Technical Article

The Boundary Element Method in Full-wave Electromagnetics

***Abstract* — The Boundary Element Method, usually called the Method of Moments in full-wave electromagnetics, has a long history which is briefly reviewed. Recent developments are extending the range of applicability in two directions: to overlap with quasi-static methods for electrically small objects and to overlap with asymptotic methods for large objects.**

I. Introduction

The use of the boundary element method (BEM) in electromagnetics is a long-established subject with a vast literature that is still growing through current research. This short article cannot be comprehensive, so the aim here is provide brief historical background and an introduction to some aspects of the current work that is widening the applicability of the method.

The original area of application of the BEM was to frequency-domain modelling of thin-wire antennas, in the frequency regime where the full Maxwell's equations are required. In this context, it is usually called the Method of Moments (MoM). Harrington [1] has provided an introduction and Jones [2] gives a more comprehensive development with an emphasis on the mathematical details.

The method is developed by deriving from Maxwell's equations surface integral equations which capture the underlying physics. The advantages of this approach are that the frequency domain gives a more natural way of describing narrow-band resonant antennas, the radiation boundary condition at infinity is built in to the method via the Green's function and only the conducting structures need to be represented in the model; there is no need for a space-filling mesh.

The method solves for the current induced on the conductors by an applied field, which may be an incident plane wave for scattering calculations or receiving antennas, or a localised source at the feed-point of a transmitting antenna. Parameters of interest, such as the antenna input impedance, radiation pattern or the bistatic radar cross-section (RCS) are derived from these currents in a post-processing step.

The most widely used thin-wire MoM code is the Numerical Electromagnetics Code (NEC) written by Burke and Poggio [3] NEC is still available and has been used in many commercial products which add value by providing user-friendly pre- and post-processing tools.

The method of moments has developed in various ways from these beginnings in thin-wire modelling. First, extended surfaces were modelled for RCS predictions using a description of the surfaces as grids of thin wires. This technique amounts to crude finite-element approximation of the surface currents.

The developments in finite-element theory presented by Nédélec [4] and Raviart and Thomas [5] introduce div- and curl-conforming representations of vector fields. These ideas were applied without acknowledgement to electromagnetic BEM (or perhaps were independently discovered?) by Rao, Wilton and Glisson (RWG) [6], who described a lowest-order triangular patch boundary element method. This work was the starting point for a large effort in improved surface modelling which continues today.

The use of boundary descriptions and surface currents can be extended to include regions of homogeneous, isotropic and linear dielectric and magnetic materials by means of Huygens' principle and a vector Green's theorem, as described by Stratton and Chu [7, 8], Poggio and Miller [9] and Harrington [10].

Inhomogeneous and anisotropic (but still linear) materials can be included using volume integral equations, as described by Schaubert *et al.* [11].

The use of integral equations has the great advantage of simplifying the mesh generation requirements and reducing the number of elements. However, the penalty for this is the production of fully-populated system matrices after discretisation.

Traditionally, direct LU solution has been used, leading to an $\mathcal{O}(N^3)$ computational cost and $\mathcal{O}(N^2)$ storage requirement for N degrees of freedom (DoFs). More recently, fast solvers such as the multi-level fast multipole algorithm (MLFMA) [12, 13] have improved this cost to $\mathcal{O}(N \log N)$ storage and operation count.

Work continues on fast solvers to push the full-wave MoM to higher frequencies to close the frequency gap and overlap with the domain of applicability of asymptotic, approximate methods such as ray-tracing.

There is also a frequency gap at the low frequency limit where traditional MoM suffers from poor matrix conditioning. This is addressed by the explicit use of solenoidal DoFs, that is loop basis functions, allowing the quasi-static limit to be approached.

The remainder of this article summarises the various types of integral equations that are used, and introduces the developments that have extended the accuracy of the MoM and its applicability to higher frequency, lower frequency and mixed scale-length problems

II. Basic Equations

Consider first the simplest case of a perfectly conducting object in an infinite, homogeneous medium, typically free space with constitutive parameters ϵ_0 and μ_0.

The starting point for the development of the MoM is an integral representation of the potentials and hence the fields. The source of the potentials is the surface current $\mathbf{j}$ induced on the object by the externally imposed incident field $\mathbf{E}^i$ and $\mathbf{H}^i$, which could be a plane wave for a scattering calculation or a localised source for a transmitting antenna.

