Technical Article

Remarks on optimal design methods in electromagnetics

***Abstract* — After various decades impressively characterised by the development of tools for field analysis, recent research in electromagnetics has been devoted to the development of tools for field synthesis. An insight on the main methodological streamlines observed in the literature is proposed; in particular, it is shown that algorithms of evolutionary computing could be interpreted in a unified way, because the same basic strategy stays behind most of them. Attention is focused on both single-objective and multi-objective optimisation, with special emphasis on optimal shape design problems; finite element analysis (FEA) is assumed to be the standard way to solve the direct field problem. After a bibliographic review of old and new trends, two methods are deepened: a non-parametric moving-boundary method, and an inferential scheme based on Bayes theorem, respectively. The former makes it possible to explore an infinite-dimensional design space, while the latter generates a distribution of probability yielding optimal solutions with given degree of certainty.**

I. INTRODUCTION

Computational electromagnetism has so advanced, since the advent of digital computers and thanks to the development of numerical methods, that in recent years it has been possible to integrate the analysis of electromagnetic field with optimisation techniques, so moving from computer-aided design (CAD) to automated optimal design (AOD) of systems and devices. Nowadays, in fact, the association of low-price and high-speed computers with numerical libraries makes it possible to identify solutions to inverse problems of various kind and complexity, so offering scientists and engineers the possibility of implementing AOD.

Optimisation of two or more conflicting design criteria – or objectives – is a more recent subject of research. Its significance, not yet completely explored, shows to be great, and many different approaches are still being developed. Papers in the field appear, regularly and quite frequently, in a number of journals; books and monographs have been published and there already exist symposia and workshops, dedicated to this topic and its manifold applications. Several approaches to the topic have been devised: modern heuristic methods, like genetic or evolutionary algorithms, as well as statistical methods have been gradually conquering a paramount role beside the more traditional gradient-based methods.

Nevertheless, the topic is a long way from being exhausted, because some theoretically-unsolved questions stand still, with the related limits in practical applications. For instance, natural and artificial evolution are opposing paradigms. In fact, in a natural system the goal of species evolution is unknown; in contrast, in a computational system, where the minimisation algorithm mimicks an evolution strategy, the objective functions are deterministically defined *a priori*, and this makes a paradox. It is not a merely theoretical difficult: in practice, given a mathematical model of the system considered, a preference function might be composed of *e.g.* a known term assigned *a priori* plus an unknown correcting term. If the latter could vary in a self-adaptive way, the intrinsic tendency of the modelled system to evolve would be exploited: a variational principle, in its essence.

II. FROM DETERMINISTIC TO EVOLUTIONARY COMPUTING: *REDUCTIO AD UNUM* ?

For a long time, deterministic methods of minimisation like e.g. steepest descent, conjugate gradient, quasi-Newton, have been a standard tool to solve local optimization problems. However, the lack of standard conditions for existence and uniqueness of the solution, like convexity and differentiability of objectives and constraints, makes it impossible to deduce the global minimum from the knowledge of the local behaviour of the objective function.

It was argued (Gottvald *et al.* 1992) that if deterministic methods are able to give occasionally good results, this happens for a paradoxical reason: the violation of convexity, smoothness and accuracy requirements makes the optimization procedure a kind of random numerical process. Actually, the slope vector, which must be updated at each iteration to identify the search direction, should be considered as a random quantity due to the inaccuracies introduced during the computation of the function and its gradient.

For instance, inaccuracy in computing the gradient of a numerically ill-conditioned objective function might originate a false minimum; in this respect, the Loney solenoid optimal design is a classical benchmark (Di Barba *et al.* 1995).

This rationale put the ground to move from deterministic to derivative-free non-deterministic methods of minimization, in which classical conditions of convexity and differentiability are replaced with principles of probability and statistics.

The basic mathematical structure governing the evolution of a random vector in the n_v-dimensional search space is a probability density function (PDF)

$$f(x,m,d) \approx \exp\left[-\sum_{i=1}^{n_v}\left(\frac{x_i - m_i}{d_i}\right)^2\right] \tag{1}$$

subject to

$$\int_{\Re^{n_v}} f(x,m,d)\,dx = 1 \tag{2}$$

where m_i and d_i are mean value and dispersion of a random sample x_i , respectively. In (1) the choice of the Gaussian law is due to the *natura non facit saltus* principle: in fact, in natural evolution small changes happen more frequently than bigger ones.

A primary advantage of evolutionary computing is its conceptual simplicity; in fact, following the Darwinian language, a very basic pseudo-code could be cast as follows:

i) initialize a population of individuals;
ii) randomly vary individuals;
iii) evaluate fitness of each individual;
iv) apply selection;
v) if the terminating criterion is fulfilled then stop, else go to step ii).

The algorithm consists of initialization, which may be a purely random sampling of feasible solutions (step i), followed by

variation (step ii), and selection (step iv) based on a preference function (the fitness, step iii). The preference function attributes a numerical value to each feasible solution, in such a way that two competing solutions can be hierarchically ranked. New solutions are generated by randomly varying existing solutions according to the given PDF; the random variation may include mutation (as in evolution strategies) and recombination (like in genetic algorithms). Selection is applied to determine which solutions will be maintained into the next generation. Unlike deterministic methods, gradient information is not required. Over iterations of random variation and selection, the population can be made to converge to the optimal solution (step v).

It can be noted that the basic algorithm behind evolutionary computing is always the same. According to (Fogel 1999), the procedure generating a new solution may be written as the difference equation

$$x(t_{k+1}) = s[v(x(t_k))] \tag{3}$$

with initial solution $x(0) = x_0$; in (3), $x(t_k)$ is the population at time t_k, while v is an operator of random variation driven by the PDF, and s is the selection operator driven by the fitness.

Several representations of a population are possible, and many variation operators as well as selection operators can be defined: in the literature, this gave rise to very many declinations of the same basic algorithm, under different names and in different contexts. The effectiveness of an evolutionary algorithm depends on the interdependence between the operators s and v applied to a given representation x of the evolving population, with initialization x_0. In practice, this interdependence gives freedom to the designer to tailor a specific evolutionary approach for a particular problem of interest. On one hand, this feature gives an extra advantage over deterministic optimisation methods; on the other hand, however, the risk of reinventing already known methods is definitely high.

In particular, the same basic algorithm could be interpreted in several ways; biology-oriented and statistics-oriented interpretations, which have become popular, are summarized in Tables I and II, respectively.

TABLE I. BIOLOGICAL INTERPRETATION OF EVOLUTIONARY COMPUTING.

m vector	parent individual
x vector	outspring individual
updating of vector m	selection
optimisation trajectory	species evolution

According to the biological interpretation, in evolution strategy algorithms, the role of the mutation operator in terms of selection, survival, death of individuals is paramount, while in genetic algorithms, the cross-over operator is paramount.

TABLE II. STATISTICAL INTERPRETATION OF EVOLUTIONARY COMPUTING.

m vector	mean value of the random set
d vector	dispersion of the random set
Gaussian law	probability density function
optimisation trajectory	stochastic process in the search space

According to the statistical interpretation, the following remark can be put forward: since the accuracy of a Monte Carlo integration depends on the number of statistical samples, it comes out that the convergence of the relevant optimization algorithm loosely depends on the number of variables.

Considering both interpretations, seemingly independent methods might be unified, possibly originating more powerful schemes of global optimisation, for a certain class of problems. This idea is not in contrast with the no-free lunch theorem (Wolpert and Macready 1997), sometimes over-emphasized a result. In fact, the simple conclusion of the no-free-lunch theorem is that there is no best evolutionary algorithm, that will always outperform all the other algorithms, *regardless of the given problem*. This result has originated a great deal of controversy in the area of evolutionary computing, and some misunderstanding too. In the nineties of the last century, there has been a considerable effort in finding the best set of operators and 'tuning knobs' of evolutionary algorithms. In genetic algorithm area, for instance, these efforts have involved the probabilities of crossover and mutation operators, the representation of a population, its size and so forth. In particular, most of this research has stimulated numerical experiments on benchmark functions.

However, the no-free-lunch theorem essentially states that conclusions based just on a set of numerical trials are limited only to the benchmark functions considered. In practice, design engineers are only interested in a subset of problems; consistent with the theorem, it makes sense to select an algorithm which outperforms other algorithms for a particular class of problems (Baritompa *et al.* 2005).

III. SINGLE-OBJECTIVE OPTIMAL DESIGN: EARLY CONTRIBUTIONS

Most of the fundamental and basic optimisation work was performed in the last decades of last century; it is important to make this point clear, because several ideas come back today under another name. Sometimes, it seems that the electromagnetic community starts working again on known results.

