Technical Article

A pragmatic approach to educating finite element methods

Abstract —**Finite element methods are widely used today by commercial organizations. Likewise, academic communities worldwide are still continuously widening its scope of applications. While preparing master students for their future career, professors are forced to make fundamental choices regarding the content of their lectures, since industry and academics tend to request different type of skills. In this paper, a pragmatic approach is presented which may be adopted to serve both worlds. By gradually increasing the theoretic level and, at the same time, treating various practical consequences, people are developed that have a global yet thorough view on finite element methods.**

I. INTRODUCTION

When being appointed to arrange a brandnew course on finite element (FE) methods in a master program for mathematics and physics students, one may be easily tempted to approach finite elements from its theoretical side. Their bachelor study programs are rather fundamentally oriented, they are thus used to deal with complicated theories and, in contrast to many master students in engineering disciplines, they are still charmed by theoretic courses. Moreover, the academic community stimulates it.

A visionary conflict immediately arises when the lecturer himself appears to be a PhD in engineering, having worked (and still working) for more than ten years in industry on real-life finite element applications in various disciplines such as magnetics, structural dynamics and fluid dynamics. Being convinced that any master program in exact or applied sciences should not only prepare for academic careers, but for industrial careers as well, he is challenged to arrange a course that prepares for both.

In this educational paper, the author describes how he brought this into practice. Though the original intention was to bridge the gap from theory to practice for students in mathematics and physics, similar ideas and concepts may be adopted to bridge the gap from practice to theory for students in various engineering disciplines.

II. BACKGROUND

Having finished their bachelor program in engineering, mathematics or physics, students should possess a thorough knowledge of the base disciplines required to become specialized in their major discipline. The question rises what the required set of base disciplines is and whether it is sufficiently/appropriately covered or not. In view of the fact that many fresh master students are about two years away from their launch into a non-academic job market, where theoretic skills are definitely not the only factor that is assessed by future employers, a huge gap is to be crossed by the students.

For example, if one asks students what differential equations are, most will give an acceptable yet mathematically inspired answer. Some of them will be eager to tell that many types of differential equations can be solved analytically. A handful of them will even respond that they have learned how to solve a partial differential equation on a unit square with some exotic boundary conditions on its edges. The author never met a student who considered his first response in view of the interest of the requestor, that is an engineer working almost full time in industry. When asking them whether they would give the same answer when sitting in front of a human resource manager while applying for a job, most students quickly realize the importance of the context. The subject of contextual awareness is extremely important, yet not often focused on, resulting in situations as the one described above.

Referring to the same question, students should actually be made conscious about the fact

- that differential equations together with their boundary conditions are merely approximate mathematical descriptions of physical, financial or other processes;
- that these models rely on parameters such as material properties that are rarely accurately known themselves;
- that what is considered the “exact solution” of the problem, often represented by a convincing set of nicely-looking colored pictures, is thus only an approximate solution of reality;
- that wrong technical conclusions might consequently be drawn, possibly resulting in loss of profit and/or reputation for the organization;
- that numerical simulation, in the end, must serve the community by yielding ever better designs in ever less time and with ever smaller prototyping costs;
- …

Another issue requiring attention in courses on finite elements, particularly for the practical sessions, is finding the right balance between commercial software and proprietary software many academic research groups possess. The latter may be best suited for illustrating all the numerical peculiarities occurring. On the other hand, commercial software is adapted to historic and actual needs of many industrial and even academic users, and addresses numerical simulation from a customer point of view. It is definitely an added value to students when they have at least shortly been exposed to the best of both worlds. Challenge the theoretic guys to generate decent meshes on solid models they assembled themselves starting from an imported CAD-model, or force the practical guys to dig into the core of a finite element solver by letting them implement a non-linear solver based on a given linear solver, … Cross-fertilization is a nice way to open people’s minds.

III. APPROACH

This section of the paper describes in which way the author organizes his lecture series on theory and practice of finite element methods. In total, there are 13 theoretic and 13 practical sessions foreseen, two hours per session. The target audience consists mainly of master students in mathematics and physics. The aim is to provide students with sufficient theoretic and practical background so that they are well prepared for both, a numerically oriented PhD study program, as well as a simulation oriented job in a private company. Therefore, theoretic sessions are always complemented with relevant examples from the lecturer's industrial experience, whereas practical sessions are always started from a theoretic baseline.

There are various approaches to initiating students in the world of finite elements. Two extreme examples:

- Start by introducing the mathematical framework of Hilbert and Sobolev spaces, variational formulations etc… ; then gradually converge to a descriptive level that allows implementation into software; eventually apply the techniques to some examples.
- Let students first play around with finite element software to give them some feeling about its practical aspects; then point out where the theory comes into play; steadily increase the level of abstraction to end with a sound mathematical framework.

Though the first approach may be the most elegant one from an academic point of view, it is the author's conviction that students are better served by the second approach. Just a handful of brilliant students may like to be immersed in an ocean of formulas, but the risk of losing other students' attention as of the first hours of the course on is too large. The aversion some may take up during this short period of time, might pursue them for the rest of their career, resulting in missed opportunities for academic and industrial society. By gradually increasing the level of difficulty, students are permanently challenged to pass small yet surveyable bridges, enhancing the confidence in their newly learned skills.

