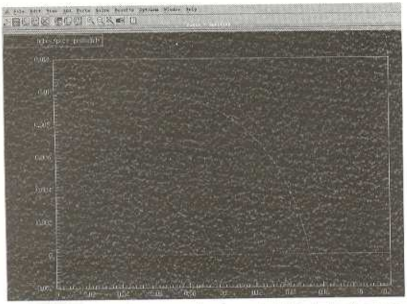

*Fig. 6. SPICE computed airgap vs. time for coupled electrohydraulic circuit of Fig. 5.*

The armature position vs. time computed by SPICE for typical circuit values is shown in Fig. 6. The closing time is changed from measurements [4], due to the back pressure created by the hydraulic fluid flow.

### Conclusion

Based on analogies between electromagnetics and hydraulics, SPICE can be used to successfully simulate hydraulic circuits. Analysis of coupled electromagnetic/hydraulic systems can be performed using dependent SPICE sources.

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*Technical article*

# The question of spurious modes revisited

It is well known that Galerkin finite element models of waveguides or resonant cavities may be affected by spurious modes [1-3]. In fact, when the variational eigenproblem modelling a waveguide or a resonant cavity is discretised by a Galerkin finite element method, it may happen that some of the obtained finite element eigensolutions have unphysical features and are strongly sensitive to the mesh size and that the situation cannot be improved by refining the mesh. From a mathematical viewpoint the approximation may fail to converge, in a sense appropriate for an eigenproblem, as the mesh size tends to zero.

Whether spurious solutions actually occur or not depends on the specific finite element used in the Galerkin approximation and it is well known that edge elements on triangular or tetrahedral meshes never suffer this drawback, while vector nodal elements do, unless very particular meshes are used [4-6].

A question naturally arise, then: why edge elements are spurious-free, while vector nodal elements are not, in general? Which is the property that makes the difference?

Till now, it has been believed the answer was: edge elements are spurious-free because the finite element space they generate contains all the gradient of the scalar nodal elements over the same mesh, a property not shared, in general, by vector nodal elements [6].

This is not the complete answer, however. As a matter of fact, we have proved that an element is spurious-free if and only if the finite element space it generates satisfies a set of three conditions [7]. The first one is fulfilled by any reasonable element; vector nodal elements included, and simply means that the finite element space is able to approximate the space where the solution is sought. The second condition means that the finite element space contains an irrotational subspace able to approximate the kernel of the curl operator and is satisfied by edge elements just because they contain the gradients of scalar nodal elements. While it is common opinion that these two conditions are sufficient to avoid spectral pollution [4-6], actually a third condition is required, which, like the second one, is satisfied by the edge elements, but not by the vector nodal elements. This is a discrete compactness condition [8], which seems to be almost unknown to the electromagnetic community. Anyway, its relevance to the question of spurious modes has never been stressed before.

But, let us start from the beginning and state the problem we are dealing with.

Let $\Omega$ be a bounded simply connected region with a connected boundary $\Gamma$ representing an empty cavity resonator with an ideally conducting wall and define the following spaces, scalar products and norms.

$$V = H_0(\text{curl};\Omega) = \left\{ \underline{v} \in L^2(\Omega)^3 \;\middle|\; \text{curl}\,\underline{v} \in L^2(\Omega)^3,\ \underline{v} \times \underline{n}\big|_\Gamma = 0 \right\},$$

$$V_0 = H_0(\text{curl}^0;\Omega) = \left\{ \underline{v} \in V \;\middle|\; \text{curl}\,\underline{v} = 0 \right\},$$

$$V_1 = H_0(\text{curl};\Omega) \cap H(\text{div}^0;\Omega),$$

where $H(\text{div}^0;\Omega) = \left\{ \underline{v} \in L^2(\Omega)^3 \;\middle|\; \text{div}\,\underline{v} = 0 \right\}$.

$(\bar{u}, \bar{v}) = \int_\Omega \bar{u} \cdot \bar{v}\, d\Omega$ and $\|v\| = (v, v)^{1/2}$ are the natural scalar product and norm of $L^2(\Omega)^3$.

$(\underline{u}, \underline{v})_{\text{curl}} = (\text{curl}\,\underline{u}, \text{curl}\,\underline{v}) + (\underline{u}, \underline{v})$ and $\|\underline{v}\|_{\text{curl}} = (\underline{v}, \underline{v})^{1/2}_{\text{curl}}$

are the natural scalar product and norm of V.

$V_1$ proves to be the orthogonal complement of $V_0$ in V.

The electric field $\underline{e}$ and the wavenumber k are determined by the eigenproblem

*Find* $0 \neq \underline{e} \in V_1$ *and* $k \in R$ *such that*

$$(\text{curl}\,\underline{e}, \text{curl}\,\underline{v}) = k^2 (\underline{e}, \underline{v})\ \forall \underline{v} \in V_1, \qquad \text{(P)}$$

which is equivalent to the Maxwell system.