Probably, the first coupling of finite-element method with non-linear mathematical programming was implemented in (Shmit 1960) for optimal design of structures. Since then, much research work has been devoted to solve problems of inverse mechanics. The impact of modern optimisation theory on electromagnetism came later. Among the pioneering contributions, it is worth mentioning (Marrocco and Pironneau 1978) who developed the optimum design of an iron-cored electromagnet with nodal finite elements for field analysis. In (Weeber and Hoole 1993) a review of structural design optimisation as a source for developments in electromagnetic is presented.

After (Tikhonov and Arsenin 1974) the importance of regularization methods to solve ill-posed inverse problems has been recognized also in electromagnetics. For instance, in (Sikora and Palka 1981) and (Adamiak 1981) regularization is used to solve the Fredholm equation of the first kind connected with the synthesis of magnetic fields. In (Rudnicki 1985) the choice of the regularisation parameter, which is a critical matter in the implementation of the method, is discussed. In turn, the singular-value decomposition method has been used *e.g.* in (Sikora et *al.* 1986) for magnetic filed synthesis: the discretisation of the Fredholm equation of the second kind leads to a rectangular system of equations solved by means of the least-square method.

In turn, when the iterative process towards the objective function minimum relies on a gradient-based method, sensitivity analysis is the most critical step for making the solution procedure work. There are two approaches to sensitivity calculation. The first method (discrete approach) differentiates the variational equations governing the minimisation problem, which have already been discretised, as shown in (Salon and Istfan 1986), (Gitosusatro *et al.* 1989),

(Sikora 1989). The second method (continuous approach) acts on the variational equations before they are discretized, like in (Il-han Park *et al.* 1992) for finite-element analysis, and (Chang-seop Koh *et al.* 1992) for boundary-element analysis. For the evaluation of the shape design sensitivity, a method based on the material-derivative concept of continuum mechanics is proposed in (Il-han Park *et al.* 1991).

As far as non-deterministic optimisation is concerned, in (Preis and Ziegler 1990) and (Preis *et al.* 1990) various evolution strategies are applied to the optimal shape design of a magnetic pole in a comparative way. An overview of evolutionary computing is presented in (Kasper 1992), where peculiarities of different algorithms are discussed. Another non-deterministic method of lowest order is simulated annealing: it mimicks the thermodynamic behaviour of a solid system that is slowly cooled in order to reach its lowest energy rate. In (Simkin and Trowbridge 1992) this method is used to solve a shimming problem, in which the finite-element method is used for magnetic field analysis.

The finite-element method is generally preferred for field analysis in devices exhibiting a complex geometry. An early contribution was in (Nakata and Takahashi 1983), where an innovative design procedure for permanent magnets, linking the finite-element method to a gradient method, was presented. The same approach is used in (Hoole *et al.* 1991) for the identification of cracks and their geometry inside a conductive bulk. In (Kadded *et al.* 1993) a strategy of active constraint set is used for the optimal synthesis of the field, in a permanent-magnet machine simulated by the finite-element method.

Moreover, the finite-element method combined with regularisation is used in (Palka 1983) and, associated with singular-value decomposition, in (Sikora *et al.* 1986).

When the field analysis is based on integral formulae, like those amenable to Green theorem, the synthesis is often reduced to the solution of a Fredholm integral equation of the first kind, which is generally obtained by means of regularisation. This is the case of the electrostatic problem discussed in (Sikora and Palka 1981) and the magnetostatic problem investigated in (Adamiak 1981) and (Palka 1984). In contrast, in (Tsuboi and Misaki 1988) the optimal shape design of an electrode is obtained by means of surface-charge simulation and gradient-based method.

The boundary-element method has proven to be convenient for electrostatic problems in homogeneous domain. It has been used with a least-square approach *e.g.* in (Sikora *et al.* 1985), for the identification of the boundary conditions in an electrostatic problem. It has also been combined with a search technique for the optimal design of an electrode in (Liu Jin *et al.* 1990). In turn, the problem of the optimal location of an electrode is solved in (Sikora 1989), using the finite-element method for the discretisation of the field region and a min-max formulation.

As far as the acceleration of stochastic methods is concerned, an early contribution is in (Hameyer and Kasper 1993), where various approaches are coupled to implement a cost-effective optimisation of a small DC motor for automotive applications; the magnetic field analysis is based on finite elements.

This short selection of early contributions simply aims at proving the long-lasting heritage of optimal design methods in computational electromagnetism.

IV. From single to multiple objectives: optimal shape design

In electromagnetics, shape design problems are particularly meaningful among AOD problems. In fact, the essential goal of shape design is that of identifying, in a completely automated way, the geometry of the device that is able to provide the prescribed performance, fulfilling a set of constraints. This is actually an inverse problem that, in general, implies the simultaneous minimisation of conflicting objectives. In fact, in engineering practice, one usually has to do with multiple objectives to fulfil at a time in the design of a device or a system, while the presence of a single objective is somewhat an exception or a simplification. Actually, there is a need for multiobjective optimisation for a number of reasons:

- in general, industrial problems have multiple solutions which fulfil objectives and constraints, thus multiple optimal solutions arise;
- often, in industrial applications, some solutions can be preferred to others, so it is better to get a spread of feasible solutions from the design procedure rather than a single solution;
- when a set of optimal solutions is available, the selection is left to an external decision maker (usually, the designer) who can express the final preference.

At the time being, single-objective models are becoming less attractive than in the past, because there is some scepticism as far as their real usefulness is concerned. In fact, having a unique solution to the design problem, which is assumed to be the optimum, is too rigid a limitation and can be unpractical, or even unfeasible, from the viewpoint of an industrial designer: thus, optimising a single objective is not seen as particularly useful and realistic too. As a consequence, the future of computational electromagnetism seems to be oriented towards, and conditioned by, the development of effective methods and robust algorithms for solving multiobjective shape design (MOSD) problems. The interest of the community on multi-objective optimisation is reflected by the increasing number of publications issued in the last few years. For instance, the biennial Conference on Evolutionary Multi-Criterion Optimization - EMO (held for the first time in Zurich, 2001) and the IEEE Transactions on Evolutionary Computation are entirely focused on the subject. However, in the area of computational electromagnetism, there is not yet a specialised forum completely devoted to multiobjective design, even if relevant papers are usually presented at major conferences, *e.g.* the biennial IEEE Conference on the Computation of Electromagnetic Fields (COMPUMAG), regularly organised since the year 1976, and the biennial IEEE Conference on Electromagnetic Field Computation (CEFC). A smaller, but more focused, biennial meeting is the International Workshop on Optimization and Inverse Problems in Electromagnetism (OIPE), founded in the year 1989 at the University of Pavia just for disseminating recent advances at various levels, from theoretical to applicative ones. It is worth mentioning also the International Symposium on Electromagnetic Fields in Electrical Engineering (ISEF), a biennial meeting about computational electromagnetism in general.

V. Classical *vs* Paretian formulation

In electromagnetic design, the traditional approach to multiobjective optimisation implies to set up a preference function, expressing a compromise among the various objectives and depending on weight coefficients or threshold values; successively, standard single-objective optimisation methods can be used to find the optimum of the preference function.

Although offering a friendly implementation, scalar formulation presents several drawbacks: first of all, the result of an optimisation gives a single solution, which is supposed to be globally optimal. A first criticism is that it is not clear whether the solution found is a non-dominated one; moreover, even if it is dominated, it is not clear where the solution is located with respect to the relevant Pareto front (PF).
Another criticism follows: solving a multiobjective optimisation problem gives rise to a variety of solutions, which are spread along the PF; knowing this variety is useful for the designer, who is provided with a wide range of possibilities for a choice *a posteriori*. In order to obtain different solutions located on the PF, one could run subsequent single-objective optimisations after modifying the preference function, and then collect results in a comparative way. This strategy might work in some particular cases; in general, however, it is unattractive for a twofold reason; in fact, it might originate dominated solutions; moreover, it makes it difficult to generate solutions uniformly spaced on the PF.
Various scalar formulations are possible (Miettinen 2000): objective-weighting, ε-constraint, min-max, goal attainment are most commonly used in electromagnetic design. However, all of them are single-objective in their essence and require some amount of problem knowledge. The main practical difficulty is that all of them need to be applied several times to find an approximation of the Pareto front of the given problem. This makes these methods loosely attractive. In general, it can be stated that classical optimisation methods suffer from various limitations:

i) an algorithm is needed to be applied several times to find multiple Pareto-optimal solutions;
ii) most algorithms require some *a priori* information about the problem being solved (*e.g.*, in the ε-constraint method both hierarchical relationship among objectives, and selection of the numerical grid fixing constraint levels, are required);
iii) some algorithms are sensitive to the shape of the Pareto front (*e.g.* objective weighting);
iv) in most algorithms it is not possible to control the spread of Pareto-optimal solutions along the front;
v) in problems involving uncertainties classical methods are not effective.