Moreover, keeping the theoretic level low in the beginning allows the lecturer to put stress on the broader context in which finite element simulation is to be viewed and on the scientific domains the examples are chosen from (e.g. electromagnetics & thermodynamics). For some students, especially those being strongly mathematically oriented, the contextual story is experienced as strange, and sometimes even difficult. After having followed the entire lecture series, most however agree upon the added value it gives to the course.

For the reasons mentioned above, the following sequence of course lectures is adopted:

- FE introduction
  - Demonstration;
  - Guided play with commercial SW;
- FE principles
  - 1D diffusion problems;
  - 1D convection-diffusion problems;
  - 2D diffusion problems;
  - 2D convection-diffusion problems;
  - Mathematical framework;
- FE spin-offs
  - Non-linear FEM;
  - Time-harmonic & transient FEM;
- FE revisited
  - Specific play with commercial SW;
  - Industrial applications (company visit).

The most important among these topics are discussed more thoroughly in the coming sections.

IV. FE INTRODUCTION

Demonstration is a convincing method for introducing the finite element method and the various issues that will be dealt with during the course. Three examples are generally considered in the first lecture.

A. Diffusion in a rod

Though not being a magnetic example, simulating the temperature distribution in a rod of length L is an ideal example to start with. Instead of beginning to analytically solve the simplest of all differential problems,

$$-k\frac{d^2T}{dx^2} = 0\,, \qquad (1)$$

$$T_{x=0} = T_0\,, \qquad (2)$$
$$T_{x=L} = T_L\,, \qquad (3)$$

where T [K] is the temperature, x [m] the position along the rod and k [W/m/K] the diffusion constant, the students are asked where such problems occur in real life. They are often surprised hearing that double glazing and cup or panhandles belong to this category. Starting from that insight, it is not a big deal to illustrate the discrepancy between modeling and reality:

- The formulation implicitly assumes that all heat is flowing along a thermally perfectly insulated rod. The panhandle is though cooled along its length by diffusion, convection and radiation.
- The Dirichlet boundary conditions may be well suited for simulating double-glazing. However, in case of a pan with boiling water in it, a volunteer will certainly not observe the boundary condition at $x = L$ as a 37°C Dirichlet condition if the length of the steel handle is rather short.

If, additionally, it is considered to have some local heating along the rod (e.g. induction heater or band heater), one may argue that the problem is no longer a 1D problem, especially when the L/D ratio of the rod is fairly small. Despite its simplicity, the educational value of this example is large.

B. Lifting magnet

The lifting magnet is a first didactical electromagnetic example. A basic model and its magnetic vector potential solution is plotted in Fig. 1.

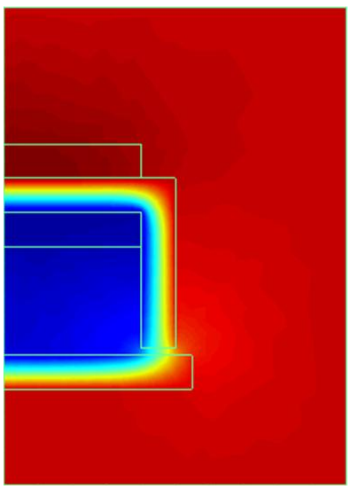

Fig. 1. Magnetic vector potential distribution in a lifting magnet.

Many relevant topics can be covered with this example:

- *2D validity.* This actuator is easily modeled in 2D Cartesian coordinates (XY-plane). Its depth is generally of the same size as its width and height. Hence, fringing also takes place in the Z-direction, but it is not considered in the analysis. An error is thus made and its magnitude additionally depends on the relative size of the airgap.
- *Symmetry.* In the end, FE requires solving a system of equations. By exploiting symmetry, the size of the system is reduced, so does the computational effort. One may take this opportunity to mention other types of symmetry (e.g. periodic symmetry and rotational symmetry), and/or to say some words about iterative solvers for systems of linear equations.
- *Discretizing, meshing.* This topic is often not adequately treated in FE courses, yet it is key for obtaining accurate solutions. The lecturer can show various strategies to obtain decent initial meshes and can briefly talk about adaptive mesh refinement. It is an ideal topic for the guided play tour with commercial SW later on, since the students will directly see the effect of their mesh related choices on the results.
- *Magnetic vector potential.* Quantities such as temperatures are easy to deal with, since they are part of our daily life. Vector potentials are not and thus require some further explanation, especially in relation to the next topic.
- *Boundary conditions.* This is the right place to convince students to think about a numerical solution prior to solving for it. One can show the effect of erroneous boundary conditions and ask the student to reason upon the validity of the result. They should be trained to consider solutions as erroneous until it is proven they are not. Next to that, the concept of natural boundary conditions can be demonstrated here as well.
- *Parameterization.* This fairly straightforward technique is often used in practice and students must at least have heard about it once.
- *Solving.* By simulating the lifting magnet as a non-linear problem with adaptive mesh refinement, the lecturer can comment on the three nested loops that are to be tackled: every adaption step requires solving a nonlinear problem; every nonlinear problem requires solving a sequence of linear problems; every linear problem is solved by an iterative sequence.
- *Post-processing.* Students are trained to solve problems and check their solutions. However, they must now be trained as well to interpret solutions: it is the only reason why simulations are to be done. Getting insight in the behavior of the actuator by looking to the field lines and flux densities, is as important as processing the solution, e.g. via the Maxwell stress method to determine the force acting on the mover. As a side remark, one can mention the concept of element order in this scope, since it easily observed in the solution, at least for first order elements.

