Technical Article

Phenomenological modeling of magnetic hysteresis

Abstract — **This paper deals with the modeling of vector fields that exhibit hysteresis. A general class of models of vector magnetizations with hysteresis that overpass some drawbacks of the Preisach-type models is defined. After reviewing some general properties of conservative fields, the particular case of unit magnitude vector fields is discussed. The paper focuses on a discussion of the properties of a general vector hysteresis operator (hysteron). In the Appendix sections some examples of vector hysterons deduced from the general definition are presented, and their properties are indicated and analyzed.**

I. Introduction

Three-dimensional modeling has many applications in the analysis of practical devices and components, such as transformers, electric motors, magnetic transducers, etc. However, the modeling of magnetic hysteresis in 3-dimensions has several open questions. Many physical approaches at micro- or even nano-magnetic scale [1] - [13] are difficult to use. This list is indicative but not exhaustive. Due to the extensive amount of computer time and memory to compute the behavior for dimensions typical of practical devices the approaches above have limited usefulness.

Phenomenological approaches [14] - [20], on the other hand, have been proposed on a macro magnetic scale for the modeling of magnetic materials with hysteresis. They have been used successfully to extend this analysis to vector hysteresis problems, since real devices are three-dimensional. The principal other vector models are: the Stoner-Wohlfart model [21], the Vector Preisach Model proposed by Mayergoyz [22] and the Coupled Hysterons Model proposed by Della Torre [23].

The Stoner-Wohlfart model has some very attractive features dealing with the physics of magnetism, but is limited to ellipsoidal, single-domain, uniaxial magnetic particles. The Mayergoyz model computes the total magnetization as the vector sum of the responses to a continuum of components of magnetization in each direction. The magnetization along each direction is obtained via a scalar hysteresis model, whose parameters are function of the direction. This model has the advantage of greater generality than the previous one, but does not compute observed dissipations due to a rotating applied field correctly. The Coupled Hysterons Model has the correct saturation and dissipation properties, but is only able to handle materials that are ellipsoidally magnetizable, and requires rotational and orientation corrections.

Other recent phenomenological approaches to the vector hysteresis that try to overpass the technical inadequacies of the previous models were presented in papers [24] – [33].

Among these, in papers [30], [31], and [32] a generalized vector model of magnetic hysteresis was introduced.

II. Basic features of the model

The model is based on the definition of a vector hysteron, described in the ***H***- space by a closed critical surface. Each vector hysteron has a unique critical surface, described by a suitable set of parameters, indicated here with the parameter vector $\boldsymbol{\Omega}$. The normalized component of the magnetization for each hysteron has unit magnitude everywhere. For fields inside the critical surface the magnetization is frozen in the direction that it had just before it entered the critical surface, and it remains constant until it exits the critical surface. When exiting the critical surface the irreversible magnetization instantly rotates so as to align itself along a new direction. This direction depends upon the different model strategies, as described above.

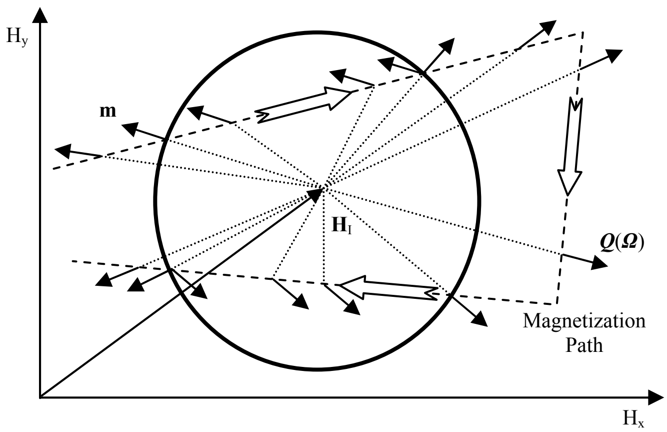


Fig. 1. Elementary circular hysteron and variation of magnetization with applied field for a particular applied field trajectory. Figure in 2-D.

The magnetization state vector of the hysteron can be denoted by $\boldsymbol{Q}(\boldsymbol{\Omega},\boldsymbol{H})$. This means that the direction of unit magnetization given by the vector hysteron is a function of the parameter vector $\boldsymbol{\Omega}$ for any point of the ***H***-space. The total magnetization is the vector sum of the magnetization due to all the hysterons. The total magnetic field that the hysteron sees is the applied field plus an interaction field $\boldsymbol{H}_I$, due to the presence of the other hysterons, and the effect is a vector displacement by $\boldsymbol{H}_I$ from the origin of the hysteron.

In the case of Fig. 1 the components of $\boldsymbol{\Omega}$, i.e. the model parameters, are the components of the interaction field and the radius of the hysteron. The hysterons are distributed in the ***H***-plane, and their density distribution can be described by a distribution function $P(\boldsymbol{\Omega})$, in analogy with the CSPM case.

This paper presents a general discussion of conservative unit magnitude vector fields, which is an integral part of this model. This leads to a definition of a 3-D basic vector hysteron, which is used as a basis for a general analysis of vector models with magnetic hysteresis.

This paper also discuss the general properties of these vector hysterons, including the energy exchanges, and the possible set of parametric systems of coordinates that define in mathematical form the general vector hysteron. These are useful as tools for numerical analysis. The properties discussed and defined in rigorous way can be generalized to any vector field with hysteresis, and are not limited to the magnetic case. Finally, some examples show that the general definition of the vector hysteron includes all the previously mentioned generalized vector models [30], [31] and [32] .

III. Conservative vector fields

In this paper is discussed a 2-D model for the sake of simplicity, but the theory can be easily generalized to the 3-D case. In this section some basic concepts and some definitions about the conservative vector fields useful for the discussion

of the next sections will be reviewed.
In general a vector magnetization, $\boldsymbol{M}$, due to an applied field, $\boldsymbol{H}$, can be defined in 2-D using a regular $\boldsymbol{H}$-Cartesian frame: $M_x(H_x,H_y)$ and $M_y(H_x,H_y)$. A vector field is defined here as *weakly conservative* if it is conservative for a given regular coordinates system. If the vector field is a continuous function of a given regular system of coordinates, with continuous derivatives, it can be shown that it is weakly conservative if in the given system of coordinates there exists a potential W, such that

$$M_x = \frac{\partial W}{\partial H_x} \tag{1}$$

and

$$M_y = \frac{\partial W}{\partial H_y} \tag{2}$$

In addition the vector field is defined here as *weakly closed* if in the given system of coordinates it has equal mixed derivatives

$$\frac{\partial M_x}{\partial H_y} = \frac{\partial M_y}{\partial H_x} \tag{3}$$

If the vector field is weakly closed and it is defined on a simply connected domain of $\boldsymbol{H}$, the line integral along any closed line γ inside the domain

$$\oint_\gamma M_x\, dH_x + M_y\, dH_y \tag{4}$$

is zero, and the vector field is weakly conservative. The adjective *weakly* is used here because the conservative property of the vector field is not general but depends upon the coordinates system. In other words, a regular transformation of coordinates can be used, whose Jacobian is non zero, and whose components in the $\boldsymbol{H}$- plane can be expressed terms of a new system of coordinates, indicated as u and v. Then

$$M_x(H_x,H_y) = M_x(u(H_x,H_y), v(H_x,H_y)) \tag{5}$$

with an analogous expression for M_y.
Symmetrically

$$M_x(u,v) = M_x(H_x(u,v), H_y(u,v)) \tag{6}$$

with an analogous expression for M_y. In Appendix I, some examples of these transformations are presented.
Using the rules for the derivative of composite functions, it follows that the necessary and sufficient conditions for the conservative vector field in the $\boldsymbol{H}$-plane can be expressed as

$$\frac{\partial M_x}{\partial u}\frac{\partial u}{\partial H_y} + \frac{\partial M_x}{\partial v}\frac{\partial v}{\partial H_y} = \frac{\partial M_y}{\partial u}\frac{\partial u}{\partial H_x} + \frac{\partial M_y}{\partial v}\frac{\partial v}{\partial H_x} \tag{7}$$

Or, more usefully as

$$\frac{\partial M_x}{\partial v}\frac{\partial H_x}{\partial u} + \frac{\partial M_y}{\partial v}\frac{\partial H_y}{\partial u} = \frac{\partial M_x}{\partial u}\frac{\partial H_x}{\partial v} + \frac{\partial M_y}{\partial u}\frac{\partial H_y}{\partial v} \tag{8}$$

IV. Unit magnitude vector fields

Some useful considerations about the properties of vector fields having unitary absolute magnitude of magnetization are introduced in this section. These field properties will be used in the discussion about a general class of models of vector hysteresis presented in the next sections.
In view of the above considerations, if the following assumptions are made

- the magnetic field must be expressed by a regular (u, v) coordinates frame;
- the curves for u = constant must be the equipotential curves of the vector field.
- the curves for v = const must be the lines of force of the vector field;

the fact that equipotential curves and lines of forces must be perpendicular can be used and the magnetization in general can be written as

$$M_x = \frac{\dfrac{\partial H_x}{\partial u}}{\sqrt{\left(\dfrac{\partial H_x}{\partial u}\right)^2 + \left(\dfrac{\partial H_y}{\partial u}\right)^2}} \tag{9}$$

$$M_y = \frac{\dfrac{\partial H_y}{\partial u}}{\sqrt{\left(\dfrac{\partial H_x}{\partial u}\right)^2 + \left(\dfrac{\partial H_y}{\partial u}\right)^2}} \quad . \tag{10}$$

Now it will be proved that the conservative condition, discussed in the previous section, leads to the condition that for v = const the abscissa of the curvilinear coordinates must be independent by v.
This necessary and sufficient condition can be expressed in mathematical form as follows

$$\frac{\partial\left[\left(\dfrac{\partial H_x}{\partial u}\right)^2 + \left(\dfrac{\partial H_y}{\partial u}\right)^2\right]}{\partial v} = 0 \tag{11}$$

Equation (11) implies that the length of two segments of lines of forces between a pair of equipotential curves must be equal for any v.
The proof of the above statement is deduced in the following of this section.
The orthogonality condition can be expressed mathematically as follows

$$\frac{\partial H_x}{\partial u}\frac{\partial H_x}{\partial v} + \frac{\partial H_y}{\partial u}\frac{\partial H_y}{\partial v} = 0 \tag{12}$$

with the assumptions made, the curvilinear abscissa of the generic equipotential surface is given by

$$\left(\frac{\partial H_x}{\partial u}\right)^2 + \left(\frac{\partial H_y}{\partial u}\right)^2 = f^2(u) \tag{13}$$

where f is a regular function of the only component u.
If (12) is differentiated with respect to u

$$\frac{\partial^2 H_x}{\partial u^2}\frac{\partial H_x}{\partial v}+\frac{\partial H_x}{\partial u}\frac{\partial^2 H_x}{\partial u \partial v}+\frac{\partial^2 H_y}{\partial u^2}\frac{\partial H_y}{\partial v}+\frac{\partial H_y}{\partial u}\frac{\partial^2 H_y}{\partial u \partial v}=0 \tag{14}$$

and if (13) is differentiated with respect to v

$$\frac{\partial H_x}{\partial u}\frac{\partial^2 H_x}{\partial u \partial v}+\frac{\partial H_y}{\partial u}\frac{\partial^2 H_y}{\partial u \partial v}=0 \tag{15}$$

Therefore

$$\frac{\partial^2 H_x}{\partial u^2}\frac{\partial H_x}{\partial v}+\frac{\partial^2 H_y}{\partial u^2}\frac{\partial H_y}{\partial v}=0 \ . \tag{16}$$