As a final remark, it should be pointed out that scalar formulations with variable weights and goals can be successfully used as front sampling techniques in few cases only: the main risk, in fact, is to generate dominated solutions.
One of the key points of methods based on Pareto optimality is the *a priori* or *a posteriori* use of higher-level information, when the decision maker has to choose among various compromises (Ringuest 1992). Often, in industrial electromagnetic design the *a priori* choice is preferred because it is easier to understand in its meaning and it is easy also to implement. On the other hand, the *a posteriori* choice implies a deeper knowledge of the problem, and gives the decision maker a more flexible approach to the design, because all non-dominated solutions must be taken into account in a comparative way before selecting one of them. A flow of this twofold logical way is shown in Fig. 1.
The Paretian formulation of a multiobjective problem, keeping conflicting objectives separate and discarding the use of scalar preference functions, seems to be more promising from both the theoretical and practical viewpoint. Following this approach, the extra knowledge - which is required in order to choose a single solution out of the theoretically infinite solutions of the multiobjective problem - is used *a posteriori* (*i.e.* after the optimisation process) instead of *a priori* (*i.e.* before the optimisation process).

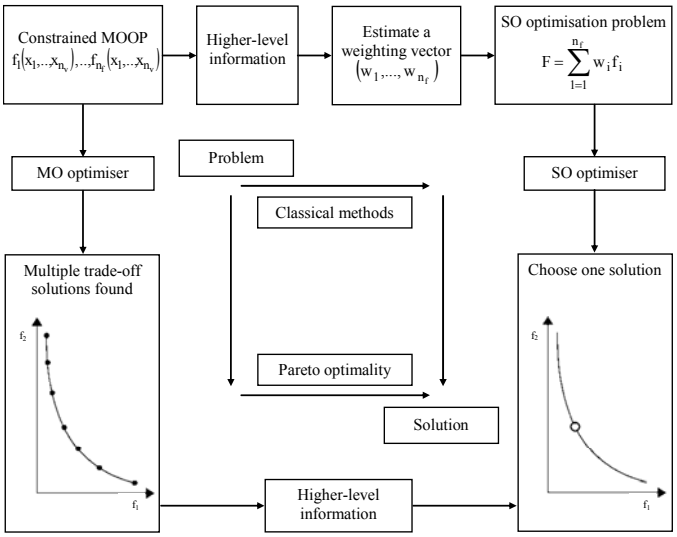


Fig. 1. Logical paths of classical and Paretian formulations.

A comprehensive review of methods for multiobjective optimisation with annotated bibliography is in (Collette and Siarry 2003).

VI. Multiobjective optimal design: a bibliographic insight

Looking back at the history of evolutionary computing, in the year 1967, Richard Rosenberg suggested a genetic search method for finding the chemistry of a population of single-celled organism with multiple properties or objectives. However, its first practical implementation was due to David Schaffer a long time after, in the year 1984; then, for almost a decade, no significant study was performed, until the year 1989 when the book by David Goldberg appeared, in which an innovative ten-line code of a non-dominated sorting procedure was presented.

Evolution strategies, in turn, are a branch of evolutionary computing which has originally developed in Germany by Ingo Rechenberg and Hans Paul Schwefel. The main feature is that the search algorithm facilitates self-adaptive strategy parameters, *i.e.* the automated adaptation of critical quantities like *e.g.* the standard deviation of the mutation operator during the optimisation procedure (Rechenberg 1973) and (Schwefel 1977). This property turns out to be the key feature, characterising evolution strategies with respect to other approaches, and making them effective algorithms of global optimisation (Bäck 1996).

The publication of early-time results showed the superiority of evolutionary methods of multi-objective optimisation over classical methods. Thereafter, many researchers developed different versions of algorithms for evolutionary multi-objective computing, with special emphasis on genetic algorithms (GA). Among the most popular of them, there are:
vector-evaluated GA (Schaffer 1984);
multiobjective GA (Fonseca and Fleming 1993);
niched Pareto GA (Horn *et al.* 1994);
strength Pareto evolutionary algorithm (Zitzler and Thiele 1998);

non-dominated sorting GA (Srinivas and Deb 1994) and (Deb *et al.* 2002).
The latter, in particular, became popular in computational electromagnetism.
Focusing now on the electromagnetic community, after various decades impressively marked by the development of tools for field analysis (think *e.g.* of fundamental contributions from the TEAM Workshop series), recent work in electromagnetics has been devoted to the development of tools for field synthesis. The whole topic of single objective optimisation in electromagnetics was exhaustively presented in (Neittaanmäki, Rudnicki and Savini 1996), a book which is still valid, at least from the methodological viewpoint. Multiobjective optimisation is a more recent subject in electromagnetics. A pioneering contribution can be found in (Russenschuck 1990), where the shape design of a permanent-magnet synchronous machine is formulated as a vector optimisation problem. The subject has been recently revisited and enhanced (Russenschuck 2010). In the last decade, in particular, evolutionary computing applied to multiobjective optimisation has proven to be successful in finding a distributed set of solutions, approximating the Pareto front of a given design problem. Evidence of this can be found in the extensive and rapidly growing literature which has already been published. Several papers are in the pipeline and new ideas emerge faster than they can be published.
Actually, it is not an easy task to review the literature on multiobjective optimisation (MOO) in electromagnetism. Due to its multidisciplinary aspects, the topic is scattered in a number of journals and conference proceedings. For instance, if the proceedings of two companion conferences like CEFC and COMPUMAG are examined, the ratio of number of papers on MOO to the total number of papers on AOD increases in time, starting from the years 1997 (CEFC, Rio de Janeiro) and 1998 (COMPUMAG, Tucson), respectively.
Anyway, this survey does not intend to be comprehensive, so that the reader should not be surprised if some important reference have been omitted; the criterion used was that a few publications were selected to represent the past and current trends in the various aspects of each topic.
From the viewpoint of methods to solve multiobjective optimisation problems, at least two main streams can be observed.
The former focuses on the use of approximation techniques to identify response surfaces representing the dependence of objective functions and constraints on the design variables. This way, field computation is required to generate surfaces, which are used to drive the search: global-optimisation oriented algorithms or nature-inspired algorithms, are then applied to find an approximation of non-dominated solutions. The surrogate model so generated is interpreted as an accurate representation of a given objective function, and the minimum of the interpolating surface is evaluated. However, this might lead to a false minimum, *i.e.* a point which minimises just the surrogate model, but not the true function. Therefore, close to a possible optimum, an assessment based on the real function evaluation is performed. The advantage in terms of computational cost is evident: field analysis is limited to the identification of response surfaces and to assessment points, while the evaluation of objectives and constraints during the search is inexpensive. A survey of main methods based on surrogate models is presented in next Section.
In turn, the latter stream of methods preserves the use of finite-element models at each iteration of the optimisation procedure (Dias and Vasconcelos 2002). Actually, the FEM is commonly employed in the AOD of electromagnetic devices, where computation of field is required for the evaluation of objectives and constraints: it is well known that this step takes most of the optimisation time, while the cost related to the optimisation procedure in itself is often negligible.
The computational burden of each function call could be lightened *e.g.* by matching accuracy and velocity of the FE analysis in a convenient fashion. The basic idea is to adjust the refinement of the mesh, according to the level of accuracy required for the objective function calculation, rather than for the field solution itself. To this aim, adaptation methods (Haslinger and Neittaanmäki 1988) are sometimes employed to improve the mesh. Similar methods were coupled, for example, with deterministic methods of single-objective optimisation, in order the mesh to be refined as the current solution gets closer to the minimum (Brandstätter *et al.* 2001) and (Gavrilovic and Webb 2002). What is common to the aforementioned approaches is that the mesh is adapted, in order to reduce the error in the objective function calculation at a given iteration. Likewise, in (Di Barba 2009) a strategy of mesh adaptation is linked with an algorithm of evolutionary MOO.
As far as applications are concerned, the suitability of the multiobjective approach to industrial electromagnetic design problems has been discussed in several contributions. In fact, procedures of optimal design are currently applied to both low and high frequency devices (Lu *et al.* 2009). In the literature, the applications of the Pareto theory to the optimal design of electromagnetic devices are growing. After early contributions (Chiampi *et al.* 1996), (Surdacki and Montusiewicz 1996) and (Borghi *et al.* 1999), a new research field is emerging, in which the traditional design, based on single-objective optimisation, is critically revisited and the development of new numerical methods and relevant algorithms is stimulated (see *e.g.* (Ho *et al.* 2002) and, more recently, (Régnier *et al.* 2005)). Low-frequency applications are mainly focused on electromechanics, with special emphasis to the shape design of magnets and rotating machines (Kim *et al.* 1998); as for high frequency, evolutionary multiobjective optimisation (EMO) is used in antenna design (Lisboa *et al.* 2006). Classical fields of design like magnetostatic systems, actuators and electrical drives are still popular.
In computational electromagnetism, MOO is still quite recent a streamline of research: it is mainly oriented to the development of cost-effective algorithms (Paul *et al* 2009) and (Wanner *et al* 2008). In areas like *e.g.* aerospace engineering, MOO is becoming more mature a subject: for instance, multiobjective design is commonly applied to the shape design of airfoils for subsonic flight (Sasaki *et al* 2001). A very basic algorithm of multiobjective evolution strategy in electromagnetics was proposed in (Di Barba 2000).
More sophisticated strategies based on parallel computing are nowadays viable. In this respect, the identification of the magnetic permeability of an anisotropic material, based on a three-dimensional field model, is presented in (Di Barba *et al.* 2011). A review of differential evolution methods suitable for a multicore processor can be found in (Alotto 2010).
Resorting to game theory as an alternative to Paretian optimality, a straightforward modification of a basic evolution strategy has proven to be successful in finding a unique optimal solution, corresponding to the Nash equilibrium of a bi-objective design problem (Di Barba 2010).
Dynamic multiobjective optimisation is another promising subject. In fact, in problems of shape design of electromagnetic devices, usually objectives and constraints do not depend on time. When either the objectives or the constraints, or both of them, depend on time, the Pareto-optimal front is time dependent too. In that case, it is not possible to identify a set of non-dominated solutions by