C. High voltage line

Most students have already heard of techno-medical investigations on the influence that electromagnetic fields in the vicinity of high voltage lines may have on the human body. Due to the emotional character often related to it, they are generally interested in this topic. From the lecturer's point of view, it complements the previous examples.

The high voltage line, with its fields ranging hundreds of meters away from the centimeter sized cables, is a good example for illustrating the effect of dimensional extremes on the meshing requirements. Should all elements be of millimeter size? Where should the boundary be placed when infinity is no option? Are there any tricks to model the far field behavior? Etc… It is also a good example for demonstrating that electric field problems and magnetic field problems are treated in a different way, since in the former case the earth surface is an equipotential surface, whereas it is no barrier at all in the latter case.

As a way to introduce some general knowledge, and scientific fun as well, a few minutes could be spent on explaining the corona effect. After having zoomed in on the electric field distribution close to the cables, students could be challenged to explain why birds never take long breaks on charged lines.

C. Guided play with commercial software

Having finished the demonstrations, students tend to be saturated with new concepts and approaches. Time has come to let them play around with a user friendly (preferably commercial) software package. They always appreciate it and are astonished by the ease of modeling, solving and post-processing.

Regarding solid modeling, some are capable of creating a 3D-model of a Greek temple or even the empire state building in less than one hour. The examples may seem weird in view of computational magnetics, but they are ideal candidates for illustrating various aspects of 3D-modelling, among which copying, mirroring, Boolean operations ... And the students are once offered a lab session they will remember for the remainder of their life.

Being sufficiently familiar with the modeling tool, the step towards modeling the lifting magnet is rather small. They can then get some feeling for the key aspects of the finite method and are sufficiently prepared for the next lectures, obviously having some more elementary content.

V. FE PRINCIPLES

As stated before, it is a fundamental choice of the author to gradually increase the level of difficulty. This is done in five steps, of which the main ideas are outlined below.

A. 1D Diffusion equations

In this first step, it is illustrated in which way the finite element method is applied to develop a numerical method for solving ordinary thermal diffusion equations as in Eq. 1, though including a heat source term q [W/m^3] in the right hand, and with at least one Dirichlet bound. In order to keep all equations as simple as possible, the domain is subdivided into n elements of the same size h [m]. The temperature profile is modeled as a sum of $n+1$ linear "hat shape" functions $N(x)$, one for each node:

$$T = \sum_{i=1}^{n+1} N_i(x) \, . \tag{4}$$

A residual is defined as

$$R(x) = k\frac{d^2T}{dx^2} + q \tag{5}$$

and the weighed residual method is introduced:

$$\int_0^L W(x)R(x)\,dx = 0\,, \tag{6}$$

where $W(x)$ is a weighing function. Next, the second derivative occurring in the diffusion term is then reduced an order by adopting the integration by parts theorem, thereby introducing a boundary term.

In order to retrieve an algebraic system of equations, the Galerkin method is introduced. For now, it is sufficient to tell the students that weighing functions are simply selected from the same set as the shape functions. It is not recommended to mention anything about the Dirichlet bounds yet. Having elaborated the integrals, the well-known (-1)—(+2)—(-1) molecule structure of the diffusion matrix is easily shown. It illustrates the close relationship with finite difference methods based on central difference approximations [1].

The right moment has arrived to let the students implement their first elementary FE program. They will typically define the system matrix **K** as

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \tag{7}$$

and find out it is not of full rank shortly after. This is the better moment to point out one has forgotten to consider the Dirichlet bound(s) at the edge(s) of the domain or, in other words, that is was not required to weigh with the hat functions in the Dirichlet nodes. They all continue by setting all off-diagonal entries on the first (and/or last) line to zero and by adding the Dirichlet value to the first (and/or last) entry of the vector at the right hand side. The numerical solution is quickly computed afterwards. However, the lecturer should then focus on the advantage of using (whenever possible) the conjugate gradient algorithm for solving the system of equations, and point out the possibility to symmetrize the system matrix here. The obvious way out is setting the first entry of the second line (and/or the last entry of the all but last line) to zero and adding appropriate values to the second (and/or all but last) entry in the right hand side vector.

The previous part is obviously experienced as fairly simple, but the lecturer should stress on the fact that it illustrates many of the key ideas behind finite elements. A profound comprehension of this basic form significantly eases the understanding of the mathematical framework later on (step 5).

Having convinced the students about the ease of finite elements, some more time is to be spent on their practical implementation. In order to retrieve the basic matrix structure of Eq. 7, it was implicitly assumed that all neighboring elements and nodes are incrementally numbered and that all elements are of the same size *h*. In view of a later 2D implementation, where such ideal circumstances cannot be met over the entire domain, one should be challenged to assemble the system matrix and right hand side vector elementwise, starting from arbitrary numbered elements and nodes, directly taking Dirichlet conditions and symmetrizing operations into account [2]. This is generally the first big obstacle to be tackled by the students, since it not only requires from them more conceptual thinking but as well more advanced programming than they are used to.

However, once their generalized software codes are properly working, they can focus on various other aspects. For example:

- Consider a domain of three different neighboring materials (e.g. glass-air-glass) with two Dirichlet bounds (e.g. inner and outer temperature) and no heat production inside.
- Consider a domain of a single material (e.g. steel) being locally heated with two Dirichlet bounds. This problem can also be solved analytically, thus it can be compared to the numerical solution. It is a nice example to illustrate the relation between accuracy and the element size that is chosen. This is illustrated in Fig. 3.

