Discrete Electromagnetics:

Maxwell's Equations Tailored to Numerical Simulations

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Abstract

The Finite Integration Technique (FIT) rewrites Maxwell's equations in their integral form into a discrete formulation. The resulting algebraic set of equations, the Maxwell-Grid-Equations (MGE), are well-suited for numerical simulation, but they represent also the theoretical basis of a discrete electromagnetic field theory. The approximation of the method lies in the construction principle of the constitutive material equations. Extensions to the so-called classical FIT such as the *Non-orthogonal FIT* or the *Conformal FIT* are presented.
In addition, the basic algebraic properties of this discrete electromagnetic field theory allow to prove conservation properties with respect to energy and charge of the discrete formulation and give an explanation of the stability properties of numerical time domain formulations. The usual restriction of the FIT to a mere spatial semi-discretization scheme is explained in response to the article of Tonti [19].

Introduction

The known macroscopic electromagnetic phenomena can be described by the solutions of the four Maxwell's equations

$$\begin{aligned}
&\int_{\partial A} \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int_A \vec{B} \cdot d\vec{A} \quad \forall A \in \mathbb{R}^3, \\
&\int_{\partial A} \vec{H} \cdot d\vec{s} = \int_A (\frac{\partial}{\partial t} \vec{D} + \vec{J}) \cdot d\vec{A} \quad \forall A \in \mathbb{R}^3, \\
&\int_{\partial V} \vec{B} \cdot d\vec{A} = 0 \quad \forall V \in \mathbb{R}^3, \\
&\int_{\partial V} \vec{D} \cdot d\vec{A} = \int_V \rho \, dV \quad \forall V \in \mathbb{R}^3,
\end{aligned} \tag{1}$$

here specifically given in their integral form. In this form, they do not require the additional mathematical smoothness properties on the field strength vectors and flux density vectors as if they would be formulated in a differential way and their macroscopic character is given with the connection to geometric objects in $\mathbb{R}^3$ such as volumes V, areas A and lines L.
Since the first formulation of this set of equations, for several generations of mathematicians, physicists and engineers only analytical methods were available to attain solutions. Thus, the continuous differential formulations of Maxwell's equations, featuring field strength and flux density vectors as state variables, have become a starting point when applying such analytical methods. Their application is however limited to cases with very simple geometries and material distributions and thus quite often the number of required simplifications and model assumptions prohibits to achieve realistic solutions of technical applications. Nonetheless,

electrical engineers and physicists during most of the last two centuries had to focus on the differential formulation of Maxwell's equations.
With the introduction of computers, new mathematical tools and engineering disciplines had to be devised to actually solve the electromagnetics equations. Since digital computer arithmetics typically just allows to process a finite number of real and integer values, the crucial step of almost any computational solution method is a discretization of Maxwell's equations. In the past decades various such methods have been devised, typically based on the continuous differential formulation of Maxwell's equations.

In the article of Tonti [19] a discrete form of the equations of electromagnetism has been developed, which specifically does not involve the vector fields of the differential formulations. In this paper we show that this approach originating from theoretical considerations on the physical structure of Maxwell's equations is closely related, if not identical, to the methodology of the Finite Integration Technique (FIT) described in [24]. We will comment on those aspects of the treatise in [19], where a difference to the FIT approach becomes visible. Our point of view is mainly focused on the practical computational solution of the discrete systems, which motivates the FIT reformulation of the equations in (1).
The paper is organized as follows: In the first part the ideas and the notation of the Finite Integration Technique are derived directly from Maxwell's equations in integral form. In section two the constitutive equations are incorporated and extensions to the classical FIT such as the *Non-orthogonal Finite Integration Technique* or the *Conformal Finite Integration Technique* are addressed; the algebraic properties of the resulting discrete formulation are shortly stated, followed by a discussion comparing the time continuous and the time discrete formulations of electromagnetism.

The Finite Integration Technique

Since its first publication by Weiland in 1977 [21], the foundation of the Finite Integration Technique are Maxwell's equations in their integral formulation (1).
For practical calculations, a first step of the FI-method consists in the spatial restriction of the electromagnetic field problem, which usually represents an open boundary problem, to a bounded space region $\Omega \in \mathbb{R}^3$ containing the space region of interest. The decomposition of the computational domain Ω into a finite number of simplicial cells V_n, such as tetrahedra, hexahedra or combinations thereof, has to fulfill the premise, that all cells have to fit exactly to each other, i.e., the intersection of two different cells is either empty or it must be a two-dimensional, simply connected facet $A_m \in G$, a one-dimensional edge $L_k \in G$ shared by both cells or a vertex $N_l \in G$. This decomposition yields the finite simplicial cells complex G, which serves as computational grid.
Each edge of the cells includes an *initial orientation*, i.e., a direction, such that the union of all these cell edges can be described as a *directed graph*. Analogously, also the facets of the cell complex are associated with an orientation, i.e., their facet normal vectors are directed.