Taking into account (13) the derivatives of the magnetization are

$$\frac{\partial M_x}{\partial u}=\frac{\frac{\partial^2 H_x}{\partial u^2}f(u)-\frac{\partial H_x}{\partial u}\frac{\partial f(u)}{\partial u}}{f^2(u)} \tag{17}$$

$$\frac{\partial M_x}{\partial v}=\frac{\frac{\partial^2 H_x}{\partial u \partial v}}{f^2(u)} \tag{18}$$

$$\frac{\partial M_y}{\partial u}=\frac{\frac{\partial^2 H_y}{\partial u^2}f(u)-\frac{\partial H_y}{\partial u}\frac{\partial f(u)}{\partial u}}{f^2(u)} \tag{19}$$

$$\frac{\partial M_y}{\partial v}=\frac{\frac{\partial^2 H_y}{\partial u \partial v}}{f^2(u)} \ . \tag{20}$$

If the conservativeness (8) is applied, using (20) the left side part becomes

$$\frac{\frac{\partial^2 H_x}{\partial u \partial v}\frac{\partial H_x}{\partial u}}{f^2(u)}+\frac{\frac{\partial^2 H_y}{\partial u \partial v}\frac{\partial H_y}{\partial u}}{f^2(u)} \tag{21}$$

and it is zero, according with (15). Again, the right side part of (8) becomes

$$\frac{\frac{\partial^2 H_x}{\partial u^2}\frac{\partial H_x}{\partial v}f(u)-\frac{\partial H_x}{\partial u}\frac{\partial H_x}{\partial v}\frac{\partial f(u)}{\partial u}}{f^2(u)}+\frac{\frac{\partial^2 H_y}{\partial u^2}\frac{\partial H_y}{\partial v}f(u)-\frac{\partial H_y}{\partial u}\frac{\partial H_y}{\partial v}\frac{\partial f(u)}{\partial u}}{f^2(u)} \ . \tag{22}$$

This again is zero, using (12) and (16). Therefore it has been shown that a unit vector field whose magnetization is perpendicular to the equipotential surface is always conservative if and only if (11) is satisfied.

Now the attention will be focused on the conditions that must be satisfied by the parametric regular coordinates system (u, v), in order to fulfill the condition (11), and consequently the conditions (12) and (13). First of all, it must be shown that for any fixed magnitude of u a fixed equipotential surface for the considered vector field is obtained. In addition, for a fixed magnitude of v is obtained a straight line (constant slope), perpendicular to the equipotential surface in the intersection point. The lines for v = constant are lines of force of the vector field.

This is shown as follows: if the partial differential system (12) and (16) is again taken into account, and $\frac{\partial H_x}{\partial v}$ and $\frac{\partial H_y}{\partial v}$ are considered as unknowns, the condition $\left(\frac{\partial H_x}{\partial v}\right)^2+\left(\frac{\partial H_y}{\partial v}\right)^2>0$ must be always fulfilled, because the family of the coordinate curves must be a regular system. Therefore the determinant of the system must be zero, and this can be written as

$$\frac{\frac{\partial H_x}{\partial u}}{\frac{\partial H_y}{\partial u}}=\frac{\frac{\partial^2 H_x}{\partial u^2}}{\frac{\partial^2 H_y}{\partial u^2}} \ . \tag{23}$$

This means that $\frac{\frac{\partial H_x}{\partial u}}{\frac{\partial H_y}{\partial u}}$ is independent on u and it can be consequently written

$$\frac{\partial H_y}{\partial u}=\breve{h}(v)\frac{\partial H_x}{\partial u} \ . \tag{24}$$

Integrating,

$$H_y(u,v)=\breve{h}(v)H_x(u,v)+\breve{k}(v) \tag{25}$$

This shows that for a fixed magnitude v_0, the curve $\left(H_x(u,v_0),H_y(u,v_0)\right)$ is a segment of straight line (for $\frac{\partial H_x}{\partial u}=0$ it is a vertical segment). It can be also deduced that the function $f(u)$ can be set equal to a constant: if the change of variable $u^*=f^*(u)$ is made, where $\frac{\partial f^*(u)}{\partial u}=f(u)$, then $f(u)=1$ in (13) and in the following.

In the remainder of this section it will be shown that the coordinates curves (equipotential curves, or surfaces in 3-d) obtained for u = constant are circles (spheres), if the definition domain is the entire ***H***-plane $(u \geq 0)$.

Substituting (24) in (13)

$$\frac{\partial H_x}{\partial u}=\pm\frac{1}{\sqrt{1+\breve{h}^2(v)}} \ . \tag{26}$$

Assuming, for the sake of simplicity, the positive sign in (26), and integrating with respect to u

$$H_x=\frac{u}{\sqrt{1+\breve{h}^2(v)}}+\phi(v) \tag{27}$$

where $\phi(v)$ is an arbitrary function. From (25) and (27)

$$H_y = \frac{u\breve{h}(v)}{\sqrt{1+\breve{h}^2(v)}} + \psi(v) \tag{28}$$

where $\psi(v) = \breve{h}(v)\phi(v) + \breve{k}(v)$.

For $\breve{h}(v) = 0$ (27) and (28) are reduced to

$$H_x = u + \phi(v)\,, \quad H_y = \breve{k}(v)\,. \tag{29}$$

Using (29) and the orthogonality condition (12) the result $\dfrac{\partial\phi(v)}{\partial v} = 0$ is derived, therefore

$$H_x = u + constant\ . \tag{30}$$

This describes the Cartesian coordinate's frame, which obviously fulfills the conditions (12) and (13).

Getting back to the general case, and introducing a change of variable $w = \arctan\left(\left|\breve{h}(v)\right|\right)$ the following result is obtained

$$H_x(u,w) = u\cos w + \phi^*(w)\,, H_y(u,w) = u\sin w + \psi^*(w) \tag{31}$$

where $\phi^*(w)$ and $\psi^*(w)$ are suitable regular functions. If the orthogonality condition (14) is again applied in the (u,w) coordinates this leads to

$$\frac{\partial\phi^*}{\partial w}\cos w + \frac{\partial\psi^*}{\partial w}\sin w = 0\,. \tag{32}$$

If a regular coordinates system is searched where u = constant is a family of equipotential curves (surfaces) defined in the whole ***H***-plane from (31) and (32) the only possible solution is $\phi^*(w) = h$ and $\psi^*(w) = k$, where h and k are arbitrary constants and the equipotential curves (surfaces) are circles (spheres). In Appendix II some examples of magnetization for this case are presented.

On the other hand, if the domain is limited to the family of equipotential curves of a part of the ***H***-plane there are solutions of (32) that originate for $u = 0$ equipotential surfaces described by a closed curve. In this case the function $\psi^*(w)$ is selected as an odd real function, defined in $(-\pi, \pi)$, continuous with its first and second derivative. The function $\psi^*(w)$ is positive only if $0 < w < \pi$ and it must have one only maximum for $w = \frac{\pi}{2}$. Incidentally, these conditions lead to $\psi^*(0) = \psi^*(\pi) = \psi^*(-\pi) = 0$ and to the fact that $\lim_{w\to\frac{\pi}{2}}\left(\frac{\partial\psi^*}{\partial w}\right) = 0$ is at least an infinitesimal of first order.

Now the existence of the integral $\psi_0^*(w) = -\int_{-\pi}^{w}\frac{\partial\psi^*}{\partial v}\tan(v)dv$ is guaranteed by the fact that the derivative in the integral has at least a first order infinitesimal. In addition $\dfrac{\partial\psi^*}{\partial w}$ is an even function, therefore the function in the integral is an odd function and $\psi_0^*(-\pi) = \psi_0^*(\pi) = 0$. The choice $\psi^*(w) = \psi_0^*(w) + a$, where a is an arbitrary real number individuates the starting (and ending) point of the equipotential curve corresponding to u=0.

The properties of the signs of the functions defined, and of their derivatives ensure that the equipotential curve found is a regular curve. If the functions $\dfrac{\partial\phi^*}{\partial w}$ and $\dfrac{\partial\psi^*}{\partial w}$ are written in polar coordinates

$$\frac{\partial\phi^*}{\partial w} = \rho(w)\sin\tau(w)\,, \ \frac{\partial\psi^*}{\partial w} = \rho(w)\cos\tau(w)\,. \tag{33}$$

The condition (32) becomes

$$\rho(w)\sin(w+\tau(w)) = 0 \tag{34}$$

and ($\rho(w) = 0$ is obviously a not interesting solution)

$$\tau(w) = -w \ \text{ or } \ \tau(w) = \pi - w\ . \tag{35}$$

In the interval (0, 2π) is

$$\frac{\partial\phi^*}{\partial w} = -\rho(w)\sin w\,, \ \frac{\partial\psi^*}{\partial w} = \rho(w)\cos w$$

or

$$\frac{\partial\phi^*}{\partial w} = \rho(w)\sin w\,, \ \frac{\partial\psi^*}{\partial w} = -\rho(w)\cos w\ . \tag{36}$$

By integrating the (36), and taking into account (31)

$$H_x(u,w) = u\cos w + A - \int_0^w \rho(v)\sin v\,dv$$

$$H_y(u,w) = u\sin w + B - \int_0^w \rho(v)\cos v\,dv\ ,$$

or

$$H_x(u,w) = u\cos w + A - \int_{-\pi}^w \rho(v)\sin v\,dv$$

$$H_y(u,w) = u\sin w + B - \int_{-\pi}^w \rho(v)\cos v\,dv \tag{37}$$

where A and B are arbitrary real constants.

These are the more general expressions of a conservative and orthogonal unit vector field defined by means of a regular parametric system of coordinates (u,w).

It is interesting to note that the equipotential curves defined in the theory of Stoner and Wohlfart [21] are included in the family described by (31) and (32). See Appendix III.

The explicit formulas of the magnetization given by this vector field are deduced by (12)

$$M_x = \cos w\,, \ M_y = \sin w\ . \tag{38}$$

Moreover from (31)

$$\cos w = \frac{H_x - \phi^*(w)}{u}, \quad \sin w = \frac{H_y - \psi^*(w)}{u} \quad . \tag{39}$$

Therefore

$$u = \sqrt{\left(H_x - \phi^*(w)\right)^2 + \left(H_y - \psi^*(w)\right)^2}\ ,$$
$$w = \arctan \frac{H_y - \psi^*(w)}{H_x - \phi^*(w)} \quad . \tag{40}$$

It is also easy to see that the magnetic scalar potential as defined in (1) and (2) for this class of unit magnetization vector fields is equal to *u*.
Using the rules of the derivation for the change of variables, from (1) and (2)

$$\cos v + \sin v = \frac{\partial W}{\partial u}\left(\frac{\partial u}{\partial H_x} + \frac{\partial u}{\partial H_y}\right) =$$
$$\frac{\partial W}{\partial u} \frac{u \cos v + \frac{\partial \psi^*}{\partial v} + u \sin v - \frac{\partial \phi^*}{\partial v}}{u + \frac{\partial \psi^*}{\partial v}\cos v - \frac{\partial \psi^*}{\partial v}\sin v} \tag{41}$$

And, taking into account the orthogonality condition (12)

$$\frac{\partial \phi^*}{\partial v} = -\frac{\partial \psi^*}{\partial v}\frac{\sin v}{\cos v} \tag{42}$$

It follows $\frac{\partial W}{\partial u} = 1$ and $W = u$.

In conclusion, if a regular real function $\phi^*(w)$ is selected the function $\psi^*(w)$ is found by integrating (32), or vice versa. Finally the expressions for $\boldsymbol{M}$ as a function of $\boldsymbol{H}$ are found by (38), (39) and (40). The equipotential curves and the related lines of force are described by the parametric equations in *u* and *w* (31), where the value of *u* is the scalar potential of the vector field.
The discussion about the properties of unit magnitude conservative vector fields presented in the previous sections allows us to define in rigorous way the general class of vector hysterons. Any critical surface must be an equipotential surface, because the vector field must be conservative out of the critical surface. Therefore the family of the critical surfaces must satisfy the conditions (31) and (32).