Exploiting linearity, the usual convention is to employ complex quantities and an implicit time dependence of $\exp(+i\omega t)$, which is always omitted from expressions. The taking of the real part to obtain physical quantities should always be understood.

The total field is the sum of the incident field and a scattered field $\mathbf{E}^s$ and $\mathbf{H}^s$, derived from the potentials using

$$\mathbf{E}^s = -\nabla\phi - i\omega\mathbf{A}, \qquad \mu_0\mathbf{H}^s = \nabla\times\mathbf{A}$$
$$\mathbf{A}(\mathbf{r}) = \mu_0\int G\mathbf{j}(\mathbf{r}')\,d\mathbf{r}', \qquad \phi(\mathbf{r}) = \frac{i}{\omega\epsilon_0}\int G\nabla'\cdot\mathbf{j}(\mathbf{r}')\,d\mathbf{r}'$$
$$G(\mathbf{r};\mathbf{r}') = \exp(-ikR)/4\pi R, \qquad R = |\mathbf{r}-\mathbf{r}'|$$

The integral equations then follow by inserting these field representations into the physical boundary conditions at the surface

of the conductor. The electric field integral equation (EFIE) enforces zero tangential electric field, while the magnetic field integral equation (MFIE) relates the tangential magnetic field to the surface current density:

$$\hat{\mathbf{n}} \times (\mathbf{E}^i + \mathbf{E}^s) = \mathbf{0} \qquad \hat{\mathbf{n}} \times (\mathbf{H}^i + \mathbf{H}^s) = \mathbf{j}$$

Note that the boundary condition on the magnetic field is a jump condition. The MFIE can be used only on closed surfaces with the additional information that the fields are identically zero inside perfect conductors, so that the jump condition produces a specific statement about the magnetic field just outside the conductor.

In contrast, the EFIE can be used on laminae with free edges and on thin wires. In the latter case the azimuthal fields are ignored and only the longitudinal field is tested.

It is well-known that both the EFIE and the MFIE are not fully equivalent to the Maxwell differential equations. They each suffer failures at a discrete set of frequencies that can be interpreted physically as the resonant frequencies of the cavity formed by regarding the conducting surface as hollow. Jones [2, pp. 324–357] provides a clear mathematical explanation of these failures.

These problems can be avoided by using the combined field integral equation (CFIE), which is equivalent to Maxwell's equations at all frequencies. The CFIE is formed by linearly combining the left and right-hand sides of the EFIE and MFIE: schematically, "CFIE = αEFIE + $Z_0(1-\alpha)$MFIE". Jones [2, pp. 348–350] establishes the existence and uniqueness proofs for all real $\alpha \in (0, 1)$.

III. Boundary Element Approximation

The numerical model is derived in two steps of discretisation. First, the continuously-varying surface current is approximated by a boundary element representation with a finite number of DoFs. Second, the integral equations are tested with a finite set of weight functions. The Galerkin method results from using the basis functions as the weight functions.

With test functions $\mathbf{f}_E$ for the electric field and $\mathbf{f}_M$ for the magnetic field, the tested integral equations can be written compactly as

$$\begin{array}{lrcl} \textbf{EFIE:} & \langle \mathbf{f}_E, \hat{\mathbf{n}} \times \mathbf{E}^s \rangle & = & -\langle \mathbf{f}_E, \hat{\mathbf{n}} \times \mathbf{E}^i \rangle \\ \textbf{MFIE:} & \langle \mathbf{f}_M, \mathbf{j} - \hat{\mathbf{n}} \times \mathbf{H}^s \rangle & = & \langle \mathbf{f}_M, \hat{\mathbf{n}} \times \mathbf{H}^i \rangle \end{array}$$

The tested right-hand side of the EFIE produces a vector of "voltages", V from the incident field, $\mathbf{E}^i$, so the numerical system of equations is conventionally written as $ZI = V$, where I is the vector of unknown, complex current amplitudes for the DoFs. Consequently, the full, complex system matrix, Z is called the impedance matrix.