Starting with this very general cell-based approach to a spatial discretization it is clear, that the FI-Technique is not restricted to three-dimensional Cartesian meshes, which were used in earlier publications on the method rather for their relative efficiency with respect to grid generation and fast data structures. The contemporary FIT framework allows to consider all types of coordinate grid meshes, orthogonal and non-orthogonal meshes [13], [20], [24]. Also so-called *consistent subgridding schemes* (corresponding to a local mesh refinement including grid line termination techniques) have been developed [17]. Theoretically, the FI-Technique

itself even extends to non-simplicial cells, as long as the resulting cell complex is homeomorphic to a simplicial cell complex. For practical application such general cell complexes, where the cell edges may be curves, only play a role if they occur as coordinate meshes.

After the definition of the grid cell complex *G*, the further introduction of the FI-theory can be restricted to a single cell volume V_n. To this end, we define a set of so-called integral state variables of the FIT:
The real number $\hat{e}_k := \int_{L_k} \vec{E} \cdot d\vec{s}$ is the *electric voltage* along the edge L_k bounding the facet $A_m \in \partial V_n$, the number $\hat{\hat{b}}_m := \int_{A_m} \vec{B} \cdot d\vec{A}$ represents the *magnetic flux* through the cell facet A_m bounding V_n. Collecting the electric voltages $\hat{e}_i$ into a column vector $\hat{\mathbf{e}} = \{\hat{e}_k\}_{k:L_k \in G}$ and of the magnetic facet fluxes $\hat{\hat{b}}_m$ in $\hat{\hat{\mathbf{b}}} = \left\{\hat{\hat{b}}_m\right\}_{m:A_m \in G}$, Faraday's law in (1) can be rewritten for all cell surfaces of the complex G in the system of linear ordinary differential equations

$$\sum_{j:L_j \in G} C_{i,j}\, \hat{e}_j = -\frac{\mathrm{d}}{\mathrm{dt}} \hat{\hat{b}}_i \quad \forall i : A_i \in G\,. \tag{4}$$

The resulting matrix $\mathbf{C} = \left\{C_{i,j}\right\}_{i,j}$ contains only topological information on the incidence relation of the cell edges within G and on the orientation of the edges and the facets, which determines the signs of its components following the right-screw-rule of Maxwell's curl equations. Hence it features matrix coefficients $C_{i,j} \in \{-1, 0, 1\}$ and represents a discrete *curl*-operator on the grid *G*. Fig. 1 depicts the situation for a cell facet A_n. The integral formulation of Faraday's law in (1) is valid for each single facet A_m of G and the discrete approach in the FIT naturally extends to larger facet areas $A = \bigcup A_i$ due to the relation $\int_A = \sum_i \int_{A_i}$. The same result will hold for line integrals. This motivates the spatial discretization approach by a finite cell complex chosen within the Finite Integration Technique.

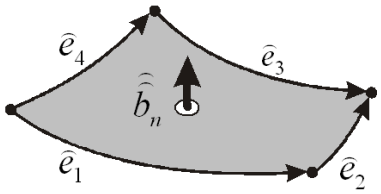

Fig.1 A cell facet A_n of a cell complex G with the allocation of electric voltages $\hat{e}_1, \ldots, \hat{e}_4$ along the edges and magnetic flux $\hat{\hat{b}}_n$ allocated on this surface. Faraday's law rewrites for this facet as $+\hat{e}_1 + \hat{e}_2 - \hat{e}_3 - \hat{e}_4 = -\frac{\mathrm{d}}{\mathrm{dt}} \hat{\hat{b}}_n$.

The second discrete differential operator to be considered is the *divergence* operator. Its derivation originates from Maxwell's equation describing the non-existence of magnetic charges which is considered for a cell $V_n \in G$ as shown in Fig. 2.

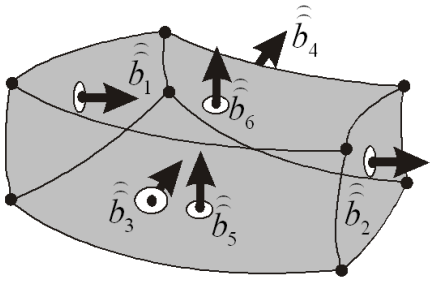

Fig.2 Mesh cell of G with an allocation of all flux quantities contributing to the closed surface integral for the non-existence of magnetic charges within a cell volume. The integral balance for this cell volume reads as $-\hat{\hat{b}}_1+\hat{\hat{b}}_2-\hat{\hat{b}}_3+\hat{\hat{b}}_4-\hat{\hat{b}}_5+\hat{\hat{b}}_6=0$.