V. Other energy requirements of the general hysteresis operator

The general hysteresis operator defined in the previous sections must be congruent with the second principle of the thermodynamics: in other words the energy of any closed path crossing the hysteron must be dissipative.

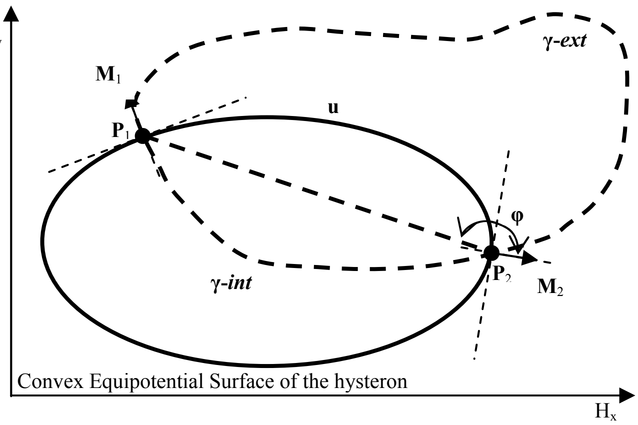


Fig. 2. Discussion of the congruence of convex critical surfaces of the vector hysteron model with the second principle of the thermodynamics. Figure in 2-D.

Referring to fig. 2, the energy exchange for any path γ-ext external to the any hysteron between any couple of points P_1 (starting point) and P_2 (end point) of the hysteron is zero, because the critical surface is an equipotential surface. Therefore the condition above can be expressed in mathematical way as follows: for any internal path *γ-int* of any hysteron between any couple of points P_1 and P_2 of the hysteron it can be written

$$\oint_{\gamma-\text{int}} M_x\, dH_x + M_y\, dH_y \le 0 \quad . \tag{43}$$

If the parametric description of the equipotential surface illustrated in the previous sections is used (43) becomes

$$\oint_{\gamma-\text{int}} M_x(H_x(u,w_1))\, dH_x + M_y(H_x(u,w_1))\, dH_y \le 0 \quad . \tag{44}$$

Where w_1 is the value of the parameter *w* of the point P_1.
$M_x(H_x(u,w_1))$ and $M_y(H_x(u,w_1))$ are constant with respect to the variable of integration. Thus

$$\oint_{\gamma-\text{int}} M_x(H_x(u,w_1))\, dH_x + M_y(H_x(u,w_1))\, dH_y =$$
$$(H_x(u,w_2) - H_x(u,w_1))\cos w_1 + (H_y(u,w_2) - H_y(u,w_1))\sin w_1 \tag{45}$$

If the conditions (38) and (39) are applied to (45)

$$\oint_{\gamma-\text{int}} M_x(H_x(u,w_1))\, dH_x + M_y(H_x(u,w_1))\, dH_y =$$
$$u(\cos w_1 \cos w_2 - \sin w_1 \sin w_2 - 1) + (\phi^*(w_2) - \phi^*(w_1))\cos w_1 + (\psi^*(w_2) - \psi^*(w_1))\sin w_1 \tag{46}$$

Where w_2 is the value of the parameter *w* of the point P_2.
First of all the congruency with the second principle of the thermodynamics will be discussed when $u \ge 0$. It is easy to see that the magnitude of $u(\cos w_1 \cos w_2 - \sin w_1 \sin w_2 - 1)$ in this case is always less or equal to zero, therefore the condition (43) becomes

$$(\phi^*(w_2) - \phi^*(w_1))\cos w_1 + (\psi^*(w_2) - \psi^*(w_1))\sin w_1 \le 0 \tag{47}$$

It follows that (47) is a necessary and sufficient condition for the second principle of the thermodynamics.
If $\phi^*(w)$ and $\psi^*(w)$ are constant in the domain of definition of w the condition (43) is verified. If $\phi^*(w)$ is constant, in fact, $\psi^*(w)$ is constant too, and vice versa. Moreover, in this case

$$\begin{aligned}&\oint_{\gamma-\mathrm{int}} M_x(H_x(u,w_1))\,dH_x + M_y(H_x(u,w_1))\,dH_y = \\ &\oint_{-\gamma-\mathrm{int}} M_x(H_x(u,w_2))\,dH_x + M_y(H_x(u,w2_1))\,dH_y \le 0\end{aligned} \tag{48}$$

where $-\gamma$*-int* means the inverse path of γ*-int*.
Now a more explicit expression for the condition (43) will be derived, and the following statement will be proved. A necessary and sufficient condition for (43) is that

$$\frac{\partial\psi^*(w)}{\partial w}\cos w \ge 0 \text{ and } \frac{\partial\phi^*(w)}{\partial w}\sin w_1 \le 0 \tag{49}$$

for any *v*. The prove of this statement can be given as follows.
If $\cos w_1 = 0$ the first part of (47) is equal to zero. If $\cos w_1 > 0$, the first part of (47) is less or equal to zero if, and only if $\left.\frac{\partial\psi^*(w)}{\partial w}\right|_{w=w_1} \ge 0$.
If the contrary is supposed, and (47) is divided by $(\psi^*(w_2)-\psi^*(w_1))\cos w_1$, where w_2 is in the domain where

$$\left.\frac{\partial\psi^*(w)}{\partial w}\right|_{w=w_1} \le 0$$

$$\frac{\phi^*(w_2)-\phi^*(w_1)}{\psi^*(w_2)-\psi^*(w_1)} \ge -\tan w_1 \tag{50}$$

For the property of the derivable functions (Cauchy theorem) there is at least a point $\breve{w}$ in the interval between w_1 and w_2 where it can be written

$$\frac{\phi^*(w_2)-\phi^*(w_1)}{\psi^*(w_2)-\psi^*(w_1)} = \frac{\left.\frac{\partial\phi^*(w)}{\partial w}\right|_{w=\breve{w}}}{\left.\frac{\partial\psi^*(w)}{\partial w}\right|_{w=\breve{w}}} \ge -\tan w_1 \tag{51}$$

In addition, from (32)

$$\frac{\left.\frac{\partial\phi^*(w)}{\partial w}\right|_{w=\breve{w}}}{\left.\frac{\partial\psi^*(w)}{\partial w}\right|_{w=\breve{w}}} = -\tan\breve{w}\ . \tag{52}$$

Therefore

$$-\tan w_1 \le -\tan\breve{w}\ . \tag{53}$$

Equation (53) can not be true, because the tangent function is monotonically increasing. So it has been proved that if $\cos w_1 > 0$, the first part of (47) is less or equal to zero if, and only if $\left.\frac{\partial\psi^*(w)}{\partial w}\right|_{w=w_1} \ge 0$. Analogously it can be shown that if $\cos w_1 < 0$, the first part of (47) is less than or equal to zero if, and only if $\left.\frac{\partial\psi^*(w)}{\partial w}\right|_{w=w_1} \le 0$.
Following the same procedure for the second part of (47), it can be seen that the sign of $\sin w_1$ and $\left.\frac{\partial\phi^*(w)}{\partial w}\right|_{w=w_1}$ must be opposite. Thus statement (49) is a necessary condition.
To show that the condition (49) is also a sufficient condition, it will assumed that (47) is not verified. Then

$$(\phi^*(w_2)-\phi^*(w_1))\cos w_1 + (\psi^*(w_2)-\psi^*(w_1))\sin w_1 \ge 0 \tag{54}$$

at least for the two magnitudes w_1 and w_2, with $w_1 < w_2$ and with $\cos w_1 \ne 0$ and $\sin w_1 \ne 0$. In this case it must be either $(\phi^*(w_2)-\phi^*(w_1)) \ne 0$ or $(\psi^*(w_2)-\psi^*(w_1)) \ne 0$. If, for example, $(\psi^*(w_2)-\psi^*(w_1)) > 0$, then it is also $\cos w_1 > 0$. Therefore, since (49) is verified (54) can be divided by $(\psi^*(w_2)-\psi^*(w_1))$ and

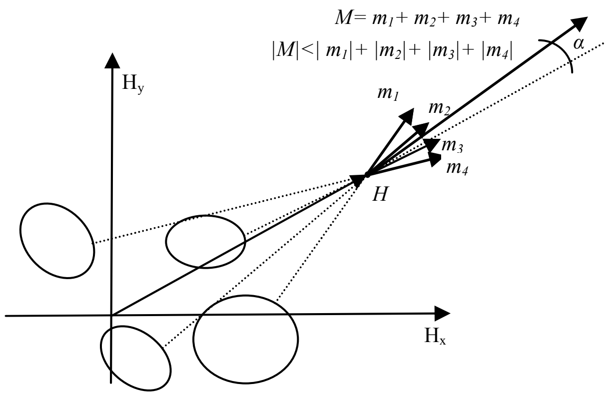
$$\frac{\phi^*(w_2)-\phi^*(w_1)}{\psi^*(w_2)-\psi^*(w_1)} \ge -\tan w_1\ . \tag{55}$$

If it is again used the Cauchy theorem, using the fact that $\frac{\partial\phi^*(w)}{\partial w}$ and $\frac{\partial\phi^*(w)}{\partial w}$ are not zero simultaneously, it is again obtained the absurd condition (53) for $w_1 < \breve{w}$.
The other cases can be treated in analogous way.
The congruency with the second principle of the thermodynamics can be discussed in *weak* form, if the value of u is assigned.
It will be proved in the following that the congruency with the second principle of the thermodynamics is guaranteed (sufficient condition) for any family of curves described by (31) and (32) if the curves are convex.
If the equipotential curves for $u \ge u_0$ are convex it follows that the angle (φ in fig. 2) between the vector $\overrightarrow{P_1P_2}$ and the vector magnetization, perpendicular to the curve is always $\ge \frac{\pi}{2}$ and $\le \frac{3\pi}{2}$, therefore the scalar product between the vector $\overrightarrow{P_1P_2}$ and the vector magnetization is always ≤ 0. This scalar product is equal to (45), therefore the statement above about the condition of weak congruency with the second principle of the thermodynamics is proved.

VI- SATURATION PROPERTY

The saturation property can be enunciated as follows
"The total magnetization due to an assembly of vector hysterons must be always less than or equal to the saturation value, that occurs when the value of the applied magnetic field go to the infinite".
Any assembly of hysterons, defined as above, obeys this property: Referring for more convenience to the Fig. 3 it is easy to check that from the Schwarz's inequality the sum of vectors comes that the maximum value of the magnetization is achieved when all the contributions to the magnetization of the

assembly of hysterons are oriented in the same direction: this occurs only when the applied field is far from the assembly of hysteron considered and tends to the infinite.



Generic assembly of Hysterons

Fig. 4. Graphic representation of the saturation and loss property

VII- LOSS PROPERTY

The loss property can be enunciated as follows

"The magnetic losses for any applied rotating magnetic field tends to zero when the magnitude of the applied rotating magnetic field go to the infinite" . Any, obeys this property: Referring again to the Fig. 4 it is easy to check that the lag angle between magnetization and magnetic field goes to zero when the applied field is far from the assembly of hysteron considered and tends to the infinite. Therefore the change of the magnetic energy density in one rotation $\oint M \bullet dH$, that

means the magnetic losses in a turn, goes to zero.

VIII- VIRGIN STATE

We want now that any assembly of vector hysterons must reproduce the virgin state, or zero-magnetization state, that means that the total magnetization must be zero for a magnetic material previously demagnetized. This can be achieved with the additional condition that the hysteron distribution in the ***H***-space must be symmetrical respect to the origin and that the direction of the magnetization is in frozen state for the hysterons whose critical surface contains the origin must be toward the origin. Fig. 4 explains the concept graphically.