The cavity resonance failures of the EFIE and MFIE appear in the numerical approximations as ill-conditioning of the impedance matrix over a range of frequencies in the neighbourhood of each resonant frequency. This is not a problem for small scattering bodies, but becomes more significant as the size (in wavelengths) of the body increases, since the density of resonances also increases with frequency. Use of the CFIE for large closed surfaces produces better conditioning of the impedance matrix.

IV. Solving the Linear System

The practical utility of the method of moments depends critically on the feasibility of solving the dense system of linear equations just described. Consider the scaling of the computational complexity of this solution.

For a perfectly conducting body, the basis functions used to describe the surface current must give sufficient resolution of the phase variation, so for a typical dimension L and wavelength λ the number of DoFs is $N \propto (L/\lambda)^2$.

A. Direct Solution

The use of a direct solution method such as LU decomposition needs the explicit computation of the full impedance matrix with $\mathcal{O}(N^2)$ storage requirement and a significant amount of arithmetic for each element of the matrix. The matrix factorisation requires $\mathcal{O}(N^3)$ complex multiplications and additions.

Subsequently, the solution for the currents requires a further $\mathcal{O}(N^2)$ complex multiplications and additions for each excitation.

For some applications, such as monostatic RCS computations, the same impedance matrix is used with many different excitation vectors. Then the high cost of the factorisation can be worthwhile.

However, this $\mathcal{O}(L/\lambda)^6$ cost provides a very steeply rising barrier to the computation for large objects. Remember that $2^{1/6} \approx 1.12$ so a doubling of processing speed yields only a marginal 12% increase in the size of calculation tractable in a fixed time. Similarly, $2^{1/4} \approx 1.19$ so a doubling of storage capacity yields only a 19% increase in the size of the maximum feasible calculation.

B. Iterative Solution

The high cost of the factorisation step can be avoided by using an iterative solver such as BiCGstab or GMRES [14]. Note that the impedance matrix is not Hermitian, so the conjugate gradient method itself is not applicable and convergence is not guaranteed.

Each iteration requires the computation of one or two ZI matrix-vector products, so a straightforward implementation still requires $\mathcal{O}(N^2)$ storage and computation.

In contrast to the direct method, each excitation vector is a fresh calculation for an iterative solver. When the excitation vectors change systematically from one calculation to the next, as in a monostatic RCS sweep, some savings in the number of iterations can be made by using one solution as the starting vector for the next.

V. Fast Solution Methods

The above considerations motivated the development of fast solvers such as the MLFMA, which dominates the subject because of the work of Chew *et al.* [15].

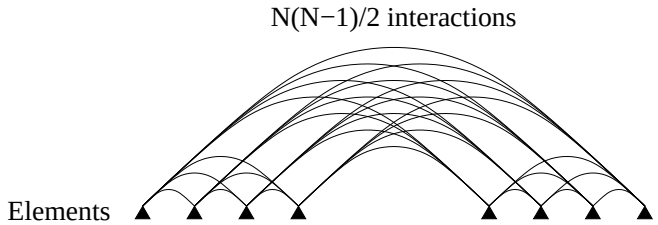
The objective of fast methods is to calculate approximately the ZI product without the need to compute all the matrix elements explicitly and with fewer operations than $\mathcal{O}(N^2)$. Then iterative solvers can run faster.

An integral equation such as the EFIE describes a balance of fields at each point on the surface of the scatterer with contributions from every other point, hence the $\mathcal{O}(N^2)$ scaling of the coupling.

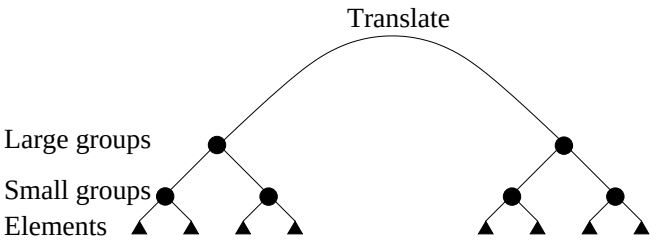
A. The Fast Multipole Method

The basic concept of the fast multipole method is to group the couplings between distant parts of the surface. The original elec-

tromagnetic fast multipole method with one level of grouping is described in an accessible form by Coifman *et al.* [16]. The concept of grouping and the origin of the computational savings it generates are illustrated schematically in Fig. 1.



(a) Direct interactions



(b) Grouped interactions

Figure 1: Grouping distant interactions reduces computational complexity.