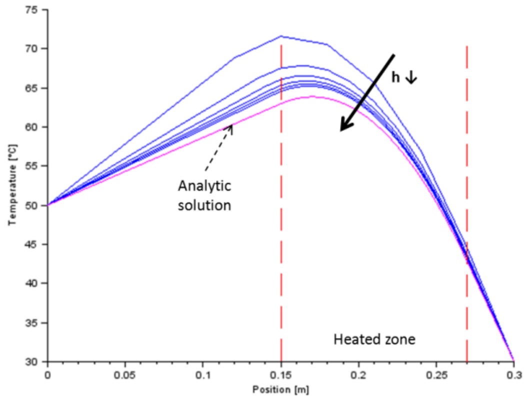

Fig. 3. Illustration of the influence of element size on the numerical accuracy, for a typical thermal diffusion problem with internal heating and two Dirichlet boundary conditions.

B. 1D Convection-diffusion equations

The treatment of 1D thermal convection-diffusion problems, e.g.

$$-\kappa\frac{d^2T}{dx^2} + v\frac{dT}{dx} = 0\,, \tag{8}$$

$$T_{x=0} = T_0\,, \tag{9}$$
$$T_{x=L} = T_L\,, \tag{10}$$

where κ [m^2/s] is the diffusivity and v [m/s] the velocity of the medium, is done in exactly the same way as before. It is to be focused only on the convection term. The analytic solution reads

$$T = T_0 + (T_L - T_0)\frac{e^{vx/\kappa}-1}{e^{vL/\kappa}-1}\,. \tag{11}$$

The software implementation is straightforward. The students are then requested to selected a parameter combination (v, κ, L) that yields a not too steep slope in the analytic solution. Subsequently they are asked to play with the size of the elements till numerical oscillations start occurring, as shown in Fig. 4.

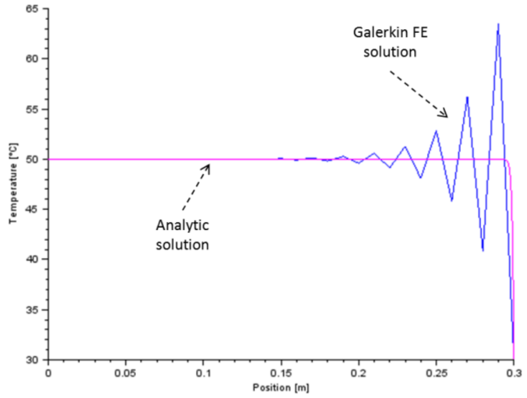

Fig. 4. Illustration of the numerical oscillations occurring when applying the Galerking FE method to convection dominated convection diffusion problems ($Pe > 1$).

The remainder of the lecture on this topic is dedicated to the analysis of the cause of these oscillations and on numerical strategies to avoid them. From finite difference theory with central differences it is shown that the oscillatory solution on an equidistant grid with $n + 1$ nodes reads

$$T = T_0 + (T_L - T_0)\frac{r^i - 1}{r^n - 1} \tag{12}$$

$$r = \frac{1+Pe}{1-Pe} \tag{13}$$

$$Pe = \frac{vh}{2\kappa} \tag{14}$$

$$i = 0, 1, 2, \ldots, n \tag{15}$$

The dimensionless quantity Pe is the so-called Péclet-number and expresses the level of convection relative to diffusion for the chosen grid. When $Pe > 1$, the ratio r is negative, causing r^i to change sign at each consecutive node, i.e. oscillations [3].

Two solution strategies are proposed. The simplest one is artificially adding diffusivity, up to a (summed) value of $\kappa + vh/2$, such that the "new" Péclet number is always smaller than one [3]. The resulting numerical solution smears out the sharp gradient. It is shown that this actually corresponds to upwind differencing of the convection term [1], causing the resulting accuracy of the method to be only of $O(h)$, even for diffusion dominated problems.

The better alternative is weighing between central and upwind differencing of the convection term, the weights depending on the relative convection dominance of the problem. In practical terms, this can be done by replacing the diffusivity κ by

$$\kappa \leftarrow \kappa + \zeta\frac{vh}{2} \tag{16}$$

$$\zeta(Pe) = coth(Pe) - \frac{1}{Pe} \tag{17}$$

One can show that this particular choice yields a numerical solution that coincides with the exact solution of Eq. 11 in all the nodes. For diffusion dominated problems, the accuracy is of order $O(h2)$, which is better than with artificial diffusion [3].

It is not necessary yet to show in which way a modification to the set of weighing functions, instead of augmenting the diffusivity, results in exactly the same system matrix. This is better considered in the 2D case.

Having finished this topic, the students are sufficiently acquainted with the basics of finite elements and its close relation to finite difference methods. They are now ready to apply similar ideas to partial differential equations.