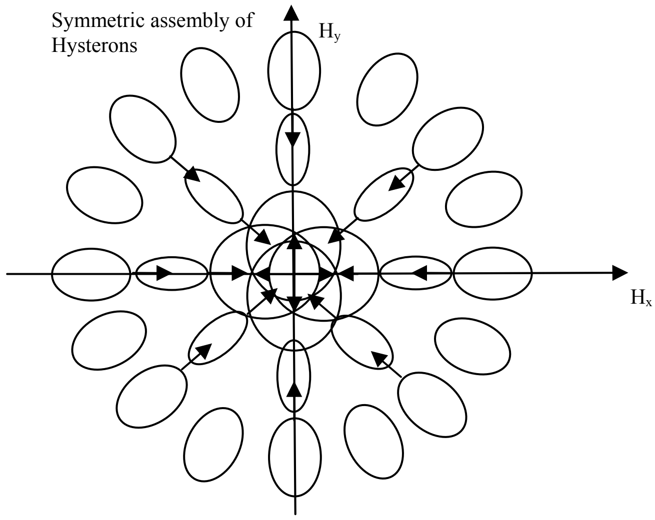


Fig. 5. Graphic representation of the virgin state property

IX. THE CONGRUENCY PROPERTY

We now show that an assembly of vector hysterons, as we discussed, has the vector congruency property: "Let γ be a directed closed curve in the applied H-space, and let Γ be the corresponding curve traced in the M-space by the vector hysteron model. Then all the Γ curves are congruent, and their displacement in the M-space is a vector function of the path from the origin of the H-space to γ". Figure 3 shows a typical curve γ and two possible input paths from the origin O to two different points A and B on γ. As shown in Fig. 3, we can divide the total population of possible hysterons in five classes: Class 1 are those completely internal to γ; Class 2 are those intersected by γ; Class 3 are those completely external to γ; Class 4 are those intersected by γ and by the input path from the origin to γ; Class 5 are those containing γ. The components of the total magnetization, at a generic point P of γ, produced by the hysterons of the class 1, 2 and 3 are independent of the input path. The components produced by the hysterons of the class 4, after one turn on γ, are also independent of the input path. On the other hand, for the hysterons of class 5 the magnetization produced is dependent on the input path, because these hysterons have a unit magnetization that is frozen in the direction that it had at the enter point. For example, in Fig. 3 the unit vector magnetization M5-OA for the input path that passes through A, and the vector unit magnetization M5-OB for the input path that passes through B are different. We can conclude that the two curves ΓA and ΓB produced in the M-space for the two input paths are congruent, and that ΓA is displaced respect to ΓB of the difference of the two integrals of M5-OA and M5-OB in the H-space, as shown in Fig. 4.

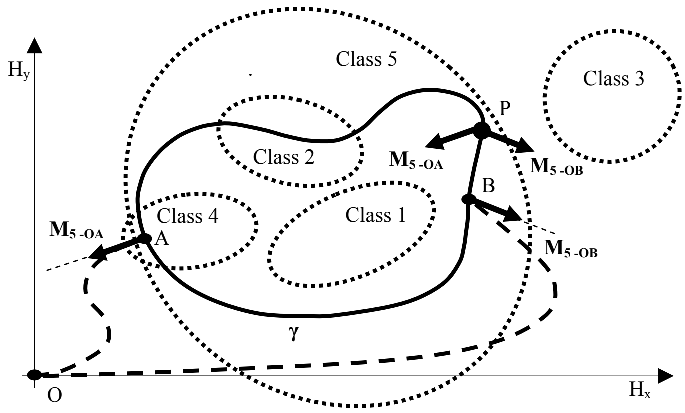


FIG. 3. An example in 2-D of classification of the hysterons (dotted lines) in the **H**-space and the typical contribution to the magnetization for the five classes of hysterons.

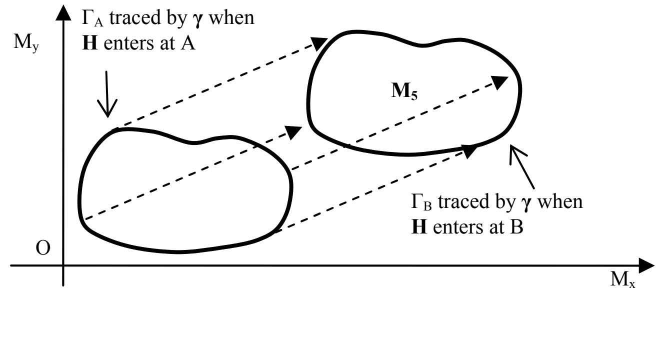


FIG. 4. Example in 2-D of the congruency property of closed paths traced in the M-space by the vector hysteron model with an input closed curve in the H-space.

X. THE DELETION PROPERTY

We will now show that the model has the vector deletion property: "Let α be a generic path from the origin to a given point P1 in the H-space. A necessary and sufficient condition, in order to cancel the magnetic memory created by the vector hysteron model in the M-space due to the input of α, is to trace in the H-space a generic path β from P1 to P2 such that all hysterons that contain β also contain α". The condition above is necessary, since if this condition is violated, for example tracing a path β1 from P1 to P1' shown in the figure 6, there exists at least one hysteron that contains β1 and not contains α. Thus, the magnetization produced by this hysteron in the point P1' depends on the path α. This condition is also sufficient, since if this condition is violated, for example by tracing a path β2 from P1 to P2 shown in the figure 6, there are no hysterons that cross α and also contain β2. Thus, the magnetization at the point P2 is independent of the path α. We can note that another way to express a sufficient condition in order to cancel the magnetic memory created by the input of α is to trace in the H-space a generic closed path β3 that surrounds α (see Fig. 5). This way is very effective and practical, either from the computational point of view, or from the experimental one.

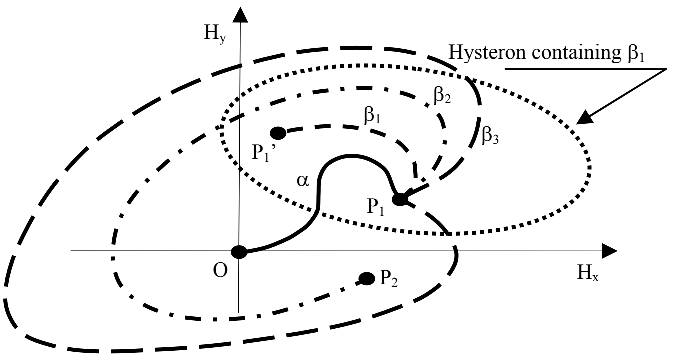


FIG. 5. Example in 2-D of the deletion property of the magnetic memory due to a generic path α from the origin to the point P1.

In the APPENDIX III and IV two choices of vector hysterons interesting for possible applications to anisotropic magnetic materials are presented and their particular properties are discussed.

XI. EXAMPLES

Here are illustrated typical results computed by the proposed vector hysteresis model.

We refer here to hysteron density function expressed in Lorentian form

$$P_{i,j,k} = \left(\frac{\sigma_x}{\pi^3 (H_x^2(i) + \sigma_x^2)} \right) \cdot \left(\frac{\sigma_y}{\pi^3 (H_y^2(j) + \sigma_y^2)} \right) \cdot \left(\frac{\sigma_u}{\pi^3 (u^2(k) + \sigma_u^2)} \right) \tag{54}$$

where σ_x , σ_y , and σ_u are the standard deviations identified for the given material. In order to keep the presentation the more general possible, the magnetization and the magnetic field are expressed in arbitrary units (a.u.).

Fig. 6 illustrates the family of symmetric loops for a scalar magnetization in the x-direction.

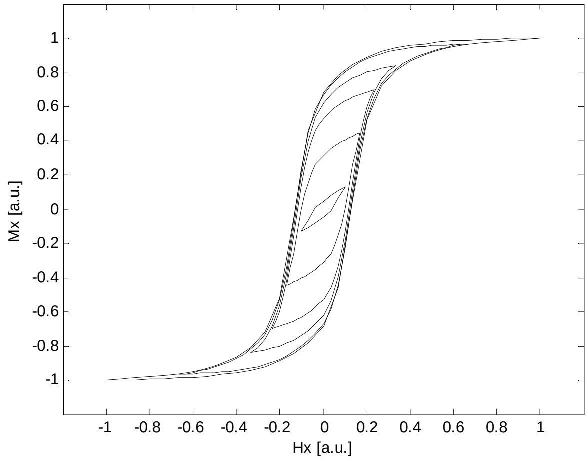


Fig. 6. The family of symmetric loops for a scalar magnetization in the x-direction.

In Fig. 7 is shown in detail the virgin curve for the scalar magnetization above.

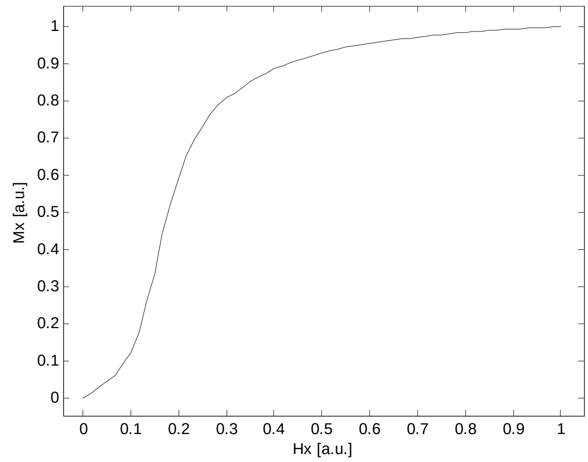


Fig. 7. The virgin curve for the scalar magnetization of Fig.6.

In Fig. 8 is pictured the family of the first reversal curves from the ascending branch of the major loop of the scalar magnetization above.

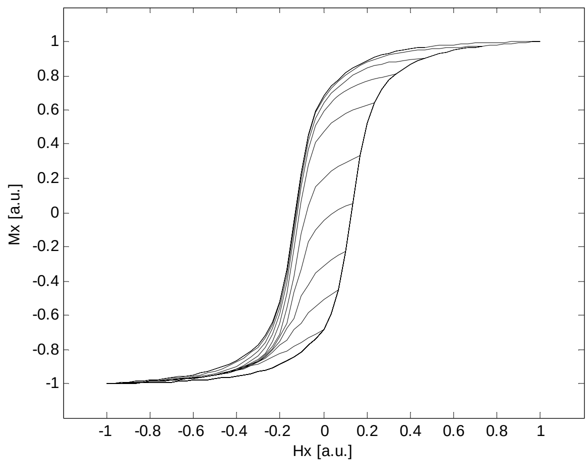


Fig. 8. The family of the first reversal curves from the ascending branch of the major loop of Fig.5.

Fig. 9 shows the family of asymmetric loops attained applying a scalar magnetic field $H_x = 0.5(1 + \sin t)$.

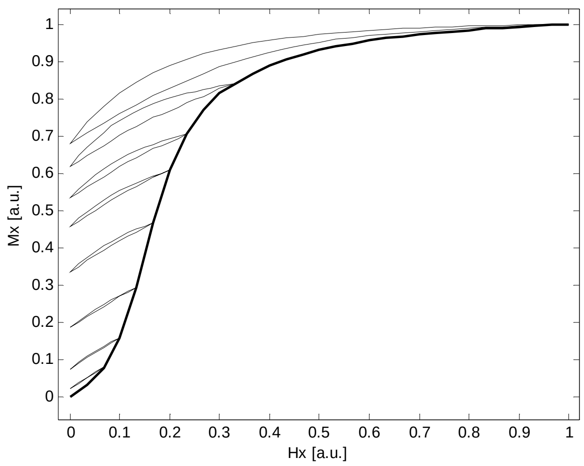


Fig. 9. The family of asymmetric loops for a sinusoidal magnetization plus a DC bias.

Fig. 10 deals with a typical magnetic path attained applying a two-harmonic scalar magnetic field.