The complexity is further reduced by repeating the grouping recursively to generate nested groups of increasing size, leading to the MLFMA.

The scattering object is embedded in a regular oct-tree of cells and the individual geometrical elements are assigned to cells at the smallest level. Interactions between nearby elements must be computed explicitly, but more distant interactions are computed in three steps: aggregate contributions in a source cell, translate to a target cell, and distribute contributions to test elements within the target cell. In the MLFMA, the aggregation and distribution steps are repeated so that the translation step transfers the largest possible collection of contributions between the largest possible pair of cells.

The name of the method originates from considering the outgoing contributions from a cell as a series of spherical waves, or multipoles. However, this description cannot be used directly, because the translation operator for spherical waves is dense. Then there would be no improvement in the computational complexity.

The method becomes feasible when plane wave expansions are used to describe the transfer of fields from source cells to target cells. Then the translation operator is diagonal; each plane wave just requires a phase shift to displace its point of reference.

However, even this technique is not sufficient to deliver the computational savings. The final insight is that small cells need fewer plane-wave directions than large cells to describe the outgoing fields adequately. At each level of the MLFMA, the aggregation of field contributions from child cells to their parent requires the interpolation of the plane wave amplitudes to yield the larger number of directions used by the parent cell. In the distribution of incoming contributions from a parent cell to its children the inverse process is required. The description must be coarsened to fewer plane wave directions.

The resulting complexity for the MLFMA is an arithmetic cost of $\mathcal{O}(N \log N)$ operations per ZI product and a storage requirement which also scales as $\mathcal{O}(N \log N)$.

There are various implementation issues to be addressed when the MLFMA is used in practice. It is usual to assume that the computer has sufficient main memory to store all the data describing the plane wave amplitudes, the sparse impedance matrix for the near interactions, the current amplitudes and the other vectors required for the iterative solution. This requirement can be satisfied with modern parallel, distributed-memory systems for problem sizes up to millions of DoFs.

The implementation needs careful housekeeping to ensure that all the pairwise interactions are accounted for, and none is counted twice.

B. The Adaptive Integral Method

The adaptive integral method (AIM) described by Bleszynski *et al.* [17, 18] provides a different way of accelerating the solution of the integral equations. As in the FMM, the fields produced by a distribution of currents are split into near and far contributions and the near contributions are calculated using selected elements of the MoM impedance matrix.

In the AIM, the far contributions are calculated by approximating the current distribution by a distribution of currents on a regular cartesian lattice. The approximation is chosen so that the original current distribution for a DoF and its regular replacement have matched multipole moments to some chosen level. This ensures that the far fields are well matched.

The regular lattice combined with the translational invariance of the Green's function allow the use of the Fast Fourier Transform (FFT) to compute the ZI product for the far contributions.

The computational complexity of the AIM depends on the type of problem being analysed. For a volume distribution of sources, where the regular lattice of approximating sources is well populated, both the operation count per iteration and storage scale favourably as $\mathcal{O}(N \log N)$. However, for perfectly conducting objects the sources are concentrated on the scatterer's surface and this non-uniform distribution degrades the scaling to $\mathcal{O}(N^{3/2} \log N)$.

C. The Block P^{3}M Method

The P^{3}M method, described by Eastwood in [19, 20, 21] and the article in this issue, also uses a splitting between explicit pairwise calculation of the near interactions and a mesh method for the far interactions.

This method deals efficiently with non-uniform distributions of sources by using nested levels of mesh refinement and also uses the FFT to achieve $\mathcal{O}(N \log N)$ computational cost per iteration. The storage requirement is reduced to $\mathcal{O}(N)$.

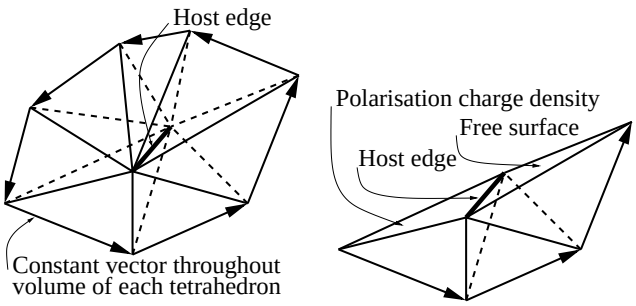
VI. Volume Scatterers

The discussion so far has concerned the MoM being used to solve for surface distribution of currents on perfectly conducting objects or the surfaces of homogeneous regions of penetrable material.