exploring the objective space at steady state, but the time evolution is to be taken into account (Di Barba 2010).

Nature-inspired algorithms offer promising strategies of search. The main popular are the following: ant colony (Dorigo 1997), swarm intelligence (Beni and Wang 1989, Reddy and Kumar 2007), artificial immune systems (de Castro and Von Zuben 2002), cultural evolution (Reynolds 1994). As far as electromagnetic design is concerned, an application of immune algorithms can be found *e.g.* in (Canova *et al* 2005) and (Batista *et al* 2009), swarm intelligence is used in (Baumgartner *et al* 2004), while cultural algorithms are employed in (dos Santos Coelho and Alotto 2009).

Recently, the extension of an artificial-immune-system algorithm to the MO case, with an application in magnetostatics, was proposed in (Freschi and Repetto 2006).

VII. Surrogate modelling and optimal design

For a surrogate model to be interpolating an objective function f, it is necessary to have m_s+n_s sampled points and to use a set of additional basis functions ϕ, each centred around one out of n_s observations. In general, the predictor can be written as

$$p(x)=\sum_{i=1}^{m_s} b_i\psi_i(x)+\sum_{j=1}^{n_s}\beta_j\phi(x-x_j) \qquad (4)$$

where ψ_i are basis functions modelling a global trend (*e.g.* a low-order polynomial), while the ϕ-dependent term can be viewed as a functional deviation; coefficients b_i and β_j are found by *e.g.* least-square fitting.

Issues of meta-modelling and global optimization are discussed in (Mullur and Messac 2006), while neural-network models of inverse electromagnetics are used *e.g.* in (Cau *et al.* 2007) and (Carcangiu *et al.* 2008).

Actually, multiple choices exist for ϕ; for instance, kriging is a surrogate model based on statistical foundations, first developed in geoscience in the sixties of last century (Jones 2001). It assumes

$$\phi(x-x_j)=e^{-\sum_{j=1}^{n_s}\theta_j|x-x_j|^{\alpha_j}} \qquad (5)$$

where $\theta_j\geq 0$ and $\alpha_j\in\left]0,2\right]$; therefore, a Gaussian stochastic process is used to model the second term in the right-hand side of (5). Again, the exponential family has a paramount role.

A certain number of points needs to be sampled before a kriging model can be constructed (Santner *et al.* 2003). This initial set is called an experimental design, and the theoretical background for selecting suitable points is known as design of experiments (Montgomery 2001). In the past, techniques derived from design of experiments were used for planning computer simulations in a cost-effective way; at the time being, the two commonly used techniques of experimental design are the Latin hypercube (Jones *et al.* 1998) and the Hammersley sequence (Kalagnanam and Diwekar 1997). In the context of statistical sampling, a square grid containing sample points is termed a Latin square, if there is only one sample in each row and each column. A Latin hypercube is the generalisation of this concept to an arbitrary number of dimensions, where each sample is the only one in each axis-aligned hyperplane containing it. Latin hypercube sampling ensures that the set of random samples is a good representative of the real variability of the function. An improvement is given by orthogonal sampling, in which the sample space is divided into equally probable subspaces: all sample points are then chosen simultaneously, in such a way that the whole set of points is a Latin hypercube and that each subspace is sampled with the same density.

From the viewpoint of surrogate modelling, optimisation methods can be classified according to the kind of model used, and the technique to select points driving the search. In two-stage methods, first the surrogate model is fitted to the observed points and, then, an utility function is used to find the next search point. In turn, in one-stage methods, a new search point is generated which would yield the most likely response surface. Almost all existing algorithms are two-stage; however, one-stage algorithms have been implemented, using both kriging and radial basis-function models (Regis and Shoemaker 2007).

Recently, a hybrid one-then-two stage algorithm has been proposed, consisting of three steps: initialization, one-stage experimental design, and two-stage search (Hawe and Sykulski 2007). Initialization allows a kriging model to be constructed by means of a set of $4n_v$ sample points, based on Hammersley sequence. Successively, the one-stage experimental design aims at deciding where to sample next. To this end, let f_{min} and f_{max} be minimum and maximum values of the objective function within the set of $4n_v$ sampled points, respectively. A design vector $\tilde{x}$ is assumed to exist, which is the inverse image of

$$\tilde{f}\equiv f_{\min}-c\left(f_{\max}-f_{\min}\right) \qquad (6)$$

where parameter $c\in[0,1]$ is updated using a cyclic scheme dependent on the iteration index. At each iteration, the design vector actually chosen for evaluation is the one which maximises the likelihood of the assumption that its image is $\tilde{f}$. This step is repeated until $10n_v$ points in total have been evaluated.

Finally, in the two-stage search, a kriging model is constructed using the $10n_v$ previously sampled points; the weighted-expected-improvement utility function (Sobester *et al.*, 2005) is then used to select new points for evaluation. The whole procedure is repeated until the stopping criterion is fulfilled.

Multiobjective optimisation methods using surrogate models can be divided into scalarizing and non-scalarizing. The former combine multiple objectives into a scalar preference function and then use one of the methods for single-objective optimisation (see previous Section V). By varying the parameters which control how multiple objectives are combined, an approximation of the Pareto front can be recovered (Knowles 2006). In turn, non-scalarizing methods consider each objective individually; the simplest approach is to evaluate the non-dominated solutions predicted by the surrogate models associated with the objectives; an application in electromechanics is in (Lebensztajn *et al.* 2005).

Evolutionary algorithms using surrogate models belong to the class of non-scalarizing methods; cost-effective algorithms have appeared rather recently (Emmerich *et al.* 2006) and (Keane 2001).

A possible alternative could rely on field diakoptics and generalized Thévenin theorem. In fact, when solving analysis problems in electricity and magnetism by means of FEM, it often happens that only a small part of the field domain incorporates the region of main interest. Nonetheless, the analysis of the whole domain should be performed, even in

subdomains of no interest. If repeated field analyses have to be performed, like *e.g.* in optimal design problems, the computational burden is remarkable. In (Di Barba and Savini 2010) an optimal design theory based on a principle of field diakoptics is presented: the region Ω_2 which is not of interest is replaced with a multi-terminal element by means of the generalized Thévenin theorem (Santini and Silvester 1996), and the optimisation takes place within the reduced domain Ω_1 only. The mesh discretising Ω_1 is updated according to the shape variation governed by the minimisation algorithm, while the Thévenin multi-pole is an invariant.

VIII. Topology optimisation and sensitivity analysis

In topology optimisation, the continuous-valued parameterization of the geometric model enables virtually all feasible shapes of the device under consideration to be explored (Kim *et al.* 2004), and this feature is definitely interesting for synthesizing a new device. The methodological background can be found in the level set method (Sethian 1996).

In the traditional approach to optimal shape design, the geometry of the region to synthesize is parametrized by means of a finite set of variables, which are updated by a minimisation algorithm according to the value of the objective function. This way proved to be effective for problems with a low or moderate number of variables; however, difficulties occur when the problem complexity increases. In fact, the performance of every minimisation algorithm deteriorates when several variables are to be handled simultaneously, because the ill-conditioning of the associated inverse problem increases. On the other hand, the description of complicated geometries necessarily asks for a large number of parameters.