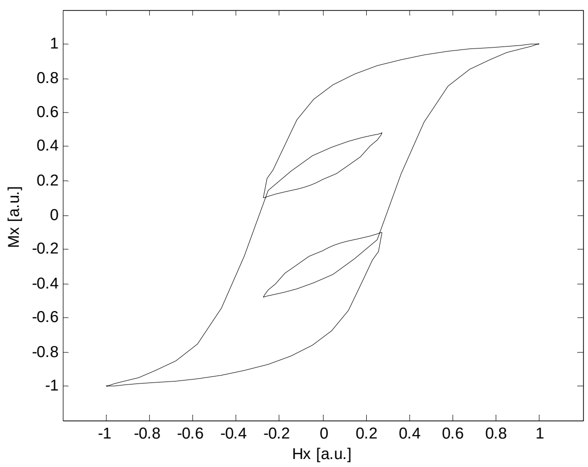


Fig. 10. Typical magnetic path for a two-harmonic scalar magnetic polarization.

Fig. 11 illustrates the family of the magnetization loci for a circular polarized magnetic excitation. In this case the material is quasi-hysotropic.

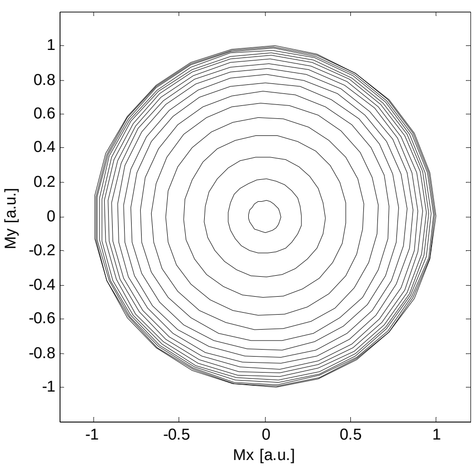


Fig. 11. Typical family of the magnetization loci for a circular polarized magnetic excitation and quasi-hysotropic material.

Fig. 12 shows a typical magnetization path for a spiroidal polarized magnetic excitation and the same material of the previous figure.

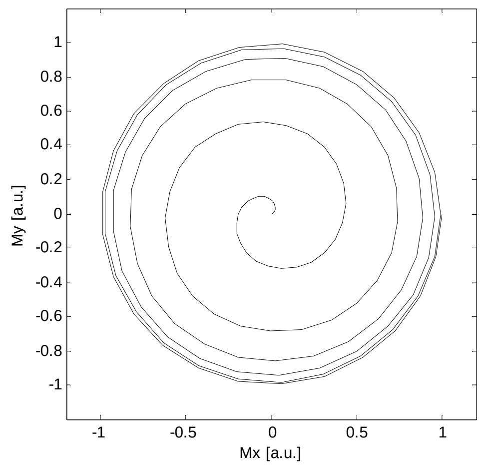


Fig. 12. Typical magnetization path for a spiroidal polarized magnetic excitation and the same material of Fig. 11.

Finally, in Fig. 13 and 14 are respectively represented the static losses in case of scalar alternate and circular excitation.

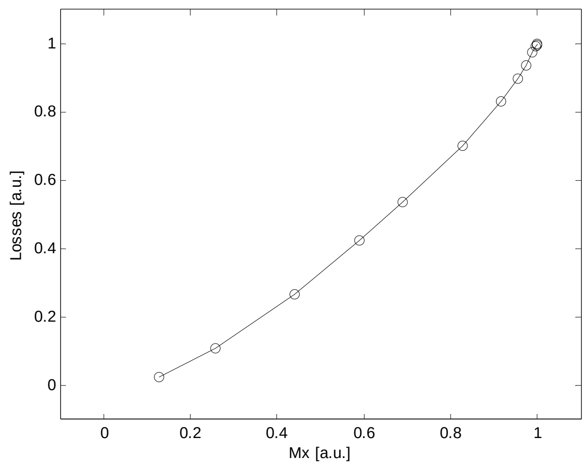


Fig. 13. Static losses for an alternate scalar excitation.

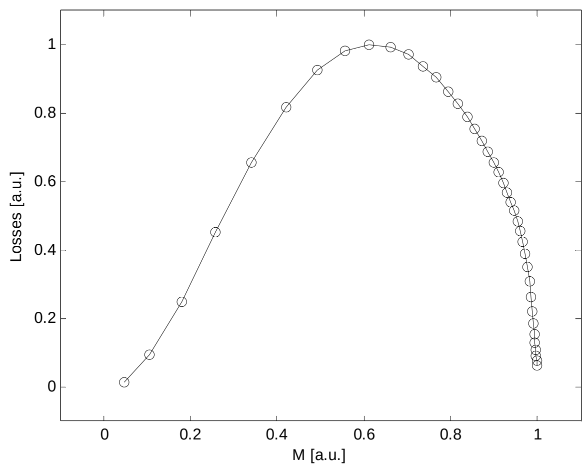


Fig. 14. Static losses for a circular vector excitation.

XII. Conclusion

- The concept of conservative unit magnitude vector fields has been introduced, and some properties of this vector fields have been discussed and defined in rigorous way.
- This properties have been applied in order to obtain the general mathematical expressions for

conservative unit vector fields as a function of a regular parametric coordinates frame, where the equipotential surfaces are traced when one parameter is constant and the lines of force are traced when the other parameter is constant. It has been proved that the lines of force of the conservative vector unit magnetic field are always straight lines.

- A necessary and sufficient condition for the conservative property of unit vector fields as a function of the parametric coordinates frame has also been derived.
- The results above have been used to define in general way a unit vector hysteresis operator. It has been proved that the equipotential curves of this operator are circles (spheres in 3-D) if their existence is extended to the entire ***H***-plane, but it has also been proved that they are a set of nested closed curves (surfaces), if the domain of definition of the model is limited to the entire *H*-plane minus a region around the origin, that can be minimized as wanted. A couple of general conditions for these families of closed curves has been derived in rigorous way.
- An additional necessary and sufficient condition for the congruency of the unit vector hysteresis operator with the second principle of the thermodynamics has been given as a function of the parametric coordinates frame. It has been proved that the congruency of the unit vector hysteresis operator with the second principle of the thermodynamics is always verified in weak form for the vector hysteresis operator proposed.
- Some examples of possible choices for the vector hysteron as a function of two parameters have been presented. The conservative condition has been proved in explicit form for these examples. The explicit mathematical expressions derived for the lines of force and for the equipotential curves of the examples presented are useful in the numeric implementation of these models.
- Other general properties of vector magnetic hysteresis models based on this class of unit vector hysteresis operator (deletion, congruency, virgin state) have been introduced and discussed.
- Finally an overview about numerical outputs possible using the vector hysteresis operator proposed has been given.

XIII. AKNOWLWDGEMWNTS

Authors wish to thank Domenico Candeloro for their interesting suggestions and the improvements to the work done.

XIV. REFERENCES

[1] Morrish, A.H., *The Physical Principles of Magnetism*, Wiley, New York, 1965.
[2] Chikazumi, S., *Physiscs of Magnetism*, Wiley, New York, 1964.
[3] Bozorth, R. M., *Ferromagnetism*, IEEE Press New York, 1994.
[4] Brown,W.F., *Magnetostatic Principles in Ferromagnetism*, North Holland, Amsterdam, 1962.
[5] Brown, W. F , *Micromagnetics*, Krieger, New York, 1978.
[6] Shtrikman, S., Treves, D., *"Micromagnetics", in Magnetism*, Vol, I-V, pp. 395-414, GT.T. Rado H. Suhl Eds., Academic Press, New York, 1963-1973.
[7] Aharoni, A., *Introduction to the Theory of Ferromagnetism*, Oxford University Press, New York, 1996.
[8] J. Miltat, "Domains and Domain Walls in Soft Magnetic Materials, Mostly", in *Applied Magnetism*, pp. 221-308, R. Gerber, C. D. Wright, and G. Asti Eds., Kluver, 1993.
[9] M. E. Schabes, "Micromagnetic Theory of Non-Uniform Magnetization Processes in Magnetic Recording Particles", *J. of Magnetism and Magnetic Materials*, Vol. 95, pp. 249-288, 1991.
[10] T. Schrefl, "Micromagnetics of Thin Films and Multilayers", in *Magnetic Hysteresis in Novel Magnetic Materials*, pp. 49-68, G. Hadjipanayris Ed., Kluver, 1997.
[11] J. G. Zhu, "Micromagnetics of Thin-Film Media", in *Magnetic Recording Technology*, pp. 5.1-5.78, C. D. Mee and E.D. Daniel Eds., Mac Graw-Hill, New York 1987.
[12] H. Kronmuller, "Micromagnetic Background of Hard Magnetic Materials", in *Supermagnets, Hard Magnetic Materials*, pp.461-498, G. J. Long and F. Grandjean Eds., Kluver, 1991.
[13] A. Aharoni, "Magnetostatic Energy Calculations", *IEEE Trans. on Magn.*, Vol. 27, NO.4, pp. 3539-3547, July 1991.
[14] F. Preisach " Uber die magnetische Nachwirkung", *Zeitschrift fur Physik*, Vol. 94, p. 277-302, 1935.
[15] G. Biorci and D. Pescetti, "Analytical Theory of the Behavior of Ferromagnetic Materials", *Il Nuovo Cimento*, Vol. 7, pp. 829-842, 1958.
[16] O. Benda, "Possibilities and Limits of the Preisach Model", *J. of Magnetism and Magnetic Materials*, Vol. 113, pp. 443-446, 1992.
[17] M. A. Krasnosel'skii and A. V. Pokrovskii, *Systems with Hysteresis*, SpringerVerlag, Berlin 1989.
[18] D.C. Jiles, and D. L. Atherton, "Theory of Ferromagnetic Hysteresis", *J. of Magnetism and Magnetic Materials*, Vol. 61, pp. 48-60, 1986.
[19] D.C. Jiles, J.B. Thoelke, and K. Devine,"Numerical Determination of Hysteresis Parameters for the Modelling of Magnetic Properties Using the the Theory of Ferromagnetic Hysteresis", *IEEE Trans. on Magnetics*, Vol. 28, NO.1, pp. 27- 35, 1992.
[20] D.C. Jiles, *Introduction to Magnetism and Magnetic Materials*, Chapman and Hall, 1991.
[21] E.C Stoner, E.P. Wohlfarth " A Mechanism of Magnetic Hysteresis in heterogeneous alloys", *Philos. Trans. R. Soc. London* A240 pp. 599-642, 1948.
[22] I.D. Mayergoyz, *Mathematical Models of Hysteresis*, Springer Verlag, New York (1991).
[23] E. Della Torre, *Magnetic Hysteresis*, IEEE Press: Piscataway, NJ, 2000.
[24] A. A. Adly, S. K. Abd-El Hafiz, "Efficient Implementation of Anisotropic Vector Preisach-Type Models using Coupled Step Functions", *IEEE Trans. on Magn.*, Vol. 43, NO.6, pp. 2962-2964, June 2007.
[25] K. Löschner, V. Rischmüller, M. Brokate, "Natural Vectorial Extension of the Preisach Operator", *IEEE Trans. on Magn.*, Vol. 44, NO.6, pp. 878-881, June 2008.
[26] J. V. Leite, P. A. Da Silva Jr., N. Sadowski, N. Batistela, P. Kuo Peng, J. P. A. Bastos, "Vector Hysteresis under Nonsinusoidal Induction Waveforms: Modelling and Experimentation", *IEEE Trans. on Magn.*, Vol. 44, NO.6, pp. 906-909, June 2008.
[27] Y. Zhang, Y.H. Eum, D. Xie, C. S. Koh, "An Improved Engineering Model for Vector Magnetic Properties", *IEEE Trans. on Magn.*, Vol. 44, NO.11, pp. 3181-3184, November 2008.
[28] T. Matsuo, "Rotational Saturation Properties of Isotropic Vector Hysteresis Models Using Vectorized Stop and Play

Hysterons”, *IEEE Trans. on Magn.*, Vol. 44, NO.11, pp. 3185-3188, November 2008.
[29] M. Trapanese, “Identification of the Parameters of Reduced Vector Preisach Model by Neural Networks”, *IEEE Trans. on Magn.*, Vol. 44, NO.11, pp. 3197-3200, November 2008.
[30] E. Della Torre, E. Pinzaglia, E. Cardelli, “Vector modeling - Part I: Generalized Hysteresis Model”, *Physica B*, Vol. 372, pp. 111-114, 2006.
[31] E. Della Torre, E. Pinzaglia, E. Cardelli, “Vector modeling - Part II: Ellipsoidal Vector Hysteresis Model. Numerical Application to a 2-D Case”, *Physica B*, Vol. 372, pp. 115-119, 2006.
[32] E. Cardelli, E. Della Torre, A. Faba, “Properties of a class of vector hysteron models”, *Journal of Applied Physics* 103, 07D927, 2008.
[33] E. Cardelli, E. Della Torre, and A. Faba, “Numerical Implementation of the DPC Model”, *IEEE Transaction on Magnetics*, VOL. 45, NO. 3, Pages: 1186-1189, March 2009.

