (integration by parts).

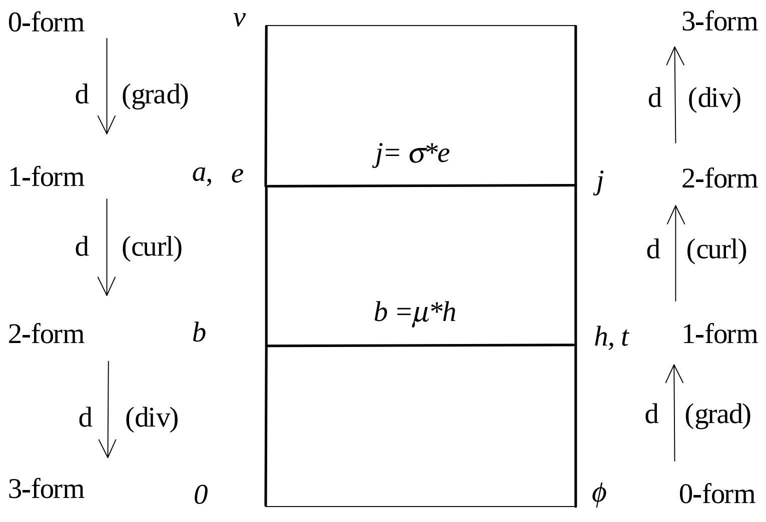

Fig.2. Tonti's dual flow diagram of eddy current problems

FUNCTIONAL SPACES

An electromagnetic field has a finite energy in a bounded region. This implies that the field quantities, at least at the stationary state, are square integrable. It follows that the Hilbert spaces of square integrable forms are natural for the electromagnetism. Let Ω be a connected and open set (a bounded domain) in a three-dimensional Euclidean space, not necessarily topologically trivial. The field quantities (differential p-forms) belong to the following functional spaces (domains of differential operator d):

$$H(d, \Omega) = \{\omega \in L_p^2(\Omega),\ d\omega \in L_{p+1}^2(\Omega)\}, \qquad p = 0, 1, 2.$$

Where $L_p^2(\Omega)$ is the space of square integrable p-forms over Ω. In terms of more familiar vector presentation, they are written as

$$H(\mathrm{grad}, \Omega) = \{\phi \in L^2(\Omega),\ \mathrm{grad}\, \phi \in IL^2(\Omega)\}$$
$$H(\mathrm{curl}, \Omega) = \{\mathbf{u} \in IL^2(\Omega),\ \mathrm{curl}\, \mathbf{u} \in IL^2(\Omega)\}$$
$$H(\mathrm{div}, \Omega) = \{\mathbf{u} \in IL^2(\Omega),\ \mathrm{div}\, \mathbf{u} \in L^2(\Omega)\}$$

where L^2 and IL^2 are the spaces of square integrable scalar and vector fields over Ω, respectively. These spaces have the following orthogonal decompositions:

$$H(\mathrm{grad}, \Omega) = \ker(\mathrm{grad}) \oplus \mathrm{cod}(\mathrm{grad})$$
$$H(\mathrm{curl}, \Omega) = \ker(\mathrm{curl}) \oplus \mathrm{cod}(\mathrm{curl})$$
$$H(\mathrm{div}, \Omega) = \ker(\mathrm{div}) \oplus \mathrm{cod}(\mathrm{div})$$

They are related by the following sequence (De Rham's complex):

$$H(\mathrm{grad}, \Omega) \xrightarrow{\mathrm{grad}} H(\mathrm{curl}, \Omega) \xrightarrow{\mathrm{curl}} H(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} L^2(\Omega)$$