C. 2D Diffusion equations

In this lecture topic, it is focused on diffusion equations. To keep the same notation as before,

$$-\nabla \cdot (\bar{\kappa}\nabla T) = q \tag{18}$$

is proposed as model equation. Magnetostatic 2D problems are covered as well when replacing the temperature by the z-component of a magnetic vector potential A_z [Vs/m ; Tm] and the diffusivity tensor with a slightly modified reluctivity tensor [Am/Vs ; m/H] [2]:

$$\bar{\nu} = \begin{pmatrix} \nu_{yy} & -\nu_{xy} \\ -\nu_{xy} & \nu_{xx} \end{pmatrix} \tag{19}$$

This lecture is the first one in which differential operators and tensors are used. Students are usually familiar with both concepts from their bachelor programs, so one can focus on interpretation of formulas instead of their derivation.

Similarities between the 1D and 2D approach are easily explained. Once it is clear for the students that integration by parts (1D) is actually a special case of Green's second identity (2D or more), they rapidly realize that the slightly more complicated way of notation in 2D is just a simple formalism that can be easily interpreted via the framework given in the 1D cases.

It is worth spending some time of the lab sessions on the concept of symmetric second order tensors for modeling e.g. magnetic anisotropy. Though students are familiar with the tensor transformation rule when switching from one coordinate system to another, they often do not well understand what it can be used for in practice. By using a rectangular core of a one-phase core constructed with oriented steel as an example, and by writing the tensors for the horizontal and vertical legs on the black board, they are easily convinced that something must be done with the tensor entries when the principal direction of the material is at some nonzero angle with respect to the chosen coordinate system. To illustrate that non-zero off-diagonal entries can occur as well, it is instructive to let the students recompute the tensors for a coordinate system that is rotated such that it does no longer coincide with the material's principal directions. A numeric solution for the latter case is shown in Fig. 5. In Fig. 6, the influence of anisotropy on the distribution of magnetic field lines in the areas where the legs of the core overlap is compared to the case in which the ferromagnetic material model is isotropic.

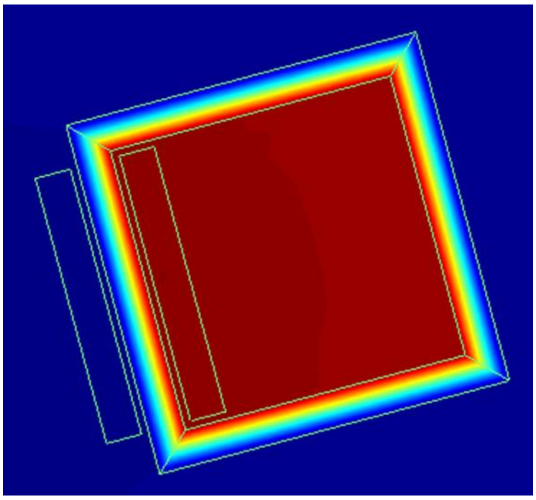

Fig. 5. Magnetic vector potential distribution in a one-phase core made of oriented steel.

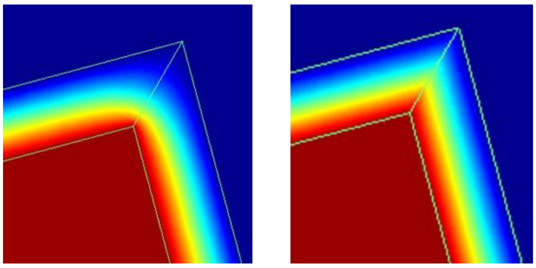

Fig. 6. Detail of Fig. 5 (right) and comparison with the magnetic vector potential distribution obtained when using an isotropic ferromagnetic material model (left).

For what is yet to come, it is important to demonstrate in which way applying the finite element principles on a single triangular element finally yields an elementary (3x3) contribution $\mathbf{K}_e$ to the system matrix, as follows:

$$\mathbf{K}_{e,ij} = -\frac{1}{4\Delta_e}(b_i \quad c_i)\begin{pmatrix}\kappa_{xx} & \kappa_{xy}\\ \kappa_{xy} & \kappa_{yy}\end{pmatrix}\begin{pmatrix}b_j\\ c_j\end{pmatrix} \qquad (20)$$

in which b [m] and c [m] are geometric quantities depending on the nodal coordinates of the element and Δ_e [m^2] the surface area of the element [3,4].

Having reached this point, the students are ready for implementing their own linear 2D finite element solver and post-processor, starting from given data structures in which the nodes, elements, current or heat sources, boundary conditions, materials, material directions, … are uniquely defined. Achieving this is a challenge for both the students and the lecturer. Most students only programmed short and basic functions before and quickly tend to lose the overview. For the lecturer, the biggest concern is the number of different codes he needs to debug for errors. One could as well group the students, but this is not advised: the risk is high that one person will do the main job, while the others are just passively looking and thus not capturing. Though it was already treated in their improved 1D implementation, students appear to have most problems with an efficient assembly of the system matrix, thereby directly accounting for boundary conditions [2]. Despite the effort from both parties, satisfaction is very large when students plot their first correct solution on the screen. Having accomplished this, students definitely have a better idea about the complexity of a finite element solver than they ever could have had from simply using software or from reading complicated theory books.

D. 2D Convection-diffusion equations

The addition of a convective term

$$\vec{v} \cdot \nabla T \qquad (21)$$

to the left-hand side of Eq. 18 first requires some explanation of concepts coming from fluid dynamics, such as streamlines, upwind, downwind and crosswind direction [3]. In a similar way as for the diffusion term, the elementary (3x3) contribution $\mathbf{C}_e$ of the convection term to the system matrix is then derived for a linear triangular finite element:

$$\mathbf{C}_{e,ij} = -\frac{1}{6}(b_i \quad c_i)\begin{pmatrix}v_x\\ v_y\end{pmatrix} \qquad (22)$$

Students are often surprised since the implementation of this formula in their software is a matter of five minutes. They should however be told that the resulting system loses its symmetry.