Volume integral equations can also be formulated for inhomogeneous regions of both dielectrics and magnetic materials. A recent description of the available choices of equations has been given by Botha [22], who also describes the conformity requirements of the finite elements.

For the simplest case of the volume EFIE applied to a dielectric medium with background permeability, the method uses a modified electric flux, $\mathbf{D}$ as the unknown field. This field is solenoidal, so there is a requirement for the basis functions to be solenoidal as well as div-conforming. Such basis functions have been discussed, for example, by Mendes and Carvalho [23]. At lowest order, they are piecewise-constant on tetrahedral elements and their assembly requires that the support of a single DoF should be continued from tetrahedron to tetrahedron until a closed loop is formed or the chain of tetrahedra terminates on free surfaces.

Closed loops can be identified around every internal edge of the mesh and open chains occur around every edge of free surfaces. On each tetrahedron, the constant flux is directed parallel to the edge opposite the edge hosting the DoF. Examples of these DoFs are shown in Fig. 2.



(a) Whole loop (b) Open chain

Figure 2: (a) a whole loop surrounding an internal edge (b) a partial loop at an edge on a free surface.

It is non-trivial to ensure that these DoFs are linearly independent. In the case of a single body without holes or internal cavities, Scheichl [24] has established (for a different application) that the loops associated with a spanning tree of the mesh's edges form a linearly-independent subset. The issue is more involved for general topologies. There can be macroscopic loops threading the handles of tori, and chains are required to connect the surfaces of cavities to the external space [25].

The DoFs are required to describe a volume distribution of sources and so many more are required than for surface sources of the same electrical size. The computational complexity of direct solution methods is prohibitive, but fast solvers make the volume methods practical.

VII. Strengths and Issues

The MoM offers several advantages over volume-filling mesh methods and, of course, also has some limitations. These strengths and issues are now briefly described. Then the remainder of this article gives an indication of some areas of current work that are addressing these issues.

A. Strengths

The principal strength of the MoM is that it is a boundary element method. Only the field sources need to be meshed. There is no requirement to place a volume-filling mesh in a computational volume surrounding the sources. This greatly simplifies the CAD requirements. Consider, for example, the challenge of generating a mesh in the space surrounding a thin-wire helical antenna!

The radiation condition at infinity is automatically built into the MoM via the Green's function. The MoM is well-suited to antenna and scattering calculations. There is no need for an artificial termination of a finite computational box.

The lack of a volume-filling mesh also greatly reduces the number of DoFs needed in a computation. Now, with fast solvers, the $\mathcal{O}(N \log N)$ operation count (per iteration) and $\mathcal{O}(N)$ storage is attractive.

The MoM allows general formulations that include thin wires, surface currents on perfect conductors, surface-equivalent currents on the surfaces of homogeneous materials and volume polarisation currents in inhomogeneous materials. These may all be mixed in a general-purpose code.

B. Issues

The formulation of the FMM (and hence the MLFMA) is based on a mathematical abuse of Gegenbauer's addition theorem. The theorem states that

$$\frac{e^{-ik|\mathbf{X}+\mathbf{d}|}}{|\mathbf{X}+\mathbf{d}|} = ik\sum_{l=0}^{\infty}(-1)^{l+1}(2l+1)j_l(kd)h_l^{(2)}(kX)P_l(\hat{\mathbf{d}}\cdot\hat{\mathbf{X}})$$

and it is used to displace the source point for the Green's function by an offset $\mathbf{d}$ which is small compared with the distance $\mathbf{X}$ from the centre of the source cell to the field point. The convergence of the infinite sum relies on $|\mathbf{d}| < |\mathbf{X}|$ so that the decay of $j_l(kd)$ defeats the growth of $h_l^{(2)}(kX)$ and their product becomes small as $l \to \infty$. However, the next step of the development replaces the Bessel function by its plane-wave representation and interchanges the order of the summation and the integral over the plane wave directions. This operation is legitimate only if the sum is truncated at some upper limit, $l = L$. Consequently, the FMM suffers from a low-frequency breakdown when the cell size at the bottom of the oct-tree becomes small compared with the wavelength and the growth of the Hankel function causes rounding error to dominate.