We now consider the relation of De Rham's cohomology groups $\mathsf{H}_1(\Omega)$ and $\mathsf{H}_2(\Omega)$ with the above functional spaces. In three dimensions, $\mathsf{H}_1(\Omega)$ and $\mathsf{H}_2(\Omega)$ are related to the loops and cavities in the domain Ω. They have the following properties [11]:

$$\mathsf{H}_1(\Omega) = \{\mathbf{u} \in IL^2(\Omega) \mid \mathrm{curl}\, \mathbf{u} = 0,\ \mathrm{div}\, \mathbf{u} = 0,\ \mathbf{n}\cdot\mathbf{u}|_\Gamma = 0\}$$
$$\mathsf{H}_2(\Omega) = \{\mathbf{u} \in IL^2(\Omega) \mid \mathrm{curl}\, \mathbf{u} = 0,\ \mathrm{div}\, \mathbf{u} = 0,\ \mathbf{n}\times\mathbf{u}|_\Gamma = 0\}$$

where Γ is the boundary of Ω. The dimensions of H_1 and H_2 are finite and are equal to, respectively, the number of loops and the number of cavities in Ω. Remind that $\mathsf{H}_1(\Omega) \subset \ker(\mathrm{curl})$ but not in cod(grad) and $\mathsf{H}_2(\Omega) \subset \ker(\mathrm{div})$ but not in cod(curl), as described by the Hodge decomposition:

$$\ker(\mathrm{curl}) = \mathrm{cod}(\mathrm{grad}) \oplus \mathsf{H}_1(\Omega)$$
$$\ker(\mathrm{div}) = \mathrm{cod}(\mathrm{curl}) \oplus \mathsf{H}_2(\Omega)$$

The functional spaces and De Rham's cohomology groups are related by De Rham's complex as shown in Fig.3 [9]. It illustrates clearly the mathematical structure behind the electromagnetic theory and is very useful in the study of electromagnetic problems.

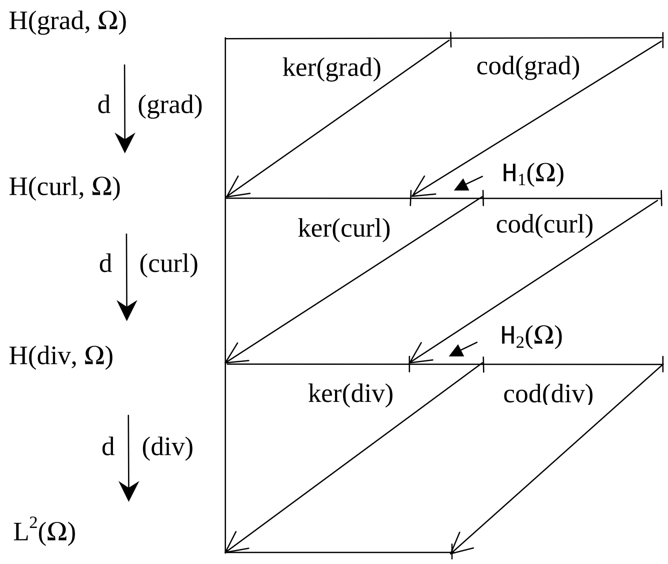

Fig.3. De Rham's complex showing mathematical structure of electromagnetic theory.

DISCRETE SPACES - ELEMENTS BASED ON DIFFERRENTIAL FORMS

In order to approximate correctly and naturally the previous functional spaces, suitable elements must be adopted. These elements are derived with the help of the calculus of differential forms. Let the domain Ω be paved with a tetrahedral (3-simplex) mesh. The number of nodes, edges, facets and tetrahedra are noted, respctively, by N, E, F and T.

We consider the general case of high order elements and note by $W_q^p(\Omega)$ the function space of q-order p-form elements over Ω. The case of the first order (q = 1) corresponds to the well know Whitney elements [6]. The functional space of p-form element W_q^p can be decomposed into a null space of differential operator $Z_q^p(\Omega)$ (set of closed forms) and a range space of differential operator $Y_q^p(\Omega)$: $W_q^p = Z_q^p \oplus Y_q^p$. The element W_q^p fulfils the following requirements: model correctly the null space Z_q^p of the differential operator and is complete to q-1 order in the range space Y_q^p under the differential operation.

The basis functions of p-form elements take the p-form bases: λ_i, $d\lambda_i$, $d\lambda_i \wedge d\lambda_j$, $d\lambda_i \wedge d\lambda_j \wedge d\lambda_k$, for p = 0, 1, 2, 3, respectively, and have polynomial coefficients, where λ_i is the barycentric coordinates of a point related the node i. The degrees of freedom of a p-form element are assigned to r-simplexes according to the order q ($p \le r \le \mathrm{Min}\{p+q-1, 3\}$) [8].