In this respect, a parameter-free approach can be promising. In practice, the material distribution – which is unknown - is forced to vary gradually from void to solid state, according to an acceptance criterion preventing the occurrence of distorted shapes (Shim *et al.* 2008). In the case of a source synthesis problem, the distribution of current-carrying conductors (or permanently magnetized domains) is unknown.

In the magnetic case, a generalized sensitivity formula, showing the total derivative of the objective function in the direction of the design vector, has been derived for both the primary and the adjoint systems (Kim *et al.* 2005) and (Kim *et al.* 2007). Despite having been introduced as a mathematical derivation, the adjoint system is physically meaningful, because geometric and material properties are found to be the same as in the primary system. Therefore, both primary and adjoint problems can be solved by means of a standard FE analysis. Then, the sensitivity formula is evaluated and the material distribution is updated, to drive the search towards the distribution corresponding to the optimal shape of the device. Alternatively, procedures of numerically derived sensitivity, like those based on the Lipschitz constant (Campelo, Watanabe and Igarashi 2008), are applicable.

IX. Multi-level optimal design

As complexity of engineering systems increases, often knowledge and experience of just a single designer are not enough to deal with the full design problem. An approach to cope with this difficulty is to decompose the corresponding optimisation problem into smaller ones (components) that are easier to solve (Brisset and Brochet 2005) and (Moussouni, Brisset and Brochet 2008). *A posteriori*, the system solution is recovered from the component solutions. In practice, when *e.g.* a multi-physics inverse problem is dealt with, a complex system can be partitioned by physical domain. Moreover, due to decomposition, a parallel-distributed computation can be implemented.

In particular, target cascading is a hierarchical multi-level design method, useful when an optimisation problem can be organised according to a tree structure (Moussouni *et al.* 2008). This way, a large-scale problem is first decomposed into levels, and levels are subsequently decomposed into sub-problems (or components), each of which acts on a reduced set of design variables. Components exchange data by means of linking variables: the application of a coordination strategy forces consistency (*e.g.* fulfilment of global contraints) among component solutions. In a multiobjective context, an approximation to the Pareto front of the original problem is obtained by composing the fronts of each sub-problem (Nguyen Huu *et al.* 2008).

A drawback of the method is that the derivatives of objectives and constraints must be calculated; in this respect, strategies for the automatic differentiation of computer codes, accurate up to machine precision, have proven to be effective (Enciu *et al.* 2008).

X. FEA tools and optimal design

In a sense, design optimisation is always a field-dependent numerical procedure, because the calculation of objectives and constraints in an irregularly-shaped field domain implies the numerical simulation of electric and magnetic fields which the device operation is based on.

In fact, software for electromagnetic analysis has become a mature tool, commonly used in design offices across all sections of an industrial company. Thinking of an industrial R&D centre, FEM-assisted optimal design is simple to implement, because it does not require a major modification to a commercial software. A basic knowledge of any programming language is enough to link a FEA code with an optimisation routine. Surrogate modelling or target cascading are interesting alternatives to FEM-assisted design because they are finalised to low-cost models of inverse problems; however, they are still at early stages of development and require substantial modifications to a commercial FE software in view of a practical implementation. Moreover, the source code of a commercial FE software is seldom available to an industrial user for making substantial modifications to the program architecture. Finally, even if the source code is available, seldom the man-time, or the know-how, is sufficient for doing it.

The very point is that commercially available FE codes are designed to help the solution of direct problems rather than inverse problems. For instance, in a procedure of AOD, it might be necessary to keep the number of nodes along the boundary constant, and to prescribe their position too. In practice, using a commercial code of FEA, the topology of the domain mesh can be hardly controlled by the user at this level. In fact, it happens that the domain boundaries are meshed first, while the domain subregions are subsequently meshed: the solution mesh is automatically generated in a recursive way to reduce the discretization error, by means of classical techniques like *e.g.* Delaunay triangulation or mesh refinement (George 1991). As a consequence, the best possible mesh for analysis is often not adequate for synthesis.

Moreover, according to a trivial criticism, it happens that optimisation procedures are considered to be nothing but

useless (a mere exercise of mathematics !) for industrial design purposes, mainly because the associated analysis models are simplified in order to keep the computational cost moderate. Again, a misunderstanding between analysis and synthesis takes place: an optimisation procedure, no matter whether deterministic or evolutionary, does not need a high degree of accuracy when the direct problem is solved, because the main goal of the optimisation itself is to identify the right trend towards an improvement of the modelled device. In practice, this means that low-density FE meshes are satisfactory enough, while denser meshes are necessary *a posteriori*, just to compare initial and final solution, so assessing the improvement found. This is a way to make the optimisation work even for complex analysis problems. And the search for robust solutions in terms of small perturbations (another requirement of industrial design) is naturally emphasized by evolutionary computing.

To the best knowledge of the author, a commercial package truly devoted to multiobjective optimisation in electromagnetism has not yet been released, even if some software houses offer tools for automated optimal design (*e.g.* OptiNet code by Infolytica, Optimetrics code by Ansoft, and, more recently, Optimizer by Vector Fields). At an academic level, the NIMBUS algorithm, implementing a non-differentiable interactive multiobjective bundle-based optimisation method, is accessible at the Jyväskylä University web-page.

XI. Beyond Parametric Multiobjective Optimisation: A Moving Boundary Approach

Another way to the optimal shape design is to consider the boundary of the region to synthesize as a front propagating with a velocity dependent on the objective function. So doing, a parameter-free moving boundary problem is originated. By treating the boundary line (in 2D) or surface (in 3D) as a functional level, changes in the propagating front are easily handled.

In natural processes, like *e.g.* crystal growth, the velocity of a propagating front, might be an arbitrary function of its curvature, and the front is passively advected by an associated flow. Likewise, in a problem of optimal shape design, the velocity is a vector field defined in each node of the moving boundary, and dependent on the value of the objective function.

The methodological background can be found in the theory of fronts propagating with curvature-dependent velocity (Osher, and Sethian 1988).

In principle, a simple algorithm could be defined as follows:

i) given a feasible shape of the boundary, solve the direct problem;
ii) compute the objective function (OF);
iii) compute the OF-dependent velocity components;
iv) solve the advection problem in a time step;
v) update the boundary position;
vi) iterate until the velocity of the boundary is zero.

In other words, the boundary of the field region to synthesize is considered as a moving boundary, and the velocity field leading to the optimal solution is unknown. On the other hand, however, the method is not based on a regularization principle; therefore, the evolution of the boundary might fall into a local minimum, and the relevant solution would depend on the initialisation.

A simplified solution to the moving boundary problem could be obtained by means of a kinematic approach. In contrast to various methods of topology optimization based on the level set theory (Kim *et al.* 2009), the kinematic approach does not require to solve the diffusion equation governing the advection problem. In order to mitigate the local behaviour of the moving boundary method, it is here proposed to synthesize the kinematic law, leading to the optimal boundary, by means of a global-minimum oriented algorithm. The key idea is defining the velocity vector *e.g.* as follows:

$$v_x(x,y,t) = v_0 \frac{x - x_0}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} e^{-\frac{t}{T}} \tag{7}$$

$$v_y(x,y,t) = v_0 \frac{y - y_0}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} e^{-\frac{t}{T}} \tag{8}$$

where speed constant v_0 and time constant T are unknown, while (x,y) are the coordinates of n_g moving nodes along boundary γ , and (x_0 ,y_0) is a reference point. So, the velocity field is radially directed towards point (x_0 ,y_0). Accordingly, the correction vectors are:

$$x_{k+1} = x_k + v_x \Delta t \ , \ x \in \gamma \tag{9}$$

$$y_{k+1} = y_k + v_y \Delta t \ , \ y \in \gamma \tag{10}$$

with k iteration index. Having prescribed Δh as the maximum displacement of γ, the time step Δt is defined according to the following constraint:

$$\left[\sup_{(\gamma,t)} \|v\| \right] \Delta t = \Delta h \tag{11}$$

where the velocity components refer to iteration k, while Δt refers to iteration $k+1$.

At each iteration, $t \in [0, 5T]$ is considered; the field analysis is updated at $t=5T$, when the boundary movement is practically expired due to the assumption of exponential time dependence. This way, only one call to the FE solver is required per iteration. Note that the definition of velocity field allows for negative components of velocity, so enabling both expansion and contraction of the boundary. Because the shape of γ is governed by $2n_g$ parameters, a multi-dimensional search, controlled by only two degrees of freedom (v_0 ,T) and requiring one field analysis per iteration, is originated. If a derivative-free algorithm of evolution computing is used, the search is global-oriented too. The user-defined objective function is implemented as a routine, according to the usual format required by the optimisation algorithm.