APPENDIX I

As an example it can be consider the vector field

$$M_x = \frac{KH_x}{\sqrt{H_x^{\,2} + H_y^{\,2}}}, \quad M_y = \frac{KH_y}{\sqrt{H_x^{\,2} + H_y^{\,2}}} \qquad \text{(I-1)}$$

And it can be easily obtained

$$W = K\sqrt{H_x^{\,2} + H_y^{\,2}} \qquad \text{(I-2)}$$

Thus, the condition (3) is satisfied, and the vector field is conservative in the plane (H_x, H_y).

On the other hand, if the some vector field is expressed in the system of coordinates (u, v):

$$H_x = u\cos v, \; H_y = u\sin v \qquad \text{(I-3)}$$

It follows that

$$M_x = K\cos v, \quad M_y = K\sin v \qquad \text{(I-4)}$$

and the given field is not closed, nor conservative in the system of coordinates (u, v).
Again, if it is taken into account the following vector field
$M_x = 2u$, $M_y = \cos v$ (I-5)
It is easy to show that in the plane (u, v) this vector field is conservative and it follows that
$W = u^2 + \sin v$ (I-6)
However, the vector field above can be written in the H-plane as follows

$$M_x = 2\sqrt{H_x^{\,2} + H_y^{\,2}}, \quad M_y = \frac{H_x}{\sqrt{H_x^{\,2} + H_y^{\,2}}} \qquad \text{(I-7)}$$

And it is easy to see that it is not conservative in this system of coordinates.

APPENDIX II

If $\phi^*(w) = h$ and $\psi^*(w) = k$ then

$$M_x = \frac{H_x - h}{\sqrt{(H_x - h)^2 + (H_y - k)^2}} \text{ and}$$

$$M_y = \frac{H_y - k}{\sqrt{(H_x - h)^2 + (H_y - k)^2}} \qquad \text{(II-1)}$$

In this case the equipotential curves are circles whose center is at the point $(H_{cx} = h, H_{cy} = k)$ and the lines of force are the radii of these circles.

In Fig. II-1, II-2 and II-3 are the graphs of the x-and y-components of the magnetic field, and the magnetization when the magnetic field is directed along a line at 45 degrees and varies from -30 to 30 (in arbitrary units). For $h = k$ there is a scalar magnetic polarization, i.e. the vectors $\boldsymbol{H}$ and $\boldsymbol{M}$ are parallel and directed along the line-path, as in the theoretical case of isotropic linear magnetic material, but when the values of the two parameters are different there is a half-plane rotating polarization of $\boldsymbol{M}$, i.e. vector M turns progressively from one sense of the direction of the line to the other.

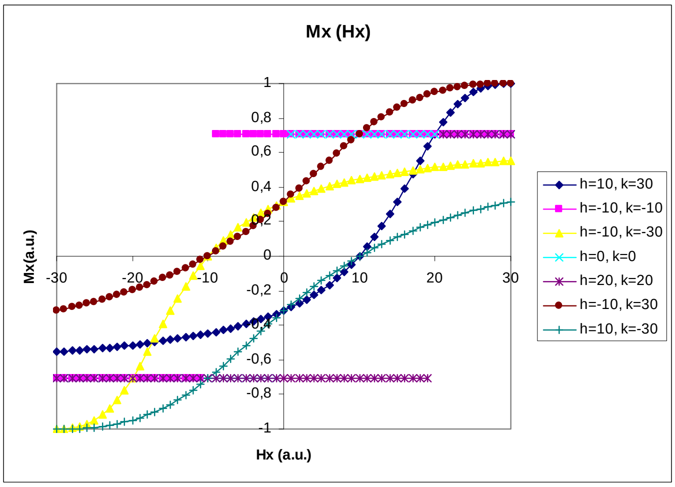


Fig. II-1 – X-components of magnetization and magnetic field for a generic conservative field with unit magnitude of magnetization as a function of the parameters h and k. The magnetic field is directed along a 45 degrees line.

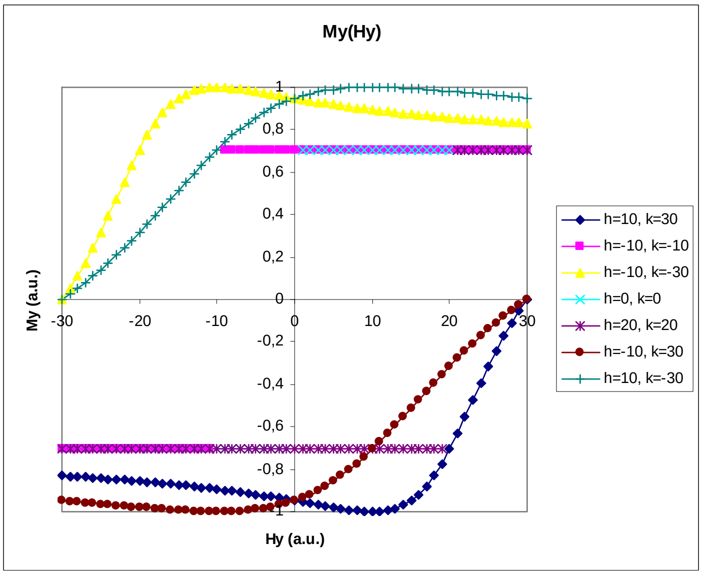


Fig. II-2 – Y-components of magnetization and magnetic field for a generic conservative field with unit magnitude of magnetization as a function of the parameters h and k. The magnetic field is directed along a 45 degrees line.

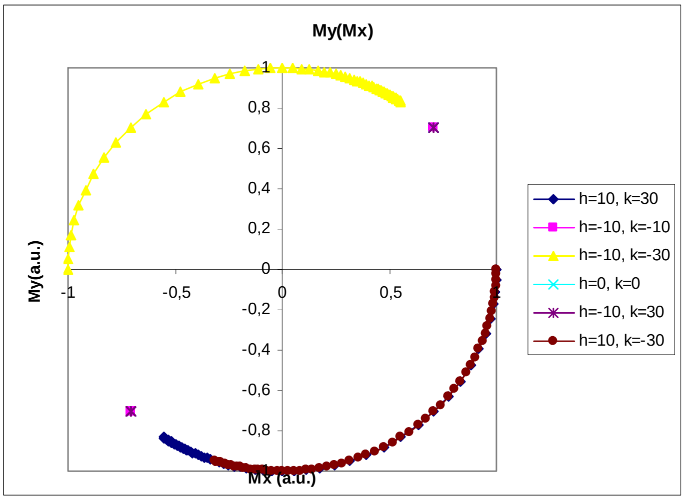


Fig. II-3 – Locii of the *x*- and *y*-component of magnetization for a generic conservative field with unit magnitude of magnetization as a function of the parameters *h* and *k*. The magnetic field is directed along a 45 degrees line.

In Fig. II-4, II-5 and II-6 are graphs of the x-and y-components of the magnetic field and the magnetization when the magnetic field rotates along a circle having the center in the origin and radius equal to 30.

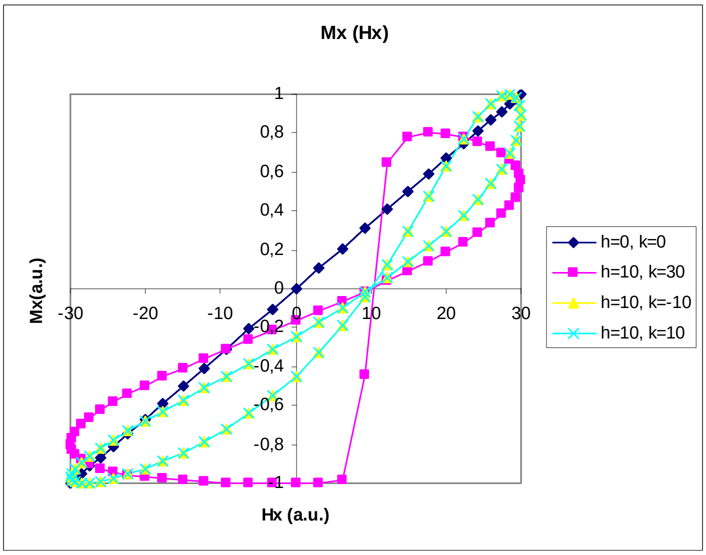


Fig. II-4 – *X*-components of magnetization and magnetic field for a generic conservative field with unit magnitude of magnetization as a function of the parameters *h* and *k*. The magnetic field rotates along a circle having the center in the origin and radius equal to 30.

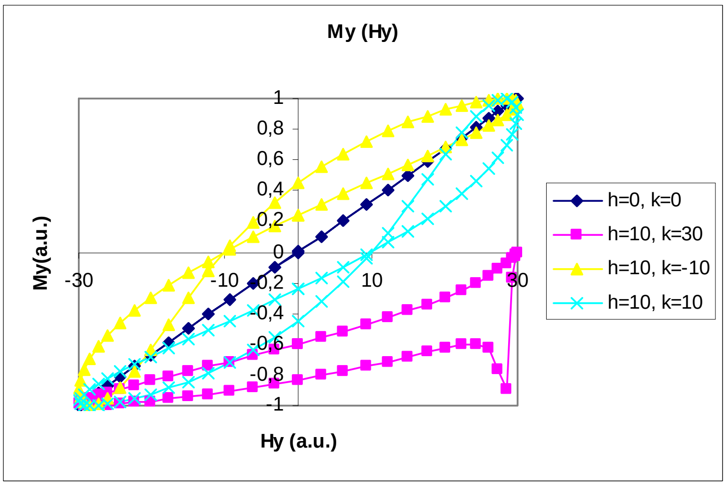


Fig. II-5 – *Y*-components of magnetization and magnetic field for a generic conservative field with unit magnitude of magnetization as a function of the parameters *h* and *k*. The magnetic field rotates along a circle having the center in the origin and radius equal to 30.