The p-form elements $W_q^p(\Omega)$ (p = 0, 1, 2 and 3) are discrete spaces of the functional spaces H(grad, Ω), H(curl, Ω), H(div, Ω) and $L^2(\Omega)$, respectively. The inclusion property $dW_q^p \subset W_q^{p+1}$ holds and is shown with the help of De Rham's complex in Fig. 4, where W_{H1} and W_{H2}, are, respectively, discrete spaces of the cohomology groups H_1 and H_2. The complex given in Fig.4 is the discrete form of the complex shown in Fig.3.

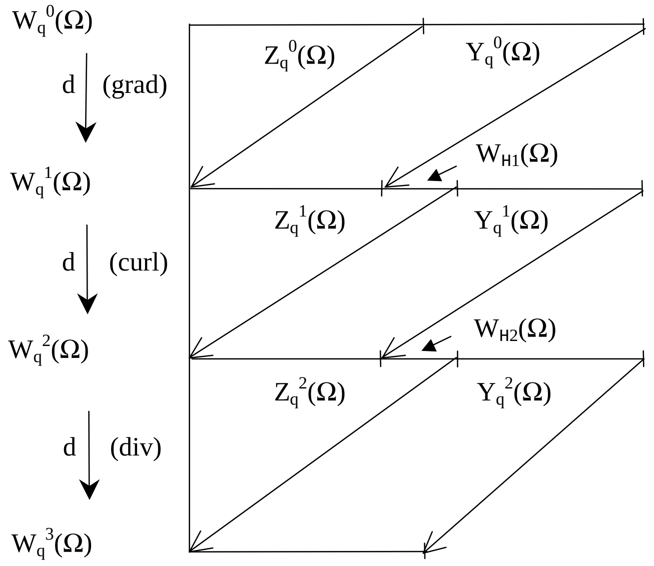

Fig.4.De Rham's complex showing the relation between p-form elements.

According to the analysis given in [8], the dimension of the function spaces $W_q^p(\Omega)$ and the dimension of the null spaces $Z_q^p(\Omega)$ are, respectively,

$$\dim(W_q^0) = N + E(q-1) + F(q-2)(q-1)/2 + T\,(q-3)(q-2)(q-1)/6$$
$$\dim(W_q^1) = E\,q + F(q-1)q + T\,(q-2)(q-1)q/2$$
$$\dim(W_q^2) = F\,q(q+1)/2 + T\,(q-1)q(q+1)/2$$
$$\dim(W_q^3) = T\,q(q+1)(q+2)\,/6$$

$$\dim(Z_q^0) = 1$$
$$\dim(Z_q^1) = E\,(q-1) + F(q-2)(q-1)/2 + T\,(q-3)(q-2)(q-1)/6 + N - 1 + N_{H1}$$
$$\dim(Z_q^2) = F\,q(q+1)/2 + T\,q(q+1)(2q-5)\,/6$$

where $N_{H1} = \dim(H_1(\Omega))$ and $N_{H2} = \dim(H_2(\Omega))$, are, respectively, number of loops and cavities in Ω.

An interest remark is that the famous Euler formula is embedded in De Rham's complex. In fact, according to De Rham's complex shown in Fig.4, the equality $\dim(Y_q^1) = \dim(Z_q^2) - N_{H2}$ holds. After the simplification, the Euler formula is straight foreword:

$$N - E + F - T = 1 - N_{H1} + N_{H2}$$

The basis functions of q-order p-form elements have to fulfill the conformity and unisolvence requirement, i.e. a p-form element must match the continuity condition of p-form field on the interface of adjacent elements, and the basis functions must be independent to provide an unique solution of the field equation. Various high order p-form elements have been developed in recent years [12-17]. They are divided into two main categories: the interpolatory basis [14-17] and the hierarchical basis [12-13]. Using the interpolatory bases, the degrees of freedom have usually a physical interpretation. However, the shape functions of different orders are all different. They are not adapted for mixing elements of different order in the same mesh in the case of *p*-version adaptive mesh generation. In addition, the separation of null space and range spacein the case of interpolatory basis is not easy. That makes the gauging condition, whenever necessary, difficult. Recent studies showed the advantages of hierarchical bases [18]. The hierarchy means that the basis functions of the high order elements include all basis functions of lower order element spaces. This property allows mixing of different order of elements in the same mesh without the difficulty of matching field continuities. It is helpful for mixed *h*- and *p*-version adaptive mesh generation or for the development of adaptive multigrid solvers. Another advantage of hierarchical bases, which is not much mentioned before, is that the basis functions belonging to the null space and to the range space, respectively, can be easily identified. This is an important advantage for the application of a gauge condition. The basis functions belonging to the null space can be easily removed.

We now consider W_{H1} and W_{H2}, the discrete spaces of De Rham's cohomology groups H_1 and H_2. They are related to the modelling of cuts and links in the case of a multiply connected domain. Detailed description about the modelling of H_1 and H_2 can be found in [10]. In this paper, the modelling of these spaces is presented in the context of De Rham's complex.

We note by Σ_i, i = 1, 2, …, N_{H1}, the cutting surfaces which make the domain simply connected, and by Λ_i, i = 1, 2, …, N_{H2} the links which connect all components of Γ. The cutting surface Σ_i cuts a layer of tetrahedra that we call the cutting domain and note by $\Omega_{\Sigma i}$. W_{H1} belongs to the null space of W_q^1 (curl free) and cannot be expressed by a gradient. It is nonzero in the domain $\Omega_{\Sigma i}$ and vanishes elsewhere. The dimension of W_{H1} equals to the number of cuts. There is one degree of freedom in each cutting domain. The 1-form elements in $\Omega_{\Sigma i}$ are correlated. Taking these conditions in to account, the space W_{H1} is spanned by the following constraint functions:

$$\mathbf{w}_{\Sigma i} = \sum_{e \in E_{\Sigma i}} \pm \mathbf{w}_e \, , \; i = 1, \ldots, N_{H1}.$$

here $\mathbf{w}_e$ are shape functions of the Whitney 1-form element and $E_{\Sigma i}$ is the set of edges across the cutting surface Σ_i. The sign depends on the orientation of edges with respect to the normal of cutting surfaces.

We note by $\Omega_{\Lambda i}$ the linking domain constituted of one bundle of tetrahedra crossed by the link Λ_i. W_{H2} is a null space of 2-form elements W_q^2 (having zero divergence), which cannot be presented by a curl field. It is non-null only in $\Omega_{\Lambda i}$. The degree of freedom is one per linking domain. The 2-form elements in $\Omega_{\Lambda i}$ are correlated. Shape functions of W_{H2} are hence given by

$$\mathbf{w}_{\Lambda i} = \sum_{f \in F_{\Lambda i}} \pm \mathbf{w}_f \, , \; i = 1, \ldots, N_{H2}$$

here $\mathbf{w}_f$ are basis functions of Whitney 2-form element, $\mathbf{w}_f \in W_1^2$, $F_{\Lambda j}$ the set of facets passed by Λ_j, and N_{H2} the number of links Λ_j. The sign depends on the orientation of facets with respect to the direction of links.

The spaces W_{H1} and W_{H2} are very useful in the computation of electromagnetic field in multiply connected regions. W_{H1} is important in the modelling of a curl free field such as the magnetic field in a multiply connected domain when there are non-zero currents flowing in the loops (conductor containing holes). Similarly, W_{H2} is useful in the modelling of a divergence free field such as the displacement field when the cavities contain non-zero charges [19].