A more sophisticated strategy would be defining a curvature-dependent velocity field, like *e.g.*

$$v_x(x,y,t) = v_0 \frac{x - x_0}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} [1 - c(x,y)] e^{-\frac{t}{T}} \tag{12}$$

$$v_y(x,y,t) = v_0 \frac{y - y_0}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} [1 - c(x,y)] e^{-\frac{t}{T}} \tag{13}$$

with *c(x,y)* curvature of the boundary at point (*x*,*y*). This way, the boundary velocity

$$\|v\| = v_0 |1 - c(x,y)| e^{-\frac{t}{T}} \qquad (14)$$

would depend on the point coordinates.

B. Case study: guard-ring optimal design

The design optimisation of the guard ring of the high-voltage winding (HVW) in a power transformer is considered as the case study. The ring is located near the ends of the HVW, where the highest electric stress occurs (Reece and Preston 2000).

For modelling purpose, a two-dimensional Cartesian model of the oil-filled (ε_r=2.2) field region Ω is developed. The test of applied voltage is simulated: accordingly, low-voltage winding (LVW), core and tank are short-circuited at the ground potential; moreover, a terminal of the HVW is open-circuited, while the other one is connected to an ideal voltage source. In terms of the analysis problem, Ω is a source-free doubly-connected domain; therefore, the Laplace equation of electric potential u in static conditions is considered, subject to u = U at the ring and HVW surface, u = 0 at the LVW surface, and $D_n u = 0$ elsewhere.

In view of the optimal field synthesis, a possible pair of objective functions, both to be minimized with respect to (v_0,T), is the maximum field strength in the oil-filled region (hot spot)

$$f_1(v_o,T) = \sup_{\Omega} \left\| \overline{E}(\gamma(t), r, v_o, T)\big|_{t=5T} \right\| \;,\; v_0 \neq 0 \;,\; T > 0 \qquad (14)$$

and the maximum field deviation (inhomogeneity) too

$$f_2(v_o,T) = \sup_{R} \left\| E_y(\gamma(t), r, v_o, T)\big|_{t=5T} \right\| \;,\; v_0 \neq 0 \;,\; T > 0 \qquad (15)$$

with $r \in \Omega$ position vector and $\overline{E} = -\overline{\nabla} u$. In turn, $R \subset \Omega$ is a controlled region defined as $[a_1, a_2] \times [a_3, a_4]$ where a_1 and a_2 are the abscissae of LVW and HVW ends delimiting the oil-channel, a_3 is set to a fixed ordinate value, and a_4 is the highest ordinate value along the moving boundary, with $a_4 > a_3$.

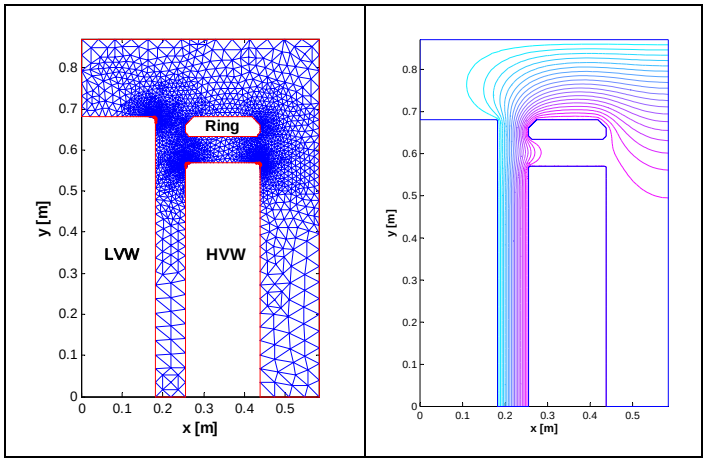


Fig. 2. Transformer model:
FE mesh (about 4,500 linear elements), and potential lines.

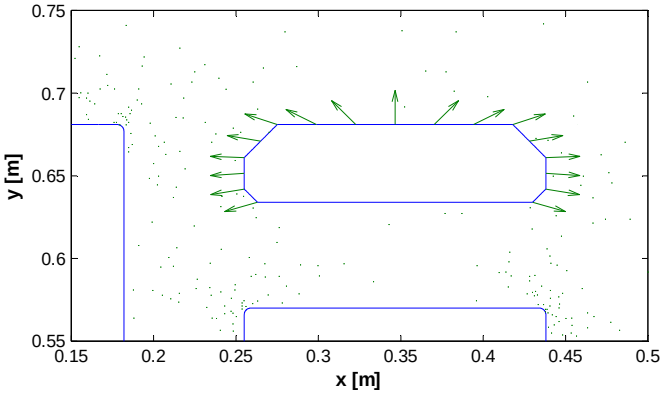


Fig. 3. An example of velocity field along the ring boundary.

In Fig. 4 a set of nine non-dominated solutions is represented in the objective space, so proving the existence of a conflict between objective f_1 and objective f_2 . Results have been obtained by means of a (1+1) evolutionary algorithm; for the sake of a comparison, the initial solution is also shown. Under a prescribed search tolerance equal to 10^{-3}, the typical number of convergence iterations is 80.

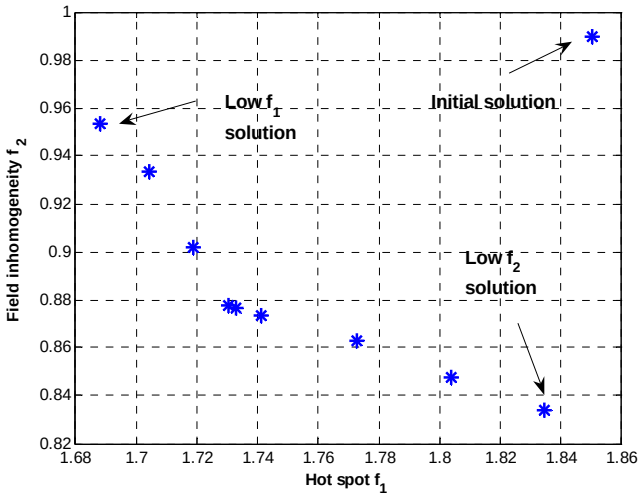
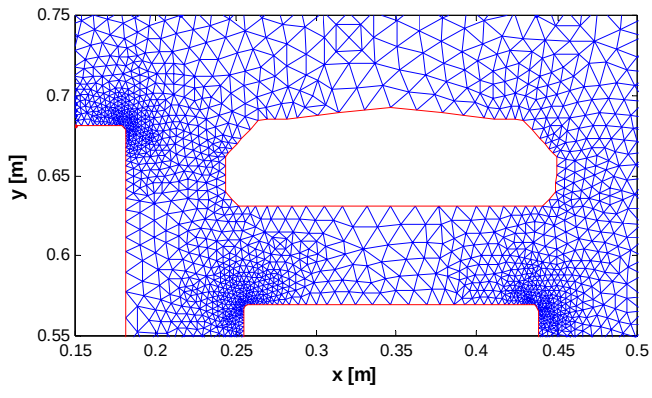


Fig. 4. A set of non-dominated solutions in the objective space (dimensionless values, referred to the uniform field in the oil channel).

Fig. 5 and Fig. 6 show the solutions corresponding to the end points of the non-dominated set represented in Fig. 4, respectively; the hot spot position is shown.



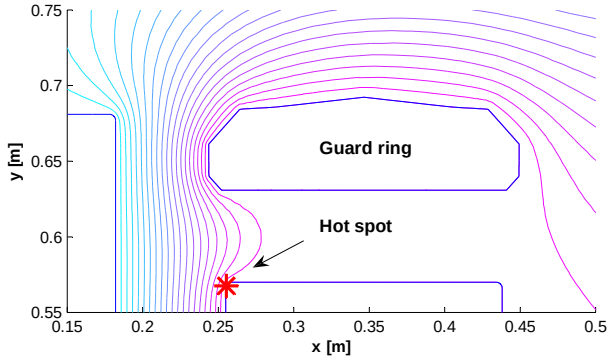


Fig. 5. Solution of low hot spot (FE mesh, and potential lines): v_0= 58 mms^{-1}, T= 123 ms (marked as "low f_1" in Fig. 4).