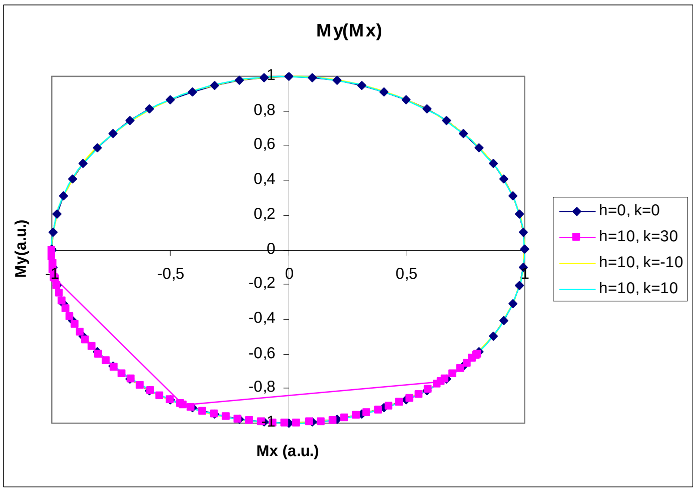


Fig. II-6 – Locii of the *x*- and *y*-component of magnetization for a generic conservative field with unit magnitude of magnetization as a function of the parameters *h* and *k*. The magnetic field rotates along a circle having the center in the origin and radius equal to 30.

In general, when the magnetic field is directed along a line passing trough the center of the equipotential curves there is scalar polarization, in the other cases there is half-plane rotating polarization. In addition, in case of rotating applied field along any circular path with the same center of the equipotential curves there is circular rotation of the magnetization and the lag angle (between ***M*** and ***H***) is always zero.

In case of rotating applied field along a circular path with center different from the center of the equipotential curves there is again a complete rotation of the magnetization if the length of the radius of the circular path is higher than the distance between the center of the circular path and the center of the equipotential circles, but the lag angle oscillates around zero. Finally, if the length of the radius of the circular path is lower than the distance between the center of the circular path and the center of the equipotential circles, there is a partial rotation of *M*, and the lag angle oscillates around zero.

It is easy to see that the energy exchange condition (49) is verified.

APPENDIX III

An interesting choice of hysterons is given by

$$\psi^*(w) = \sin^3 w\,, \quad \phi^*(w) = 3\cos w - \cos^3 w \qquad \text{(III-1)}$$

The Figures III-1, III-2, III-3, III-4 and III-5 show some of the equipotential curves obtained with this choice of variables, and the related lines of force.

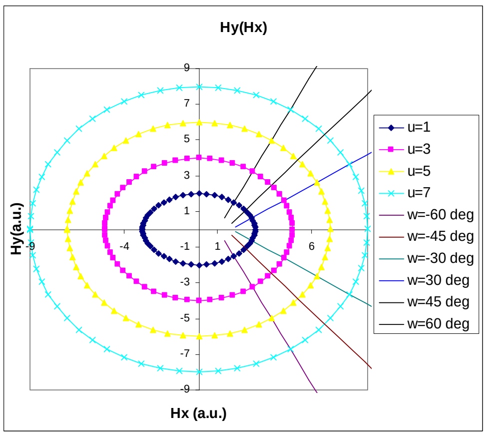


Fig. III-1 - Example of equipotential curves and lines of force generated by the (III-1) for u≥0.

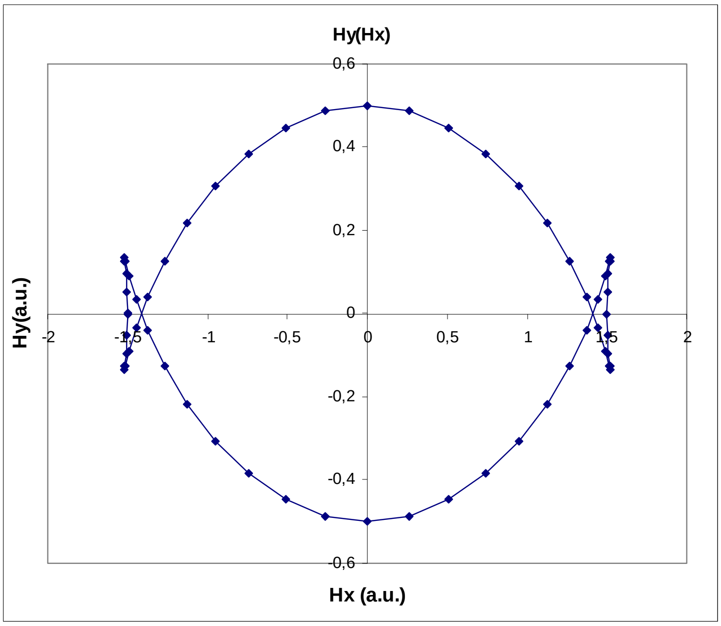


Fig. III-2 - Degeneration of the equipotential curve generated by the (III-1) for u = -0.5.

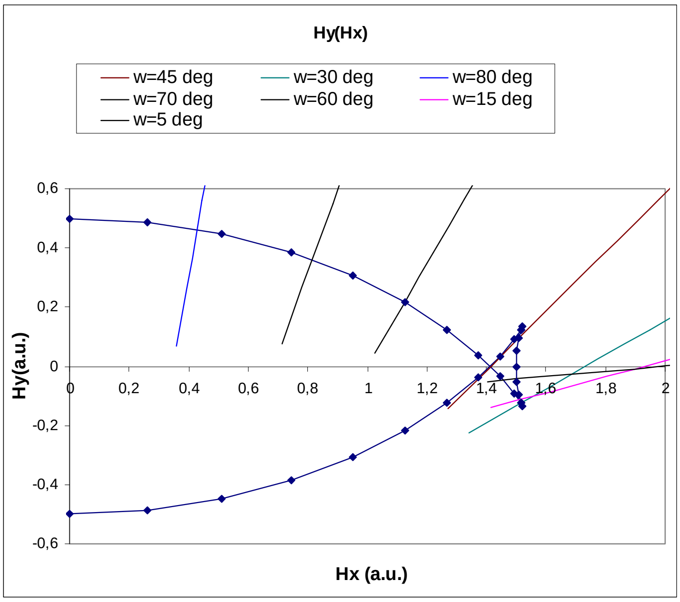


Fig. III-3 – Lines of force generated by the (III-1) for the equipotential curve with u = -0.5.

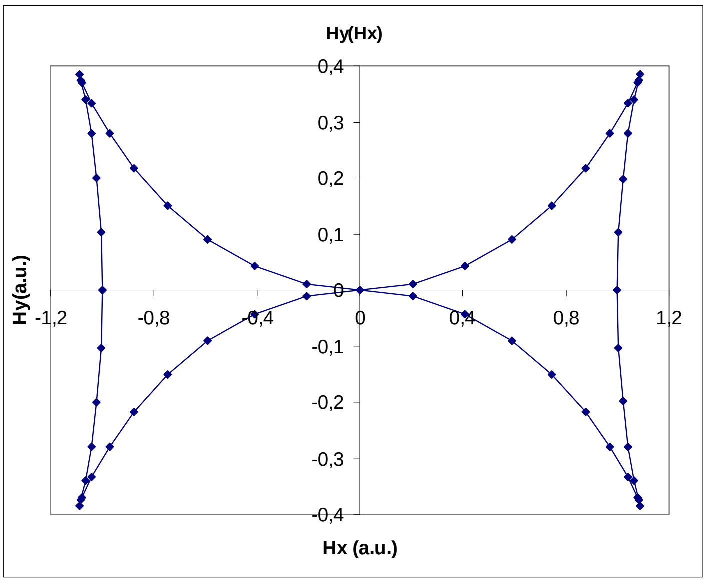


Fig. III-4 - Degeneration of the equipotential curve generated by the (III-1) for u = -1.

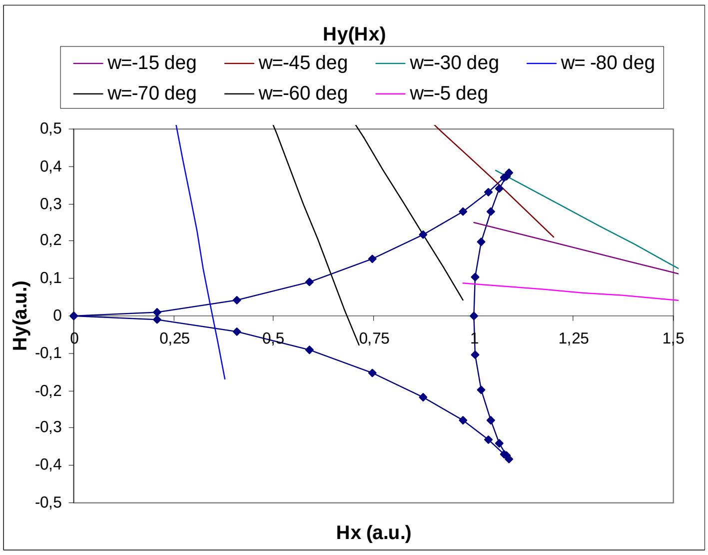


Fig. III-5 –Lines of force generated by the (III-1) for the equipotential curve with u = -1.

It is noted that these curves are not circles, nor ellipses. Moreover, for negative values of u the curves obtained are not more regular, and have singularities and double value points. Thus, the practical usability of the hysterons derived is limited to the case for $u \geq 0$, and therefore the hysterons cannot be used in the space between the origin and the critical surface for u = 0. This is not a problem, because it is sufficient to introduce a simple auxiliary variable ***H'*** = c ***H***, with c is an arbitrary constant that can minimize this region. A check about the conservativeness of the vector magnetic field (intrinsically verified by definition) can be also given in this way.

If it is examined the closed curve γ in the ***H***-plane, it follows that γ_1 is a portion of equipotential curve defined for $u=u_0$ and limited by the points (u_0, w_0) and (u_0, w_1); (see also Fig. III-6). Then γ_2 is the straight line defined for $w=w_1$ and limited by the points (u_0, w_1) and (u_1, w_1); γ_3 is a portion of critical curve defined for $u=u_1$ and limited by the points (u_1, w_1) and (u_1, w_0); and γ_4 is the straight line defined for $w=w_0$ and limited by the points (u_1, w_0) and (u_0, w_0). Then

$$\oint_{\gamma} M_x\, dx + M_y\, dy = 0 \qquad \text{(III-2)}$$

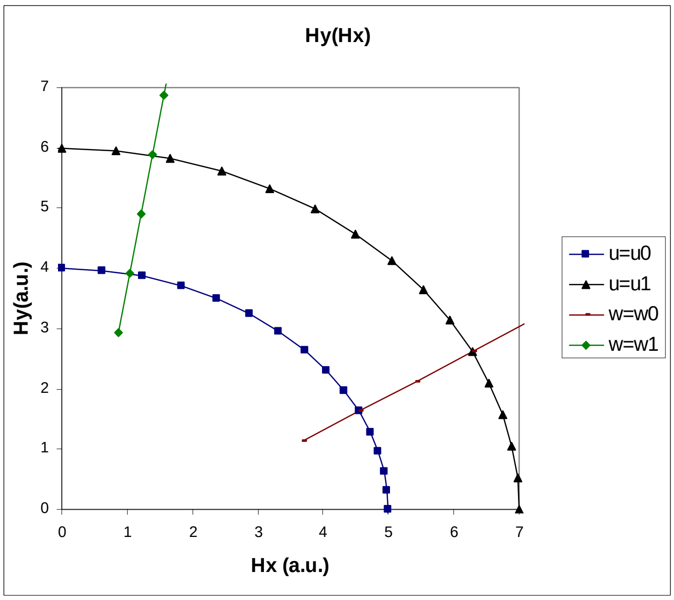


Fig. III-6 – Portions of lines of force and equipotential curves used to prove the conservativeness of the hysterons originated by (III-1).

If the energy exchange condition (49) is applied to this case

$$\frac{\partial\psi^*(w)}{\partial w}\cos w = 3\cos^2 w \sin^2 w \geq 0 \text{ and}$$

$$\frac{\partial\phi^*(w)}{\partial w}\sin w_1 = -3\sin^2 w - 3\sin^2 w \cos^2 w \leq 0 \qquad \text{(III-3)}$$

and the condition is always verified in *strong* way.