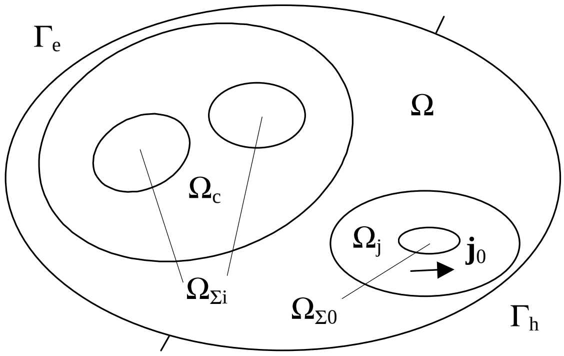

Fig.5. A typical eddy current problem

As an application, we consider the case of an eddy current problem shown in Fig.5. The study domain Ω contains a conducting region Ω_c and an excitation coil Ω_j with a given current density $\mathbf{j}_0$. The boundary of Ω is split into two parts $\partial\Omega = \Gamma_h \cup \Gamma_e$ with $\Gamma_h \cap \Gamma_e = 0$. On Γ_h we have $\mathbf{n} \times \mathbf{h} = 0$ and on Γ_e, $\mathbf{n} \times \mathbf{e} = 0$. Without loosing the generality, we consider the case where the conducting region Ω_c and the excitation coil Ω_j have holes (no trivial domains).

Before the derivation of formulations, we express the excitation current $\mathbf{j}_0$ in Ω_j by a source field $\mathbf{t}_0$ (current vector potential) such that curl $\mathbf{t}_0 = \mathbf{j}$. It can be noted that $\mathbf{t}_0$ defined in this way is not unique, but any field fulfilling curl $\mathbf{t}_0 = \mathbf{j}$ works. The most convenient way is to set $\mathbf{t}_0$ in a simply connected region Ω_t composed of the excitation coil Ω_j and the cutting domain $\Omega_{\Sigma 0}$, with the boundary condition $\mathbf{n} \times \mathbf{t}_0 = 0$ on $\partial\Omega_t$ and to calculate it using a finite element approximation [20]. After doing that, the magnetic field $\mathbf{h}$ in Ω_t is split into a curl free field $\mathbf{h}_r$ and the source field $\mathbf{t}_0$: $\mathbf{h} = \mathbf{h}_r + \mathbf{t}_0$. The excitation coil Ω_j is then removed from the study domain. In what follows, we will use a unified notation for the magnetic field $\mathbf{h}$ and keep in mind that $\mathbf{h} = \mathbf{h}_r$ in Ω_t.

Application of differential form based elements to the two dual systems of Maxwell equations as shown in Tonti diagram (Fig.1) leads to two dual formulations. The magnetic formulation is obtained by a conformal approximation of the Ampere's theorem using differential form based elements and by solving the Faraday's law using the weak variational principle (integration by parts). Taking the magnetic field (the magneto-motive force) as the working variables, the variational formulation in terms of differential forms is written as:

Find $h \in W_h^1 = \{h \in W_q^1 \mid dh = 0 \text{ in } \Omega \setminus \Omega_c,\ h = 0 \text{ on } \Gamma_h\}$

$$\int_\Omega \frac{1}{\sigma} * dh \wedge dh' + d_t \int_\Omega \mu * h \wedge h' + d_t \int_{\Omega_j} \mu * t_0 \wedge h' = 0\,, \qquad \forall\, h' \in W_h^1$$

With the more familiar vector notation, the formulation is

Find $\mathbf{h} \in W_h^1 = \{\mathbf{h} \in W_q^1 \mid \text{curl}\, \mathbf{h} = 0 \text{ in } \Omega \setminus \Omega_c,\ \mathbf{n} \times \mathbf{h} = 0 \text{ on } \Gamma_h\}$

$$\int_\Omega \frac{1}{\sigma}\mathrm{curl}\mathbf{h}'\cdot\mathrm{curl}\mathbf{h}\,\mathrm{d}\Omega + \mathrm{d}_t\int_\Omega \mu\mathbf{h}'\cdot\mathbf{h}\,\mathrm{d}\Omega + \mathrm{d}_t\int_{\Omega_j}\mu\mathbf{h}'\cdot\mathbf{t}_0\,\mathrm{d}\Omega = 0\ , \qquad \forall\ \mathbf{h}' \in W_h^1$$