A conforming finite element approximation of problem (P), however, would require a divergence-free finite element space, which is difficult to obtain.

Hence, problem (P) is substituted by the modified problem

*Find* $0 \neq \underline{e} \in V$ *and* $k \in R$ *such that*

$$(\text{curl}\,\underline{e}, \text{curl}\,\underline{v}) = k^2 (\underline{e}, \underline{v})\ \forall \underline{v} \in V, \qquad \text{(MP)}$$

which is then approximated by the finite element problem

*Find* $0 \neq \underline{e} \in V_h$ *and* $k_h \in R$ *such that*

$$(\text{curl}\,\underline{e}_h, \text{curl}\,\underline{v}_h) = k_h^2 (\underline{e}_h, \underline{v}_h)\ \forall \underline{v}_h \in V_h, \qquad \text{(FEP)}$$

where $V_h \subset V$ is the finite element space and, as usual, the mesh parameter $h > 0$ determines a specific member of a family of meshes and has the meaning of maximum element diameter of that particular mesh. The usual assumptions against element degeneration as $h \to 0$ are understood [9].

We have then this peculiar situation:

- Problem (MP) has the same solutions of problem (P) plus the infinite dimensional eigenspace $V_0$ belonging to $K = 0$.
- Problem (FEP) is an approximation of problem (MP) and some of its solutions will be approximations of $(0, V_0)$.
- Our aim is obtaining approximate solutions of (P) from (FEP).

In spite of their similarity, problems (P) and (MP) are very different in character. This difference, which is revealed by the presence of an infinite dimensional eigenspace in problem (MP), originates from the fact that $V_1$ is *compactly embedded* in $L^2(\Omega)^3$, while V is not. Unfortunately, we cannot avoid using this technical concept in our discussion, since compactness properties do play a crucial role in the whole question of spurious modes.

$V_1$ <u>*is compactly embedded in*</u> $L^2(\Omega)^3$ <u>*means that any bounded sequence*</u> $\{\underline{v}_n\}_{n=1}^{\infty}$ <u>*of*</u> $V_1$ *(i.e. such that* $\underline{v}_n \in V_1$ *and* $\|\underline{v}_n\|_{\text{curl}} \leq C$ *for some constant C and any n)* <u>*contains a sub-sequence*</u> *(still denoted by* $\{\underline{v}_n\}$*)* <u>*converging in*</u> $L^2(\Omega)^3$ *(i.e. such that* $\lim_{n\to\infty} \|\underline{v}_n - \underline{v}\| = 0$ *for some* $v \in L^2(\Omega)^3$*).* (CP)

Each of the problems (P), (MP) and (FEP) is equivalent to an eigenvalue problem for a suitable operator. Property (CP) implies that we can associate to problem (P) an eigenvalue problem for a *compact operator*, namely an operator mapping bounded sequences to sequences containing converging sub-sequences (but this defining property is not important in this context). What is remarkable is that compact operators have a simple spectrum, consisting of a countable set of isolated eigenvalues of finite multiplicity, which is less critical to be numerically approximated than a general one. Unfortunately, as V is not compactly embedded in $L^2(\Omega)^3$, instead, the operator associated to problem (MP) is not compact, but just continuous.

Hence, by substituting problem (MP) for (P), a difficulty (the divergence-free constraint) has been eluded, but new ones have been introduced: the extraneous solution $(0, V_0)$ and the noncompactness of the underlying operator.

In order that (FEP) can be really used to approximately solve (P), it is now clear that we need two things:

- (FEP) must be a good approximation of (MP) in spite of the noncompactness of the underlying operator.
- All the solutions of (FEP) approximating $(0, V_0)$ must be easily identified to be discarded.

When both the above requirements are satisfied, we will say by definition that "(FEP) *is a spurious-free approximation of* (P)."

Each of these requirements can be made precise and set in mathematical language, but doing so would require very cumbersome and technical statements. This has been worked out in [7]. Here, we will just try, by partially relying upon an intuitive understanding, to make clear which properties we require in order to call spurious-free an approximation.

As we already proposed in [10], the most natural way to make precise the first requirement is to ask that, as $h \to 0$, the operator associated to (FEP) converges to the operator associated to (MP) in the sense of the spectral approximation of general continuous operators developed in [11]. This kind of convergence is defined through four conditions that in our simple case (we do not have any continuous spectrum, for instance, which would be allowed, in general, by the theory in [11]) can be described as follows.

The first condition says that for any eigenvalue of (MP) we can find a sequence of eigenvalues of (FEP) converging to it as $h \to 0$. In other words, no eigenvalue of (MP) is missed by its approximation (FEP).