This problem with the FMM is avoided in the Block P^{3}M method which does not suffer from such a low-frequency breakdown and has a continuous quasi-static limit tending to the solution of Laplace's equation.

There is another low-frequency problem with any variant of the MoM that uses the EFIE. The contribution to $\mathbf{E}$ from $\nabla\phi$ dominates $i\omega\mathbf{A}$ as $\omega \to 0$. This can lead to local rounding-error problems with the evaluation of the elements of the impedance matrix. However, there is a more serious difficulty. Any discrete representation of the EFIE should be sufficiently accurate to yield a zero integral of $\nabla\phi$ around a closed loop. (Early ones were not.) Hence any DoFs that can be arranged to form a closed loop yield a set of rows of the impedance matrix that are almost linearly dependent and Z is ill-conditioned.

This problem can be avoided by recognising solenoidal, loop basis functions as the basic unknowns, rather than carrying them as sums of other non-solenoidal basis functions. More explanation of this technique is provided in Sec. VIII below.

The fast methods achieve their speed-up by accelerating the ZI product. This calculation is just one part of an iterative solution process and the overall performance depends on an efficient iterative scheme. The number of iterations depends critically on the conditioning of the impedance matrix, Z. The CFIE should be used when it is applicable (i.e., for closed surfaces) since it has a lower condition number than the EFIE and avoids the internal resonance problem for large bodies.

In practice, some form of preconditioning is essential to improve the rate of convergence. The choice of preconditioner is greatly constrained since the fast methods do not provide access to the entire matrix Z. The explicitly available elements of Z for near interactions can be used to produce a sparse approximate inverse preconditioner that is more effective than the simplest use of only the diagonal elements of Z.

The MoM is commonly implemented using the lowest-order RWG basis functions which employ a piecewise-linear description of the surface current and have a piecewise-constant surface charge density. This level of description is widely observed to be acceptable in practice, although the rigorous numerical analysis by Bendali [26, 27] fails to prove consistency and convergence. Improved accuracy and convergence with refinement of the mesh can be achieved by using higher-order representations of the geometry and the surface current, as described in Sec. IX below. Then Bendali's analysis does prove consistency and convergence.

VIII. MoM at Low Frequency: Loop-tree DoFs

As indicated in the previous section, the low-frequency breakdown of the MoM due to ill-conditioning of the impedance matrix can be avoided by adopting solenoidal, loop basis functions as DoFs. Every internal vertex of a triangulated surface can host a simple loop. The RWG basis functions give the correct number of DoFs, so each loop requires the discarding of a RWG DoF.

The process of identifying which RWG DoFs to discard involves global aspects of the model and cannot be approached naïvely. The surviving RWGs must visit every triangle to allow a distribution of surface charge, but cannot be allowed to form closed loops; they must form a spanning tree connecting triangle centres. So the surviving subset of RWGs are called tree basis functions.

The identification of the loop and tree basis functions can be achieved using standard graph theory algorithms: the identification of spanning trees and their fundamental cycle bases. Eibert [28] described the simplest case of a surface homeomorphic to a disc. Recently, Morgan [29] has summarised extensions to more complex topologies, including multiple junctions.

The transformation from RWG to loop-tree basis functions can be regarded as a change of basis (in the vector-space sense) in the finite element space used to approximate the surface current. The transformation can be applied retrospectively to the impedance matrix and current and excitation vectors in an existing MoM code, but it is preferable to employ the loop basis functions directly, so that the $\nabla\phi$ terms can be omitted entirely when it is known that they are not needed.

When loop and tree basis functions are used, the system of linear equations can be partitioned into a 2×2 block system in which the blocks of Z exhibit different scalings with frequency [30, 31]. Then a rescaling of these blocks and corresponding parts of the current and excitation vectors demonstrates an approximate Helmholtz decomposition into quasi-magnetostatic and quasi-electrostatic parts.

In this way, a good condition number for Z can be obtained from the EFIE down to very low frequencies, producing a continuous limit as $\omega \to 0$.

IX. Higher-order Elements

The use of higher-order elements in wire and surface MoM models offers improvements in two distinct aspects of the method.