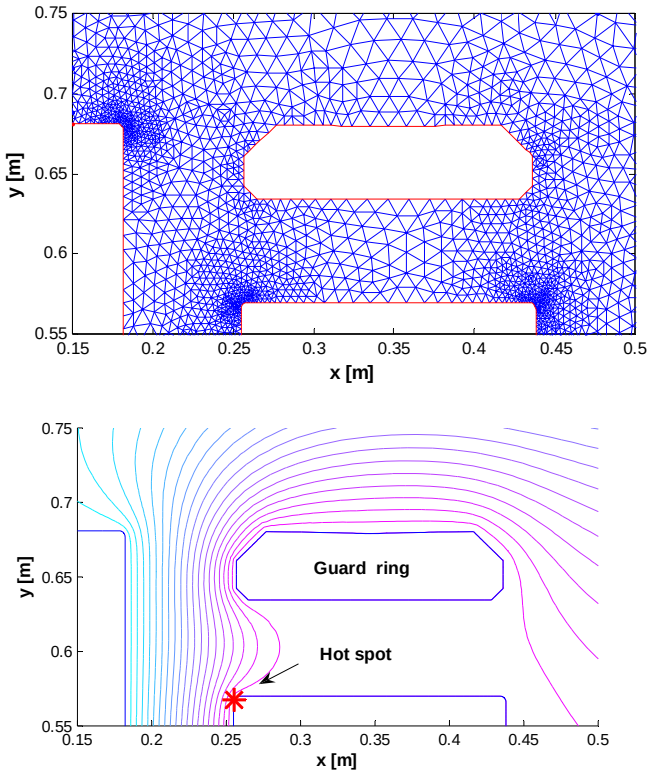


Fig. 6. Solution of low inhomogeneity (FE mesh, and potential lines): v_0= -39 mms^{-1}, T= 31 ms (marked as "low f_2" in Fig. 4).

Comparing Fig. 5 and Fig. 6, it can be noted that an expansion (v_0>0) leads to a reduction of the hot spot value, while a contraction (v_0<0) gives rise to a reduction of the field inhomogeneity, starting from the same initial solution.

XII. Beyond Evolutionary Multiobjective Optimisation: A Bayesian Approach

Though being powerful, EMO methods are affected by some inherent limits, the most crucial of which is the absence of a theoretical proof of convergence. However, there is an even more cogent reason of dissatisfaction: seldom does a configuration fit the requirements of a designer, even if it is Pareto-optimal, because there exist some constraints – or objectives – difficult to be formulated and included in the optimization problem, but not less important than those accounted for. Actually, there are some design considerations that are easily made only *a posteriori* and, usually, become evident when not fulfilled: sensitivity of the solution might be one of them. In evolutionary computing, individuals of a population-based method of optimisation run towards improvement through a randomness guided by a set of possible heuristics in the objective space *Y*. An alternative way could be developing a statistical method to identify the regions of the design space *X* which are more likely to map onto Pareto-optimal solutions. The designer, then, should be provided not with a large collection of individuals assumed to be optimal, but with a distribution of probability in the design space, which yields optimal configurations with a given degree of certainty. A possible formulation could rely on the Bayes theorem, the goal being the computation of some probability surfaces, to identify the most promising candidate regions for Pareto-optimal solutions.

From the Bayesian viewpoint, probability quantifies the level of belief about an event. Rather than expressing any physical property of the event itself, it just represents how much an event is trusted to be true. Now, beliefs are the mixture of prior assumptions - prior probability distributions, in the Bayesian language - and (incomplete) experimental information. Bayes theorem formalizes this learning chain, from a prior to a posterior through experimental observation.

A. Local formulation

The Bayesian local formulation considers two, priorly unrelated elements *x* and *y* in the design space *X* and objective space *Y*, respectively. The two following propositions can then be assigned a probability level:

$\mu(x)$: "*x* belongs to the Pareto set (PS)"

$\phi(y)$: "*y* belongs to the Pareto front (PF)"

Some prior information is available about the search space, and it could be called *I*. From the design viewpoint, *I* includes at least bounds and constraints defining the feasible search region.

What is interesting to know, for solving a multiobjective problem, is which is the PF, escaping possible local fronts, and which is the corresponding PS - or, which elements of *X* map to the PF in *Y*. Clearly, this implies relationships of conditioned probabilities between the aforementioned propositions, subject to the background of *I*. According to Bayes theorem, it turns out to be:

$$p(\phi \mid \mu, I) = \frac{p(\mu \mid \phi, I)\, p(\phi \mid I)}{p(\mu \mid I)} \qquad (17)$$

The term on the left represents the probability that a given *y* lies on the PF, conditioned to the fact that *x* is on the PS, and to prior information *I*. For one is conditioned to the other, a relationship between *x* and *y* must be implied in the evaluation of this term. A way to see it, is to take an element of *X*, say *x**, and generate the probability surface $p(\phi(y) \mid \mu(x^*), I)$, *i.e.* the left-hand side of (17) specialized to the particular value *x**, and still function of the independent variable *y*. The left hand side of (17) could be called *stopping term*, because it indicates when the search can be stopped during the optimization, according to the established degree of certainty about the probability surface.

As far as the right hand side of (17) is concerned, the denominator could be viewed as a normalization constant. The terms in the numerator indicate the two conditions

which must be true at the same time, for stating that an element x^* is in the PS in what it maps to an element y^* in the PF. Basically, x^* must be the counter-image of a y^* lying on the current front of non-dominated points, and the current front must definitely be the PF. The relationship between elements x and y is summarized by the term $p(\mu(x)|\phi(y),I)$, which represents a kind of probabilistic inversion of function F, *i.e.* the function mapping a vector from the design space to the objective space; therefore, it could be called *inverse-mapping term.* The other term, $p(\phi(y)|I)$, evaluates the probability of proposition $\phi(y)$: it is zero for y already known to be dominated - *i.e.* dominated by points in the front - and, for points lying on the front, it corresponds to the probability that it is the Paretian one. Because of this meaning, it could be named *front-mapping term.*

B. Integral formulation

The local formulation implies inverting function F, in the inverse mapping term, and this might be quite complicated a task. On the other hand, what is really important to identify the PS, is knowing which elements x do map on the front, rather than which point of the front their image falls onto; in fact, all points belonging to the same front are equivalent, according to Paretian optimality. This, in equation (17), is equivalent to taking the integral of both members, with respect to y. As far as the left-hand side is concerned, the term

$$\int p(\phi(y)|\mu(x),I)dy \tag{18}$$

is the probability that, given one $x \in X$ belonging to the PS, at least one $y \in Y$ belongs to the PF. In the right-hand side of (17), only the numerator depends on y : carrying out the integration on it yields the probability that x maps to at least one element of the PF. Now, considering the distance of any element y from the front, is a simplification, for the inverse relationship F^{-1} is not to be found out. This, however, implies finding a suitable metric, in the objective space, for expressing the distance of any point to the front. The details of the metric used – called Paretian distance (Fig. 7) – can be found in (Di Barba 2010).

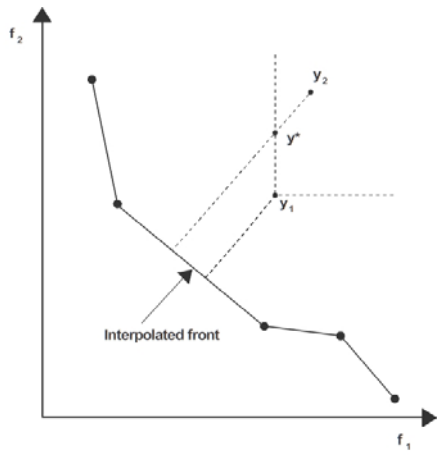


Fig. 7. Geometric interpretation of Paretian distance in the two-dimensional case.

Moreover, points will be classified as Paretian if falling within a tolerance distance from the front.
The remark that the term

$$\int p(\mu(x)|\phi(y),I)\,p(\phi(y)|I)dy \tag{19}$$

is a function of x means that it is a probability density with respect to x: the probability that the distance of $y=F(x)$ to the front is zero, wherever y might be. Such probability is weakly regular, because the objectives are supposed to be weakly regular: under this frame, the probability can be considered to be an image. When looking at the image with the appropriate scale, an subregion Ω could be detected as sufficiently homogeneous in its inner structure, and well defined with respect to the surrounding patches. Therefore, the inner structure of Ω can be described in terms of the cumulative distribution function (CDF) of the Paretian distance over Ω itself. The probability that any point inside it maps to a point of the Ω is equal to the probability that the distance associated to the points inside Ω is zero within the prescribed tolerance. This information is extracted just from the CDF, assumed as the probability value for any point of Ω.
The computation of the probabilistic terms is fully described in [6]. In particular, a central problem is that of insulating good or promising subregions of the search space, in order to generate a probability distribution function around the PS. The task can be split in three steps:

i) insulating a set of subregions;
ii) analysing the probabilistic structure of each subregion;
iii) choosing the best looking subregion for subsequent search refinement.

A Bayesian imaging method for probability computation here used was developed in (Bramanti 2002).