The second condition says that, by sufficiently reducing h, we can simultaneously push all the eigenvalues of (FEP) out of any given closed and bounded subset of the complement of the spectrum of (MP) to the real axis. This conditions means that no bounded sequence of eigenvalues of (FEP) can have a nonvanishing distance from the spectrum of (MP) as $h \to 0$. In other words, (FEP), as an approximation of (MP), does not introduce any eigenvalue extraneous to the original problem.

These first two conditions together say that the set of all the accumulation points of the union of the spectra of (FEP) obtained for all $h > 0$ is just the spectrum of (MP).

As the eigenvalues of (MP) are not necessarily simple, the remaining two conditions do not separately concern single eigenvectors, but rather whole eigenspaces. Owing to the first two conditions, for any given eigenvalue of (MP) and for any h less than some sufficiently small value, the eigenvalue(s) of (FEP) approximating this specific eigenvalue of (MP) can be identified. The span of the eigenvector(s) belonging to this (these) eigenvalue(s) of (FEP) will be the "approximate eigenspace" corresponding to the eigenspace belonging to the given eigenvalue of (MP).

The third condition says that, for any element of any eigenspace of (MP) we can find a sequence of elements of the corresponding approximate eigenspace converging to it as $h \to 0$. In other words, no eigenspace of (MP) is missed by (FEP), not even partially.

The fourth condition says that, for any eigenspace of (MP), the greatest distance a normalized element of the corresponding approximate eigenspace can have from the eigenspace of (MP) itself vanishes as $h \to 0$. This conditions means that no sequence that consists of normalised eigenvectors of (FEP) corresponding to a bounded sequence of eigenvalues can have a nonvanishing distance from the union of all the eigenspaces of (MP) as $h \to 0$. In other words, (FEP), as an approximation of (MP), does not introduce any eigenvector extraneous to the original problem.

If we have just convergence in the sense of [11] (i.e. the above four conditions), some approximations of $(0, V_0)$, the sole eigensolution of (MP) not satisfying (P), will appear as sequences of positive eigenvalues of (FEP) converging to zero as $h \to 0$. When (FEP) is regarded as an approximation of (P), however, any eigensolution of (FEP) approximating $(0, V_0)$ is extraneous to the original problem. In order that the elements of the sequences approximating $(0, V_0)$ can be easily identified and discarded on any finite mesh, the best situation occurs when, for any $h \to 0$, all of them are actually exact eigensolutions of (MP) (i.e. they satisfy $k_h = 0$, $e_h \in V_0$). In order to enforce this situation, we add the following fifth condition, which rules out any sequence of positive eigenvalues of (FEP) converging to zero, to the set defining a spurious-free approximation of (P): $k_h = 0$ is an isolated point of the union of all the spectra of (FEP) obtained for any $h > 0$.

Now that what we mean by spurious-free approximation should be sufficiently clear, we can state the three conditions we alluded to at the beginning.

First of all, we need to define the irrotational subspace $V_{0h} = V_h \cap V_0$ of the finite element space $V_h$, its orthogonal complement $V_{1h} = \{\underline{v}_h \in V_h \mid (\underline{v}_h, \underline{w}_h)_{curl} = 0 \;\; \forall \underline{w}_h \in V_{0h}\}$ in $V_h$ and notice that, while $V_{0h} \subset V_0$, we have $V_{1h} \not\subset V_1$.

The following three independent conditions are necessary and sufficient in order that (FEP) is a spurious-free approximation of (P) [7].

Completeness of the approximating subspace:

$$\lim_{h \to 0} \inf_{\underline{v}_h \in V_h} \|\underline{v} - \underline{v}_h\|_{curl} = 0 \quad \forall \underline{v} \in V \qquad \text{(CAS)}$$

Completeness of the discrete kernel:

$$\lim_{h \to 0} \inf_{\underline{v}_h \in V_{0h}} \|\underline{v} - \underline{v}_h\|_{curl} = 0 \quad \forall \underline{v} \in V_0 \qquad \text{(CDK)}$$

Discrete compactness property:

*Any sequence* $\{\underline{v}_h\}_{h>0}$ *such that* $\underline{v}_h \in V_{1h}$ *and* $\|\underline{v}_h\|_{curl} \le C$ *for some constant C and any h contains a sub-sequence (still denoted by* $\{\underline{v}_h\}$*) converging in* $L^2(\Omega)^3$ *(i.e. such that* $\lim_{h \to 0} \|\underline{v}_h - \underline{v}\| = 0$ *for some* $\underline{v} \in L^2(\Omega)^3$*).* (DCP)

About the meaning of (CAS) and (CDK) we have already said at the beginning. As for the third condition, (DCP) is the discrete counterpart of (CP), but is not a trivial consequence of it, because $V_{1h}$ $V_1$. Since (CP) is the property ensuring that problem (P) has a discrete spectrum, it is hardly surprising that (DCP) is essential for a well behaved approximation of (P). In practice, (DCP) implies that any sequence of normalized eigenvectors has an essentially convergent behaviour and aimlessly wandering sequences can be obtained only in artificial ways (e.g. by alternately picking elements out of two sequences converging to different limits).