The magnetic brake is a nice 2D example of an application in which this convective term plays a role. The lifting magnet example is easily turned into a magnetic brake example by mirroring it along the mover, by making the latter somewhat longer, and by attributing a speed to the mover. Another didactical example, easier to understand for students not having a profound background in electromagnetics, is the parallel and laminated flow of two fluids entering a tube at different temperatures. Whatever example selected, it is interesting for students to confirm what they already know, that is the occurrence of oscillations when the velocity distribution in the moving parts becomes too dominant.

Next, it is focused on techniques to avoid such wiggles. The basic idea of adding artificial diffusion is discussed first. For an isotropic medium, the following artificial diffusion scheme is proposed:

$$\kappa_{xx} \leftarrow \kappa + \frac{v_x h}{2} \qquad (23)$$

$$\kappa_{yy} \leftarrow \kappa + \frac{v_y h}{2} \qquad (24)$$

where v_x [m/s] and v_y [m/s] are the components of the velocity vector in an element and h [m] the element “size”. It is to be mentioned that various alternatives exist for the element size [3]. Exact definitions of these, with their advantages and disadvantages, are not that relevant in this scope and are not treated any further. The most important issue to discuss here is proving that the proposed scheme not only adds undesired diffusion to the crosswind directions, but that the amount of additional diffusion depends on the coordinate system definition as well. No one wishes to use a solver whose results depend on the reference frame.

From the finite difference treatment of convection-diffusion equations, it is known that upwinding the convective term yields stable solutions [1]. The question is how to realize upwinding in a finite element context. This is the right point to address a fundamentally different approach via modified weighing functions (Petrov-Galerkin methods), in which more weight is added to the upwind direction [3]. This is done as follows:

$$N_i(\Omega) \leftarrow N_i(\Omega) + \tau\vec{v} \cdot \nabla N_i(\Omega) \qquad (25)$$

$$\tau = \frac{h}{2\|\vec{v}\|}\zeta(Pe) \quad (26)$$

The weighing effect of the second term can be easily visualized, as illustrated in Fig. 7.

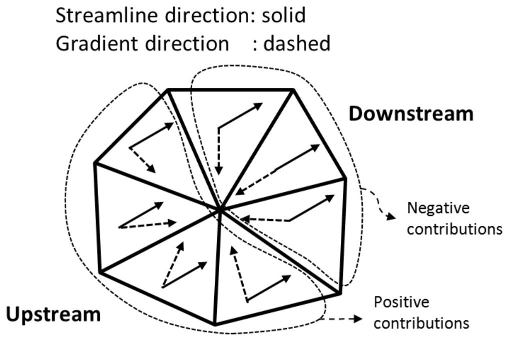

Fig. 7. Illustration of additional weight added to the upstream direction by the second term in Eq. 25.

It is obvious that the definition of the so-called *intrinsic time scale* τ [s] finds its origin in the analytical treatment of the 1D model problem, as outlined before. The students are then asked to prove that this method only adds diffusion in the direction of the streamlines, by computing the elementary (3x3) contribution **S**e to the system matrix originating from the extra term in the weighing function:

$$\mathbf{S}_{e,ij} = -\frac{\tau}{4\Delta_e}(b_i \quad c_i)\begin{pmatrix} v_x^2 & v_x v_y \\ v_x v_y & v_y^2 \end{pmatrix}\begin{pmatrix} b_j \\ c_j \end{pmatrix} \quad (27)$$

This is simply done by rotating the coordinate system axes into the streamline and crosswind direction, causing the additional diffusivity tensor to transform into

$$\tau\begin{pmatrix} \|\vec{v}\|^2 & 0 \\ 0 & 0 \end{pmatrix} \quad (28)$$

Once again, it takes not more than five minutes to implement this in the software, so the students quickly see the effect of this stabilizing term. Slight oscillations still occur, but they are localized around the typical sharp gradients observed in convection dominated problems. The mathematical proof of this (Godunov theorem) is out-of-scope.

E. Mathematical framework

Being familiar with the more complicated notations in 2D, using vectors, tensors, differential operators and integrals, the step towards a more formal description of a finite element problem is straightforward. For example:

Let H^{1h} be a subset of the Hilbert space H^1 of square integrable functions with square integrable 1^{st} order derivatives, such that

$$H^{1h} = \{T^h | T^h \in C^0; T_e^h \in P^1 : \forall e \in M_h\} \quad (29)$$

where M_h is the triangulation, e an element, P^1 the space of first order polynomials and C^0 the space of piecewise continuous polynomial base functions over the triangulation M_h. Let S^h be the set of shape functions

$$S^h = \{T^h | T^h \in H^{1h}; T^h = T_D \text{ on } \Gamma_D\} \quad (30)$$

and V^h be the set of weighing functions

$$V^h = \{T^h | T^h \in H^{1h}; T^h = 0 \text{ on } \Gamma_D\} \quad (31)$$

where Γ_D is the Dirichlet constrained part of the domain boundary. Let $(f,g)_\Omega$ be an inner product defined as

$$(f,g)_\Omega = \int_\Omega f(\Omega)g(\Omega)d\Omega \quad (32)$$

Then the Galerkin finite element solution for a convection diffusion problem consists in finding a function $T^h \in S^h$ such that $\forall W^h \in V^h$ the following holds:

$$\begin{aligned}(\nabla W^h, \bar{\kappa}\nabla T^h)_\Omega + (W^h, \vec{v}\cdot\nabla T^h)_\Omega \\ = (\nabla W^h, \bar{\kappa}\nabla T^h \cdot \vec{n})_\Gamma + (W^h, q)_\Omega\end{aligned} \quad (33)$$