The domain $\Omega \setminus \Omega_c$ is multiply connected since the conductor contains holes. Let $\Omega_{\Sigma i}$ be cutting domains which make $\Omega \setminus \Omega_c$ simply connected and note by $\Omega' = (\Omega \setminus \Omega_c) \setminus \cup\Omega_{\Sigma i}$ the simply connected domain. We now can determine the rank of the system with the help of De Ram's complex shown in Fig.5. Since $\mathbf{n}\times\mathbf{h} = 0$ on Γ_h, the degrees of freedom related to the simplexes on Γ_h are removed. We note by Ω'_h the domain Ω' excluding the boundary Γ_h, and by Ω_{c0} the conducting domain Ω_c excluding the boundary $\partial\Omega_c$. According to De Rham's complex given in Fig.4, the condition curl $\mathbf{h} = 0$ in the simply connected domain Ω'_h can be satisfied by using a scalar potential ϕ such that $\mathbf{h} = \mathrm{grad}\,\phi$, $\phi \in W_q^0(\Omega'_h)$. The gauge condition for the scalar potential is assured by the fact that the degrees of freedom related to ϕ on Γ_h are omitted (Here we suppose that Γ_h is connected. Otherwise, ϕ cannot be put to zero on all components of Γ_h. Constraints have to be introduced [21]). One can also work directly with the variable $\mathbf{h}$, by taking basis functions belonging to the null space of the 1-form element, i.e. $\mathbf{h} \in Z_q^1(\Omega'_h)$. In the case of first order (Whitney) element, it consists to set the degrees of freedom of $\mathbf{h}$ on a tree constituted by a set of edges. When the elements of higher order are used, identification of the null space is not that easy unless elements of hierarchical basis are used. In the cutting domains $\Omega_{\Sigma i}$, according to the analysis of the previous section, $\mathbf{h} \in W_{H1}(\cup\Omega_{\Sigma i})$. The rank of the whole system is $\dim(W_q^0(\Omega'_h)) + \dim(W_q^1(\Omega_{c0})) + N_{H1}$.

This formulation ensures the tangential continuity of the magnetic field. The results give the circulation of magnetic fields along edges, and hence the currents across facets.

In the conducting domain, the magnetic field $\mathbf{h}$ can be written as the sum of a current vector potential $\mathbf{t} \in W_q^1(\Omega_c)$ and the gradient of a scalar potential $\phi \in W_q^0(\Omega_c)$. We get a formulation in terms of combined vector and scalar potentials. This constitutes an alternative of the previous field formulation. A gauge condition excluding the null space from the 1-form element is then necessary to ensure a unique solution of the vector potential. The dimension of the null space to be excluded is the same as the number of unknown scalar variables introduced. The rank of the system is $\dim(W_q^0(\Omega_h)) + \dim(W_q^1(\Omega_{c0})) - \dim(Z_q^1(\Omega_{c0})) + N_{H1}$, same as the formulation in terms of $\mathbf{h}$ in Ω_c.

The dual formulation, the electric one, is obtained by using p-form elements to approximate the variables in Faraday's system and solving Ampere's theorem in weak sense. Working with the time integral of the electric field in the conducting region and the magnetic vector potential in non-conducting region, the variational formulation in terms of differential forms is

Find $a \in W_a^1 = \{a \in W_q^1 \mid a = 0 \text{ on } \Gamma_e)$

$$\int_\Omega \frac{1}{\mu} * \mathrm{d}a \wedge \mathrm{d}a' + \mathrm{d}_t\int_\Omega \sigma * a \wedge a' + \int_{\Omega_j} t_0 \wedge \mathrm{d}a' = 0\ , \qquad \forall\ a' \in W_a^1$$

or using vector notation:

Find $\mathbf{a} \in W_a^1 = \{\mathbf{a} \in W_q^1 \mid \mathbf{n}\times\mathbf{a} = 0 \text{ on } \Gamma_e)$

$$\int_\Omega \frac{1}{\mu}\mathrm{curl}\mathbf{a}'\cdot\mathrm{curl}\mathbf{a}\,\mathrm{d}\Omega + \mathrm{d}_t\int_\Omega \sigma\mathbf{a}'\cdot\mathbf{a}\,\mathrm{d}\Omega + \int_{\Omega_j}\mathrm{curl}\mathbf{a}'\cdot\mathbf{t}_0\,\mathrm{d}\Omega = 0\ , \qquad \forall\ \mathbf{a}' \in W_a^1$$