Collecting this relation for a single cell for all the flux variables in the whole cell complex G, this yields the system of linear equations

$$\sum_{j:A_j\in G} S_{i,j}\,\hat{\hat{b}}_j = 0 \quad \forall i : V_i \in G\,. \tag{5}$$

The matrix coefficients of the discrete divergence ("*source*") operator $\mathbf{S}=\{S_{i,j}\}_{i,j}$ depend only on the grid incidence relations just as the discrete *curl*-matrix $\mathbf{C}$, i.e., $S_{i,j}\in\{-1,0,1\}$.

The discretization of the remaining two Maxwell equations within the Finite Integration Technique is executed on a second, so-called *dual* cell complex $\tilde{G}$, which at this point is only required to fulfil the same theoretical prerequisites as the *primal* cell complex *G*. Further constraints on $\tilde{G}$, however, will become necessary when introducing the constitutive material relations.

Along the edge cells $\tilde{L}_k$ of $\tilde{G}$ we integrate the magnetic field intensities resulting in a magnetomotive force $\hat{h}_k := \int_{\tilde{L}_k} \vec{H}\cdot d\vec{s}$, called *magnetic grid voltage* with the physical unit *Ampère*.
On the cells surfaces $\tilde{A}_k$ of $\tilde{G}$ the *dielectric fluxes* $\hat{\hat{d}}_k := \int_{\tilde{A}_k} \vec{D}\cdot d\vec{A}$ and the *electric currents* $\hat{\hat{j}}_k := \int_{\tilde{A}_k} \vec{J}\cdot d\vec{A}$ are allocated in analogy to magnetic facet fluxes on *G*.

The discretization of Ampère's law in integral form in (1) can be performed for an arbitrary facet $\tilde{A}_k$ of a dual grid cell $\tilde{V}$ in complete analogy to Faraday's law by summing up the magnetic grid voltages in order to obtain the displacement current and the conductive current through the considered cell facet.

Finally, Gauss' law in integral form can be discretized for the dual grid cells, introducing the electric charge content $q_{E,k} := \int_{\tilde{V}_k} \rho\, dV$ of the dual grid cell $\tilde{V}_k$. Both reformulations of Maxwell's equations for the dual grid cell complex $\tilde{G}$ will result in matrix equations containing the topological grid operators $\tilde{\mathbf{C}}$ for the dual discrete curl and $\tilde{\mathbf{S}}$ for the dual discrete divergence, both featuring a tilde sign to show that they belong to $\tilde{G}$. For the cell complex pair $\{G,\tilde{G}\}$ the complete set of discrete matrix equations, the so-called *Maxwell-Grid-Equations* (MGE) are

$$
\begin{aligned}
\mathbf{C}\hat{\mathbf{e}} &= -\frac{\mathrm{d}}{\mathrm{dt}}\hat{\hat{\mathbf{b}}}, \\
\mathbf{S}\hat{\hat{\mathbf{b}}} &= \mathbf{0}, \\
\tilde{\mathbf{C}}\hat{\mathbf{h}} &= \frac{\mathrm{d}}{\mathrm{dt}}\hat{\hat{\mathbf{d}}} + \hat{\hat{\mathbf{j}}}, \\
\tilde{\mathbf{S}}\hat{\hat{\mathbf{d}}} &= \mathbf{q}_E ,
\end{aligned}
\tag{6}
$$

Irrotational electromagnetic fields in Ω can be represented as gradient-fields of scalar potentials according to Poincare's lemma. Within the context of the FIT one deals with electric grid voltages allocated on the cell edges. To represent these as difference of two nodal potential values, discrete potential values ϕ_l are allocated at the grid mesh points $N_l \in G$, such that the relation $\phi_{l+1} - \phi_l = \hat{e}_k$ holds for every oriented edge L_k of *G,* which connects node N_l to N_{l+1}.

Collecting these discrete potential values and their relations into a vector $\mathbf{\Phi} = \{\phi_i\}_{i:N_i \in G}$ for the cell complex *G,* one obtains the grid vector relation

$$
\hat{\mathbf{e}} = -\mathbf{G}\mathbf{\Phi}, \tag{7}
$$

where the discrete *gradient* matrix $\mathbf{G}$ indeed is the negative transpose of the dual discrete divergence operator.