It should be obvious that only extremely brilliant students are capable of translating such type of formulations into a correctly functioning finite element solver. By using the approach adopted in this paper, the lecturer can easily relate any detail of this formalism to theoretical and practical items that have already been covered earlier, yet in a more comprehensible way. Nevertheless, it is important that students get acquainted with this abstract formalism. They should be made aware of the fact the finite element method rests on a beautiful mathematical framework and that advanced research requires a thorough understanding of it. In that respect it is referred to [5] for a brilliant book on the foundations of finite elements.

VI. FE SPIN-OFFS

With the basic of finite element methods in mind, the students are ready for some extensions, which are briefly outlined below.

A. Nonlinear FEM

In a certain sense, treating nonlinear problems is not a finite element issue, but one of solving systems of nonlinear equations. Students normally have studied this in their bachelor program. Therefore, the principles of iterative gradient based line search and/or trust region Newton methods should normally only be discussed briefly [6].

The main purpose of treating this topic in a finite element context is to show in which way a solver needs to be modified to include non-linear information to speed up the nonlinear convergence rate of a solver. Students are first requested to implement a non-linear iterative loop around the linear solver they already have, i.e. implementing a successive substitution algorithm with some relaxation. Despite prior knowledge, this step appears to be difficult for some students. Then it is outlined in which way derivative information from the material model translates into an additional term in the Jacobian matrix, next to the linear system matrix [2]. As with adding a convective term or stabilization term, the implementation of this particular enhancement is a matter of a few minutes programming. Students can then focus on comparing convergence rates of both non-linear solvers they now have implemented.

B. Time-harmonic and transient FEM

Regarding time-harmonic FEM, it is only conceptually focused on the main items:

- the method of describing phasors using complex numbers such that problems can be studied in frequency domain;
- the derivation of the contribution of the phase-shifted eddy current term to the system matrix;
- the properties of the resulting system of equations and its consequences on the iterative linear and non-linear solvers that can be used [7];
- the conflict that rises in the material modeling, when systems are driven into their nonlinear range, e.g. due to saturation at the flux density peaks [8];

The second point in this list is interesting for the lecturer willing to test a student's level of abstraction: ask him to derive the contribution of a reactive term to the system matrix and check whether he sees the relation with an eddy current term or not.

Transient FEM is not thoroughly discussed in the author's lecture series, since it is a huge topic and since the total number of contact hours is simply too small. Therefore, only the basics of time-stepping are explained and some real-life examples from the lecturer are shown.

VII. FE REVISITED

A. Guided play with commercial software

Being close to the end of the lecture series, students not only have received sufficient theoretic background on finite elements, but they also have learned what the complexity of a finite element solver is. The moment has arrived to take them back to the beginning and have them play once again with the same (commercial) software package as before. They now look with completely different eyes on the various parameters to be set in pre-processing or to be judged upon in post-processing. They realize the many bridges they have passed, but as well the bridges that are still in front of them. This session should not be long, but it gives a lot of satisfaction to the students.

B. Industrial applications

The last session is entirely dedicated to industrial applications. For that purpose, the lecture does not take place at the university but in a company nearby: Atlas Copco. Simulation experts from that company are asked to talk about their experience with finite element methods or other related techniques such as finite volume methods. And it is not only focused on magnetic simulation, but on structural analysis and fluid dynamics as well.

This session is once again an eye-opener since many practical issues are addressed that could not be covered during the lecture series. For example: the need to modify CAD-models (removing bore holes, flattening small rims, …) before letting the mesher do its work; the lack of accurate material models and its consequences on accuracy; the element types in structural FE; the company's interest in simulation (cost and design cycle time reduction) … However, the specialists all bring a similar message, that is the fact that FE software is not black box software, thus thorough knowledge of the finite element fundamentals is required to successfully retrieve meaningful results from it.

VIII. LAB SESSIONS

As mentioned before, the students are asked to write their own finite element solver for 2D convection-diffusion equations, starting from a generic description of the problem. To do so, they are given the following set of text-files

- *nodes.txt* → A matrix containing the (x,y) coordinates for all mesh nodes. The line number corresponds to the node number.
- *elems.txt* → A matrix containing the node numbers for all triangular elements, together with a number representing the region to which each element belongs. The line number corresponds to the element number.
- *sources.txt* → A matrix containing the value of the source (i.e. current density or loss density) for all regions. The line number corresponds to the region number. Obviously, source free regions are given a zero value.
- *ucons.txt* → A matrix containing the node numbers of all Dirichlet constrained nodes, together with a number representing the constraint label assigned to it.
- *uconsvals.txt* → A vector containing the values for each label to which a Dirichlet condition is assigned.
- *orientations.txt* → A vector containing the angular positions of the principal material direction for all regions.
- *velocities.txt* → A matrix containing the x and y components of the local velocities for all elements.
- *materials.txt* → A vector containing the material names for all regions.