Let us notice also that, in general, when the operator associated to the problem under consideration is compact, (CAS) alone is sufficient to ensure a well-behaved spectral approximation [12].

It is worthwhile to consider also the following condition, which is equivalent to the fifth condition characterising (FEP) as a spurious-free approximation of (P). Hence, necessary for a spurious-free approximation.

Discrete Friedrichs Inequality:

$$\exists \alpha > 0 \text{ such that } (\operatorname{curl} \underline{v}_h, \operatorname{curl} \underline{v}_h) \ge \alpha \|\underline{v}_h\|^2_{curl} \quad \forall \underline{v}_h \in V_{1h} \;\; \forall h > 0 \qquad \text{(DFI)}$$

Also (CAS), (DFI) and (DCP) constitute a set of independent conditions necessary and sufficient in order that (FEP) is a spurious-free approximation of (P) [7]. In spite of what might be superficially thought, however, (DFI) and (CDK) are not equivalent, in general. In fact, for the sake of preciseness, we have the following situation: (CAS) and (DFI) together imply (CDK), while (DCP) and (CDK) together imply (DFI). [7].

Let us also point out that if only (CAS) and (DCP) are satisfied, then just the fifth condition characterising (FEP) as a spurious-free approximation of (P) is not fulfilled, while the other four conditions still hold true [7]. In this case, spurious eigenvalues may occur only in a neighbourhood of zero that can be narrowed by mesh refinement. However, we do not know any practical finite element showing spurious eigenvalues that behave in this way.

Now we are in a position to discuss the behaviour of some known elements in the light of the above theory.

Nodal vector element approximations, in general, satisfy only (CAS). As it is well known, these approximations show something like a flow of spurious eigenvalues ceaselessly entering the spectrum from above and nonuniformly converging to zero as $h \rightarrow 0$. This behaviour violates both the second and the fifth conditions characterising (FEP) as a spurious-free approximation of (P). Violation of the second condition, implies that the union of the spectra of (FEP) obtained for all $h \rightarrow 0$ has some accumulation point which is not in the spectrum of (MP), even though this may be difficult to observe in numerical experiments. A simple abstract example of Galerkin approximation with global basis (not a finite element one) that satisfies only (CAS) and has spurious eigenvalues showing this behaviour has been built in [7] (Example 7.5).

It has been recently found a case (2D nodal vector elements on a so called criss-cross mesh [13, 14]) in which the spurious modes exhibit a different behaviour: some eigenvalues of (FEP) converge to positive values not in the spectrum of (MP) (so violating the second condition defining a spurious-free approximation), while the corresponding eigenvectors do not converge. What is the reason of this unusual behaviour? In this case, (CAS) and (DFI) are satisfied [15], but (DCP) is surely not, since for this case the existence of spurious eigenvalues has been analytically proved [14]. Condition (DFI), being equivalent to the fifth condition characterising (FEP) as a spurious-free approximation of (P), prohibits any sequence of positive eigenvalues of (FEP) converging to zero [7]. Hence, the more usual behaviour of the spurious modes cannot take place. On the other hand, wandering sequences of eigenvectors are permitted, as (DCP) is not satisfied.

It is worthwhile to stress that, as (CAS) and (DFI) together imply (CDK), this is a counterexample to the aforementioned common opinion that (CAS) and (CDK) are sufficient to avoid spurious modes [7, 15]. It should be clear, however, that the unusual behaviour shown by the spurious modes in this case is not caused by the fact that (CDK) is satisfied, but rather by the fact that (DFI) holds true. In fact, (CDK) does not prohibit sequences of positive eigenvalues converging to zero [7, Example 7.1], unless corroborated by (DCP) (remember that (DCP) and (CDK) together imply (DFI)).

Edge element approximations, do satisfy (CAS), (CDK), (DFI) and (DCP) [7, 15, 16] and are spurious-free in the sense of [7], as expected.

Last, but not least, let us point out that in [7] the above theory has been developed under more general assumptions allowing inhomogeneous, anisotropic and discontinuous material properties, topologically nontrivial problem domains and mixed boundary conditions arising from symmetry exploitation, so covering most of the situations occurring in real-life applications. By using results in [7], we have proved in [16], for first time in this very general setting, that Nedelec's edge elements of any fixed order on tetrahedral meshes [17, 18] give rise to spurious-free approximations of electromagnetic eigenproblems.

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