First, the use of curvilinear elements, rather than straight wire segments and flat, faceted triangulations gives a more accurate representation of the physical geometry. This offers improved accuracy in two ways: the positions of the current sources are better matched to their true positions, improving the accuracy of the phase of the fields produced by the sources, and the direction of the surface currents varies with position in a more accurate manner, improving the contributions to the vector fields. Bendali [27] identified the latter point as the weakest aspect of the lowest-order RWG approximation.

Second, the use of higher-order shape functions to describe the variation of current along wires and over surfaces allows a more accurate representation of the true current distribution.

Hierarchical elements allow the selective use of higher-order descriptions on critical parts of the model, with easy matching to lower-order descriptions on less-critical parts. Their use has recently been described by Eastwood and Morgan [32].

They obtain curl- and div-conforming elements following the criteria identified by Nédélec [4]: the curl- (div-) conforming elements have continuous tangential (normal) component; the highest-order irrotational (solenoidal) terms in the polynomial representation of the current are set to zero; and the curl- (div-) conforming shape functions are complete polynomials only to the same order as their curl (div).

They regard surface and wire elements as limiting cases of 3D elements. This simplifies the concept of element assembly to obtain the correct continuity of the current, particularly in the mixed-dimensional case of a wire attached to a surface. This viewpoint also allows the use of standard finite-element terminology: the div-conforming elements are "face" elements while the curl-conforming elements are "edge" elements.

The detailed development in [32] starts from curvilinear coordinate transformations to describe the higher-order geometry. Then the shape functions describing the variation of current over the surface are defined in terms of polynomials in the parametric coordinates. This clearly separates the geometry definition from the shape functions. Finally, judicious use of scaling by the Jacobian of the coordinate transformation greatly simplifies the integrals occurring in the definition of the MoM impedance matrix elements.

The RWG basis functions are zeroth-order div-conforming; they have piecewise constant divergence on flat elements. Their normal current is constant along the edge ("face") of the element. The first-order generalisation has a linearly-varying divergence and a normal current which varies linearly along the edge of the element.

The integrals occurring in the MoM impedance matrix elements are singular whenever source and test functions overlap because of the singularity in the Green's function. These singularities require careful numerical treatment. One standard technique is singularity subtraction [33], in which a simpler singular integrand with known closed-form integral is subtracted from the required integrand to leave a less-singular remainder which can be integrated adequately with a numerical scheme.

Polynomial shape functions of any order can be integrated over flat triangular elements using the recurrence relations provided by Järvenpää *et al.* [34], but these do not generalise directly to curved elements. This problem can be resolved by using the known integral on the tangent plane at the point of closest approach between the source element and the field point. The radius vector occurring in the required integrand is then different from that in the integrand for the tangent plane integral, and care is needed to ensure that the match between the two integrands is sufficiently close for the singularity subtraction to be effective.

X. Final Remarks

The boundary element method for full-wave electromagnetics has a long history. Initially it was limited to the niche where scatterers were composed of conducting wires and surfaces, and had sizes of the order of the wavelength. Its strengths were the relative simplicity of meshing only surfaces and its automatic treatment of isolated boundary conditions. The absence of volume meshes also made the treatment of moving conductors relatively simple.

It was not without limitations. It could not handle electrically very small objects because it became ill-conditioned at very low frequency. It could not handle electrically very large objects or extensive volume scatterers because of steeply rising computational costs at high frequency.

The more recent developments have built on its strengths and greatly reduced its limitations. At high frequency, fast solvers have reduced the operation count and storage requirements to make BEM superior to other methods for large scatterers. At low frequency, application of graph theory to identify loop basis functions has led to well-conditioned matrices all the way down to statics. The introduction of Block-P^{3}M allows the same fast solver to be used for all frequencies, simplifying the treatment of multi-scale problems. Faster computers and developments of improved preconditioners for the Krylov iterative schemes used in these fast solvers continue to extend the range of feasible calculations.

The widening range of accessible frequencies has been complemented by an increase in the accuracy of the method. Consistent usage of higher-order div- and curl-conforming elements and the higher-order treatment of singular integrals allows higher accuracy treatment of both geometry and function support, making possible the introduction of hp-refinement in the MoM.

The results of the developments in boundary integral methods over the past decade have led to the BEM moving from its original niche to become the first choice for an increasingly wide range of electromagnetic calculations.

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