Materials and their properties are defined in separate files. Each material file contains electric, magnetic and thermal properties, indicated by one of the following headings: *CONDUCTIVITY*, *PERMEABILITY* or *DIFFUSIVITY*. Such headings are followed by generic descriptions of the particular property. Properties are characterized by two flags: isotropic or anisotropic, and linear or nonlinear:

- Linear isotropic properties are represented by a single number, e.g. the permeability;
- Linear anisotropic properties are represented by two different numbers, e.g. the permeability tensor entries in the principal coordinate system of the material;
- Nonlinear isotropic properties are represented by a set of points on a curve, e.g. the magnetization curve;
- Nonlinear anisotropic properties are out-of-scope.

The lecturer has a set of problems and solutions to his disposal which students are free to use. The given solutions are initially used for programming two visualizations routines: one for showing the temperature or magnetic vector potential distribution, the other for showing the distribution of the temperature gradient or the flux density. The given solutions are of course used as well for ultimately checking the correctness of their solver implementation.

One of the main hurdles the students need to tackle, is implementing an efficient matrix assembly routine, which directly accounts for Dirichlet conditions on one or more element nodes, and which automatically symmetrizes the system matrix [2]. They are given the flowchart of Fig. 8 as an aid.

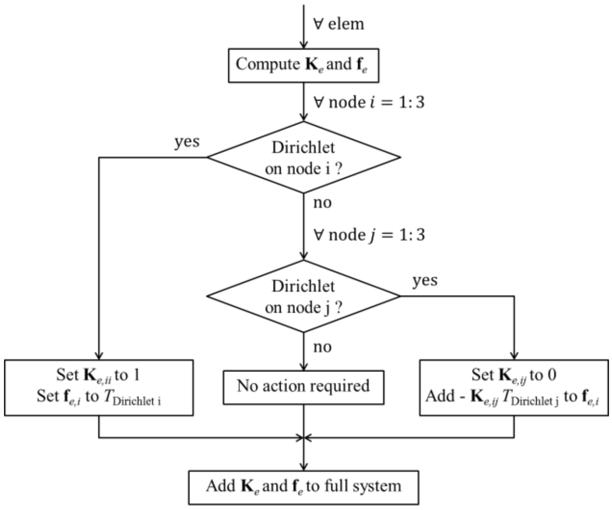

Fig. 8. Matrix assembly scheme.

Having completed their linear FE solver for simulating 2D diffusion problems that only contain linear isotropic materials, students are requested to add some extra functionalities to their software implementation:

- The first, and easiest, is adding a convective term and stabilization term for convection dominated problems.
- The second is extending the program for linear problems in which anisotropic materials are used. In this respect, the concept of a modified reluctivity tensor (Eq. 19) requires special attention, for it needs a slightly different treatment than the thermal diffusivity tensor.
- The third, and toughest for most students, is implementing a nonlinear loop around the linear solver, such that nonlinear problems can be solved as well.

It must be said that not all students succeed in all these, but they have at least thought about it once.

IX. Exam

The presented approach on educating finite element theory and practice is highly interactive. After a few sessions, the students are getting familiar to this system and are being convinced about its added value to their master program. After 50 hours of intensive contacts, the lecturer has obtained a very good idea about the skills of each individual student and actually does not need an exam to make his final judgments. However, some students still like the idea of having an official exam. Below, four of the author's favorite questions:

- Give the students a copy of a non-linear FE solver. Ask them to discuss the main blocks. Ask them to find a bug in the implementation, but tell them what the output of the erroneous program was (loggings, field distributions, typical observations, …) so that they can logically reason towards the solution.
- Show the students some nicely looking, yet erroneous, thermal and/or magnetic vector potential distributions. Ask them to discuss what is wrong.
- Let students assemble the system of equations for solving a 1D diffusion problem on a equidistant grid of not more than five elements. Yet, randomly number the nodes, set a non-homogenous Neumann boundary at one side, and provide one element in which the source term is nonzero.
- Ask the students to derive the contribution of a reactive term (in 2D) to the system of equations, obviously not telling them the similarity with the eddy current term occurring in time-harmonic FE.

X. Conclusion

Educating finite element methods can be done in a variety of ways. The approach presented in this paper finds it origin in the experience the author has acquired with finite element simulation in both industrial as well as academic contexts.

The finite element method is introduced in a very simple way, but the level of difficulty and abstraction is steadily increased. Meanwhile, lots of attention is paid to the practical aspects of finite elements. Students are not only asked to program from scratch a nonlinear FE solver for 2D convection-diffusion problems.

They are as well introduced to commercial software and to industrial applications of finite element simulation. In this way, master students deciding for an industrial career in this field are definitely well prepared, whereas others choosing for an academic career are given sufficient base material to autonomously explore the more fundamental landscape of finite elements.

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Authors Name and Affiliation

Hans Vande Sande, University of Antwerp, Department of Mathematics & Informatics, Middelheim Campus, building G, Middelheimlaan 1, B-2020 Antwerpen, Belgium, Tel. +3232653901, hans.vandesande@uantwerpen.be

Hans Vande Sande, Atlas Copco Airpower n.v., Airtec division, Boomsesteenweg 957, B-2610 Wilrijk, Belgium, Tel. +3234019390, hans.vande.sande@be.atlascopco.com