APPENDIX IV

Another possible choice of hysterons is given by

$$\psi^*(w) = D_y \sin w\,, \quad \phi^*(w) = D_x \cos w \qquad \text{(IV-1)}$$

where D_x and D_y are suitable constant, and it is described the family of hysterons of the kind shown in Fig. IV-1, calculated for $D_x = 0.1$ and $D_y = 0.45$. In Fig. IV-2 are shown the lines of force of this family of hysterons.

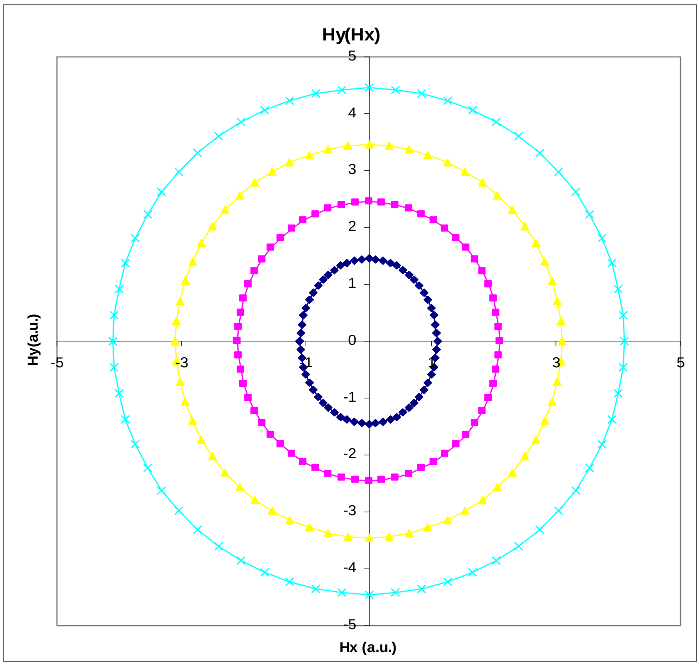


Fig. IV-1 Hysterons of the family $D_x = 0.1$ and $D_y = 0.45$.

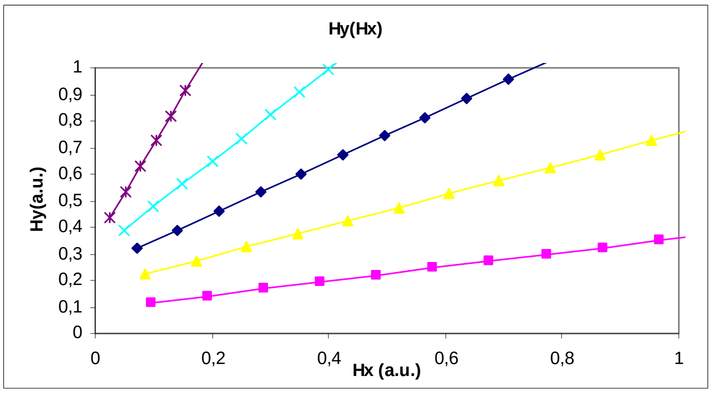


Fig. IV-2 Lines of force calculated for a family of hysterons having $D_x = 0.1$ and $D_y = 0.45$.

As $|\boldsymbol{H}|$ increases the hysterons become more spherical, and the lag angle tends to zero.
Let us write

$$H_x = (u + D_x)\cos w$$
$$H_y = (u + D_y)\sin w \qquad \text{(IV-2)}$$

Equation (IV-2) proves that the equipotential curves in this case are ellipses (ellipsoids in 3-D) having the principal half-axis length equal to $u + D_x$ and $u + D_y$ respectively. We will show in the following that the vector $\boldsymbol{M}$, as defined in this example, has slope closer to the minor axis of the critical curve (easy axis) with respect to the slope of the vector $\boldsymbol{H}$.

Let we assume, for example, the case of easy axis corresponding to x axis; in this case $D_y > D_x$ and from (IV-1) and (IV-2)

$$|\tan\alpha_H| = \frac{|u + D_y|}{|u + D_x|}|\tan w| > |\tan w| \qquad \text{(IV-3)}$$

where α_H is the slope of the vector magnetic field. In case of easy axis corresponding to y-axis, with analogous way, we get the same result. When $\boldsymbol{H}$ goes to infinity α_H goes to w.
Another system of parametric coordinates interesting for numerical applications can be obtained for v = *tan w* and it follows

$$H_y = vH_x + (D_y - D_x)\frac{v}{\sqrt{1+v^2}} \qquad \text{(IV-4)}$$

and

$$M_x = \frac{1}{\sqrt{1+v^2}}\,, \quad M_y = \frac{v}{\sqrt{1+v^2}} \qquad \text{(IV-5)}$$

If (IV-4) is solved with respect to v it can be seen that the implicit function $F(v) = (D_1 - D)\frac{H_v}{\sqrt{1+H_v^2}}$ makes weakly closed the vector field expressed by (IV-5), in all the domains outside the equipotential curve (surface in 3-D).
The change of coordinates system used here can be rewritten as

$$H_x = u\,, \qquad H_y = uv + (D_1 - D)\frac{v}{\sqrt{1+v^2}} \qquad \text{(IV-6)}$$

The curves (IV-6) with v= const are again straight lines in the $\boldsymbol{H}$-plane but not perpendicular to the equiparametric lines with u= const. This can be proved by the fact that is in general

$$\frac{dH_x}{du}\frac{dH_x}{dv} + \frac{dH_y}{du}\frac{dH_y}{dv} = \frac{dH_y}{du}\frac{dH_y}{dv} = v\left(u + \frac{dF(u)}{dv}\right) \neq 0 \qquad \text{(IV-7)}$$

In Fig. IV-3 equiparametric lines with u=const and v=const that can be used to handle the model are plotted.

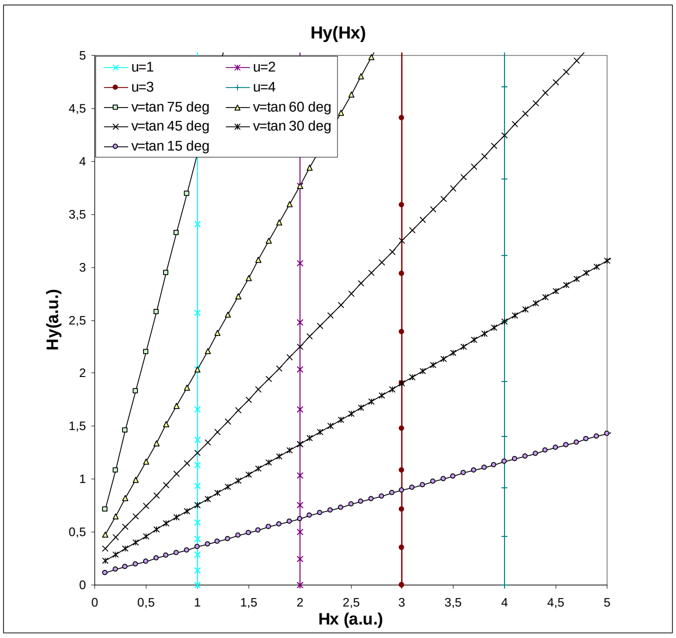


Fig. IV-3 – Portions of equiparametric lines (u=const and v = const) generated by (IV-2).

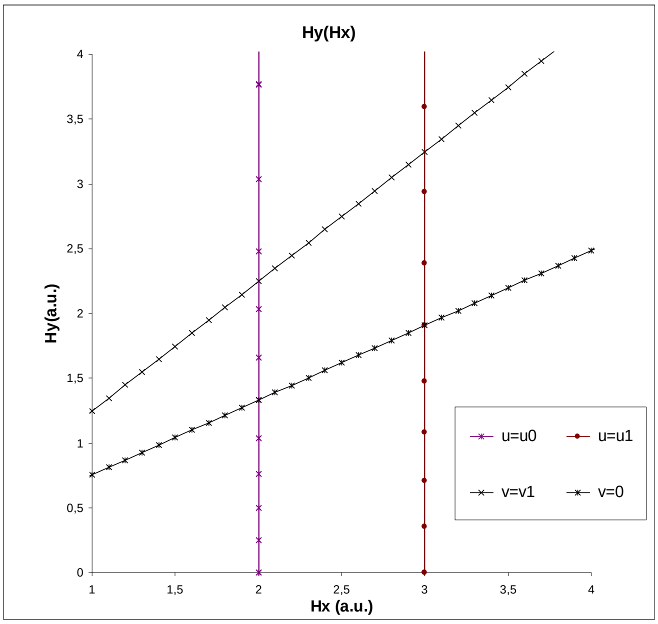


Fig. IV-4 – Portions of lines of force and equipotential curves used to prove the conservativeness of the hysteron generated by (IV-2).

Now it will be checked that the vector magnetic field as defined using (IV-2) is conservative (Remember that the conservativeness condition is intrinsically verified). The condition (8) becomes

$$\frac{\partial M_x}{\partial v}+v\frac{\partial M_y}{\partial v}=0 \qquad \text{(IV-8)}$$

and this condition is verified because

$$\frac{\partial M_x}{\partial v}=-\frac{v}{\left(1+v^2\right)^{\frac{3}{2}}} \quad \text{and} \quad \frac{\partial M_y}{\partial v}=\frac{1}{\left(1+v^2\right)^{\frac{3}{2}}} \qquad \text{(IV-9)}$$

As an example the system of parametric coordinates u,v can be used in order to prove the conservativity of the vector magnetic field more explicitly. If the closed curve γ in the ***H***-plane defined as in Fig. IV-4 is taken into account, where γ_1 is a portion of critical curve defined for $u=u_0$ and limited by the points (u_0, v_0) and (u_0, v_1); γ_2 is the straight line defined for $v=v_1$ and limited by the points (u_0, v_1) and (u_1, v_1); γ_3 is a portion of critical curve defined for $u=u_1$ and limited by the points (u_1, v_1) and (u_0, v_0); and γ_4 is the straight line defined for $v=v_0$ and limited by the points (u_1, v_0) and (u_0, v_0); it can be written

$$\oint_{\gamma} M_x\,dx+M_y\,dy=0 \qquad \text{(IV-10)}$$

It can be derived, in analogy with the theory presented in the a previous section a useful parametric representation of the vectors ***M*** and ***H***. For the sake of simplicity it will be treated the case in which ***H*** and ***M*** lie in the first quadrant. From (IV-3)

$$M_x=\frac{u+D_y}{\sqrt{\left(u+D_x\right)^2\tan^2 w+\left(u+D_y\right)^2}} \qquad \text{(IV-11)}$$

$$M_y=\frac{(u+D_x)\tan w}{\sqrt{\left(u+D_x\right)^2\tan^2 w+\left(u+D_y\right)^2}} \qquad \text{(IV-12)}$$

And from (IV-2)

$$H^2=(u+D_x)^2\cos w+(u+D_y)^2\sin w \qquad \text{(IV-13)}$$

and, therefore

$$H^2=\frac{(u+D_x)^2(u+D_y)^2}{(u+D_x)^2\tan^2 w+(u+D_y)^2}(1+\tan^2 w) \qquad \text{(IV-14)}$$

Equation (IV-14) can be split in the components

$$H_x=\frac{(u+D_x)(u+D_y)}{\sqrt{(u+D_x)^2\tan^2 w+(u+D_y)^2}} \qquad \text{(IV-15)}$$

$$H_y=\frac{(u+D_x)(u+D_y)\tan w}{\sqrt{(u+D_x)^2\tan^2 w+(u+D_y)^2}} \qquad \text{(IV-16)}$$

The mathematical expressions (IV-11), (IV-12) (IV-15) and (IV-16) describe the vectors ***M*** and ***H*** as a function of the two parameters u and $\tan w$.

It is easy to see that the congruency with the second principle of the thermodynamic in this case is verified in *strong* way for $D_x \geq 0$ and $D_y \geq 0$.

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