C. Case study: optimal design of a small actuator

The device considered is composed of a U-shaped magnetic core, and a mobile magnetic plunger, with an air-gap. The plunger is able to alternatively move, under the attractive action of the magnetic field, determined by a pulse current flowing in the copper winding, and the opposite force exerted by a spring. Fig. 8 shows the domain chosen for the analysis thanks to symmetry. The magnitude of the longest dimensions is ten through twenty mm, while the shortest side of the core is 1 through 2 mm: an electromagnet of small size is dealt with. In the past, a similar device was applied in electrical typewriters as a linear actuator: a character is printed out when the plunger impacts against the paper sheet. The system is energized by the excitation winding, which carries a pulsed specific current equal to $2 \cdot 10^5$ A m^{-2}.
The upper part of the core, and correspondingly the facing part of the plunger, are shaped in the way shown, where the profile is controlled by the angle α the tilted section forms with the reference system, and the horizontal projection $2x$ of the section itself. The reason is twofold: on the one hand, the air-gap is smaller, so reducing flux line dispersion; on the other hand, the mechanical

positioning of the plunger is improved, especially with respect to its alignment with the core.

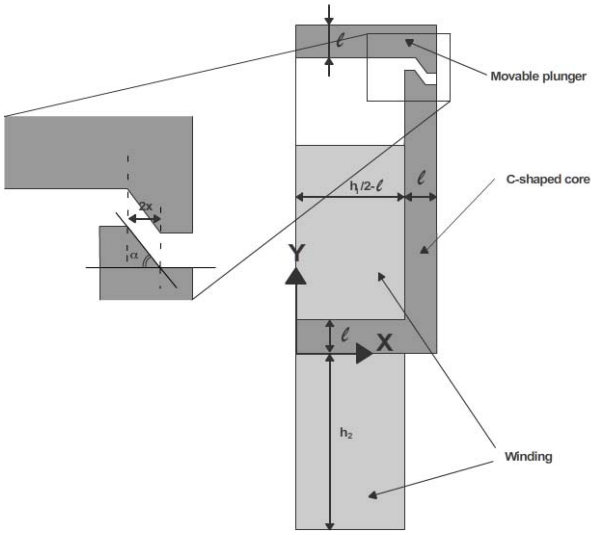


Fig. 8. Geometry of the actuator (with a detail of the air-gap), and design variables.

At a fixed current density, the geometry can be varied, in particular, acting upon the following five design variables (Fig. 8): h_1 and h_2 , height and width of the winding, respectively; ℓ , cross-sectional side of the core; α and x, angle and semi-horizontal projection of the tilted section, near the air-gap.
The bi-objective problem can be cast as follows:
find

$$\inf_{a\in\Omega} C(a)\ ,\ C(a)=c_{iron}V_{iron}(a)+c_{copper}V_{copper}(a) \qquad (20)$$

and

$$\sup_{a\in\Omega} F_y(a)\ ,\ F_y(a)\approx \left.\frac{\Delta W'(a)}{\Delta y}\right|_{g=g_0} \qquad (21)$$

with g_0 = 6 mm and W' co-energy of the system, subject to

$$\sup_{\Omega_w(a)}\left|B_y(a)\right| \le B_0 \qquad (22)$$

where B_0 = 10 mT is assumed.

The following definitions hold:
$a=(h_1,h_2,\ell,x,\alpha)\in\Omega\subset\Re^5$ is the design vector;
F_y is the force in the direction of plunger displacement, evaluated for an air-gap width $g_0=6$ mm;
C is the material cost, analytically derived from the material volume, assuming per-unit specific costs $c_{iron}=1$ and $c_{copper}=3$ respectively;
Ω_w is the area of the winding cross-section while $\sup|B_y|$ is a fringing field indicator.
Optimisation results were compared with a distribution of 1,250 random samples, randomly generated in design space X : Fig. 9 shows the whole sampling in the F space, with the PFs obtained from both sampling and optimisation.

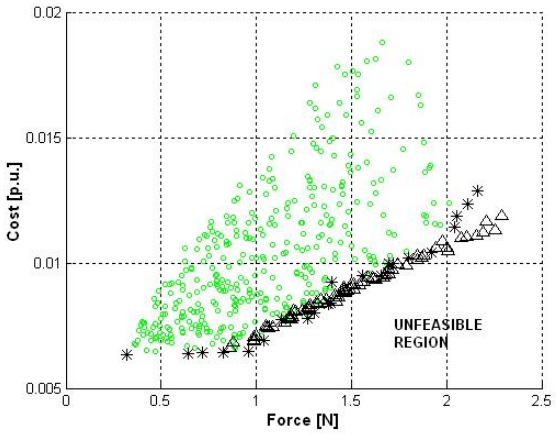


Fig. 9. F-space sampling (1,250 points, circle), with relevant PF (star), and PF derived after optimisation (triangle).

The device geometries corresponding to the two front ends are shown in Fig. 10.

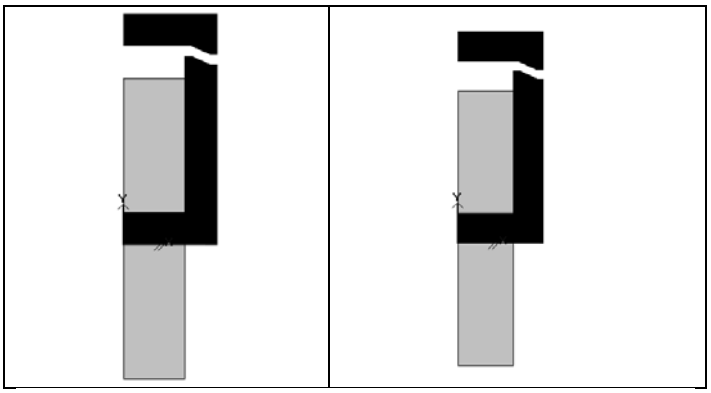


Fig. 10. Geometry of the linear actuator, for the maximum force F_y (left) and the minimum cost C configurations (right); the objective values are F_y = 2.1997 N, C = 0.0113 and F_y = 0.8639 N, C = 0.0066, respectively.

From the design viewpoint, it is interesting to investigate the projection of the PS on two significant variables, just controlling the winding cross-section, *i.e.* h_1 and h_2 . To this end, in Fig. 11 the projection of the PS, derived after the optimisation procedure on the (h_1,h_2) plane, is compared with the orthogonal projection of the sampled PS on the same plane. The PS behaves just as conjectured: there is a single island covering the region, which shows quite clearly a linear dependence between the two variables.

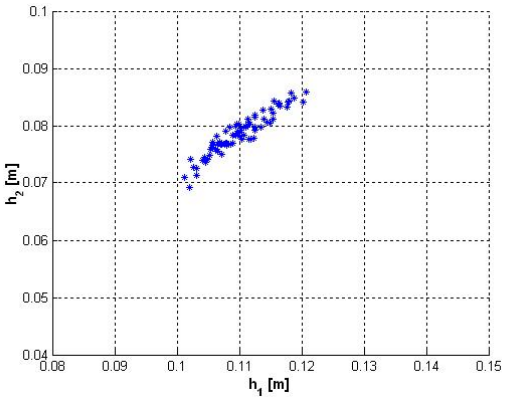


Fig. 11. PS projected on the (h_1,h_2) plane after optimisation, left, and the same projection after 1,250 point sampling, right.

The ease in the result interpretation is one of the outstanding benefits of the method. Actually, the true benefit of the Bayesian imaging method is that the PS is well determined, and other optimal solutions can be inexpensively generated, by means of new extractions, until the requirements of the designer are met. In fact, identifying the boundary of the whole Pareto set in a reliable way, rather than determining a finite set of optimal solutions, is the goal of the design space exploration driven by a Bayesian inferential scheme.

XIII. Conclusion

At the end of this short excursion in the domain of optimal design methods, looking back at the pathways run through, the author realizes that it was possible to highlight just some basic topics of the domain and to describe them in a very concise and essential way.

A general remark is that the impact of optimisation theory on electromagnetic design has stimulated completely new methods and techniques, which might revolutionise traditional design strategies and certainly deserve to be considered, even if not all of them become ultimately successful.

A final remark on industrial electromagnetic design can be put forward. While there have been significant improvements in the capabilities in the area of MO design, the uptake by industrial designers has been somewhat limited. There are, possibly, two reasons for this.

The first is that the evidence, at the industrial level, that computer-based optimisation processes can actually enhance a designer ability to create a better product has been lacking.

The second relates to the fact that most optimisation packages currently available only handle a single objective and a limited number of design variables. In fact, suitable optimisation systems, with no restriction in the size of the design space to be explored, and with simple and flexible expressions of objectives and constraints, would help match the needs of the designer.

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