The variable (1-form) **a** (such that curl **a** = **b**) must be seen as the time integral of electric field **e** in Ω_c. In the non-conducting region $\Omega \setminus \Omega_c$, we solve a divergence free field div **b** = 0. Since the surface integral of normal component of **b** over the boundary of $\Omega \setminus \Omega_c$ is identically zero, the space W_{H2} has no use in this application. We are not concerned by the trouble of non-simply connected region. The divergence free condition is simply assured by writing **b** = curl **a**. It can be seen from De Rham's complex that, the flux density **b** is in the null space of divergence operator and the vector potential **a** is in the co-domain of the curl operator. The rank is hence the dimension of the range space of 1-form element. Appropriate gauge condition removing the null space from the functional space of 1-form element should be introduced to ensure the uniqueness of **a**. Considering $\mathbf{n}\times\mathbf{a} = 0$ on Γ_e, the degrees of freedom related to the simplexes on Γ_e are omitted. We note by Ω_e the domain Ω excluding the boundary Γ_e. Supposing the intersection of $\partial\Omega_c$ and Γ_e is empty, the rank of the whole system is $\dim(W_q^1(\Omega_e)) - \dim(Z_q^1(\Omega_e \setminus \Omega_c))$. Where $\dim(Z_q^1(\Omega_e \setminus \Omega_c)$ corresponds to the number of unknowns removed in non-conducting region when a gauge condition is imposed.

This formulation ensures the tangential continuity of electric field and gives the circulation of vector potential along edges, and hence the fluxes across facets.

The electric formulation has also an alternative in terms of combined vector and scalar potentials. The field **a** (the time integral of the electric field **e**) in the conducting region can be replaced by the sum of a magnetic vector potential $\mathbf{a} \in W_q^1(\Omega_c)$ and the gradient of a scalar potential $\psi \in W_q^0(\Omega_c)$. A gauge condition excluding the null space from the 1-form element is then necessary to ensure a unique solution of the vector potential. The scalar potential is gauged by imposing the value of ψ at one node if the intersection of $\partial\Omega_c$ and Γ_e is empty. When this is done, the number of unknowns of the system is $\dim(W_q^1(\Omega_e)) + \dim(W_q^0(\Omega_c)) - 1 - \dim(Z_q^1(\Omega_e))$, same as the previous field formulation.

It can be noted that, in this formulation, the current density $\mathbf{j}_0$ is replaced by curl $\mathbf{t}_0$ and the vector potential $\mathbf{t}_0$ is projected on the space of 1-form element with the help of integration by part. This projection ensures the compatibility of the equation and improves considerably the convergence behavior [20].

Above formulations show that, in the conducting region, we can work with either the field variable or the potential variables. In the case of potential formulations, gauge condition has to be introduced to ensure the uniqueness. The rank of the matrix system is the same in both situations. It has to be pointed out that the gauge condition for the vector potential is not indispensable when an iterative solver is used. It is found that the convergence behavior is even better without the gauge condition [22]. According to the analysis given in [23], removing the null space of the 1-form elements diminishes the minimal non-zero eigenvalue and leads to a worse conditioning of the matrix system.

CONCLUSION

Differential forms present considerable advantages over the classical vector or tensor calculus not only in the electromagnetic theory analysis but also in the numerical computation of electromagnetic field. The dual flow diagram (Tonti diagram) is helpful for the derivation of dual finite element formulations of electromagnetic problems and makes clear the duality and the complementarity of dual approximation schema. Elements based on differential forms of different degrees constitute natural discrete spaces of different scalar and vector variables and ensure naturally the continuity requirement of different field quantities. De Rham's complex reveals the mathematical framework behind the electromagnetic theory, including the geometric and topologic properties of the study domain. De Rham's cohomology groups (spaces of closed but not exact forms) enables a better understanding of the modelling of curl-free or div-free field in multiply connected regions. With the help of De Rham’s complex, the link between the functional spaces of the elements based on differential forms is clearly illustrated. It is helpful to determine the rank of the matrix system and to understand the gauge conditions when potential variables are employed.

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