Analogously, the same procedure can be applied using magnetic scalar potentials on the cell vertices of the dual cell complex to derive the discrete gradient matrix $\tilde{\mathbf{G}}$ for the irrotational dual magnetic grid voltages with $\hat{\mathbf{h}} = -\tilde{\mathbf{G}}\mathbf{\Psi}$, where $\mathbf{\Psi}$ is a magnetic scalar nodal potential vector.

A discretization has been performed for Maxwell's equations only in the form of a restriction of the equations (1) to a finite number of areas A and volumes V and the computational domain has been artificially bounded. The information that these equations hold is only about integral state variables which are either allocated on points (potentials), edges (voltages), surfaces (fluxes) or the cell volume (charges). The resulting matrix equations are an exact representation of Maxwell's Equations in integral form on a grid doublet $\{G, \tilde{G}\}$, where so far no relations between the cell complexes *G* and $\tilde{G}$ have been defined.

The Constitutive Material Equations

For the calculation of solutions, the four equations in (1) have to be coupled with the constitutive material relations. In order to be able to couple flux and voltage degrees of freedom, additional constraints concerning the relation between *G* and $\tilde{G}$ have to be specified, including especifically the *duality conditions*, i.e. the identical orientation and labeling of the intersecting edges and facets:

- Each edge $L_m \in G$ intersects one (and only one) facet $\tilde{A}_m \in \tilde{G}$, the same holds vice versa for $\tilde{L}_p \in \tilde{G}$ and $A_p \in G$.
- Each node $N_l \in G$ is located in exactly one dual cell volume $\tilde{V}_l \in \tilde{G}$, the same holds vice versa for $\tilde{N}_n \in \tilde{G}$ and $V_n \in G$.

- The unified volume of all grid cells $V_n \in G$ equals the union of all dual grid cells $\tilde{V}_l \in \tilde{G}$, which represents the computaional domain.

A reordering and, if necessary, a reversal of the predefined orientations of the edge and facet degrees of freedom is performed in the component vectors with two aims: the numbering indices of each pair of intersecting geometrical objects are required to be identical and the orientations of the edges of G and orientations of their intersecting facets of $\tilde{G}$, i.e., their flux directions, should be identical. The resulting *duality* of the cell complex pair $\{G, \tilde{G}\}$ is responsible for essential properties of the final discrete formulation. Starting directly with (1), the topic of orientation reduces in practical applications of the FIT to the obligation "to get the signs right".

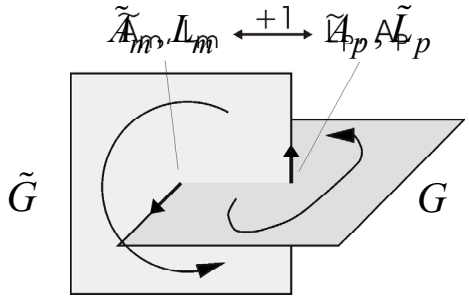

Fig.3 The edges $L_m \in G$ and $\tilde{L}_p \in \tilde{G}$ each contribute with a factor +1 to the summation of the local curls, i.e., $C_{pm} = +1$ and $\tilde{C}_{mp} = +1$. Here, the intersecting edges and facets have the same enumeration index and the same orientation, following a right-screw rule for the facet orientation and the orientation of the corresponding curls.

Within the classical FIT, the grid doublet $\{G, \tilde{G}\}$ consists of a dual-orthogonal grid pair, i.e., it represents a *Delaunay-Voronoi* grid pair, where each edge $L_m \in G$ *perpendicularly* intersects with only one facet $\tilde{A}_m \in \tilde{G}$, and vice versa. This allows to couple the related edge and dual facet degrees of freedom in a one-to-one relation as e.g. depicted in Fig. 4, where the electric current $\hat{\hat{j}}_m$ through a facet $\tilde{A}_m \in \tilde{G}$ is coupled to a grid voltage $\hat{e}_m$ along $L_m \in G$ to yield the discrete version of Ohm's law

$$\frac{\hat{\hat{j}}_m}{|\tilde{A}_m|} = \bar{\kappa}_m \cdot \frac{\hat{e}_m}{|L_m|}, \tag{8}$$

featuring the edge length $|L_m|$ and the area of the dual facet $|\tilde{A}_m|$ and an averaged electric conductivity $\bar{\kappa}_m$. This approach is used in FIT mainly for reasons of numerical efficiency: It results in diagonal material matrices and the required coordinate axis parallel Delaunay-Voronoi cell complex pairs are easy to generate, which is typically does not hold for more general dual-orthogonal cell complexes.

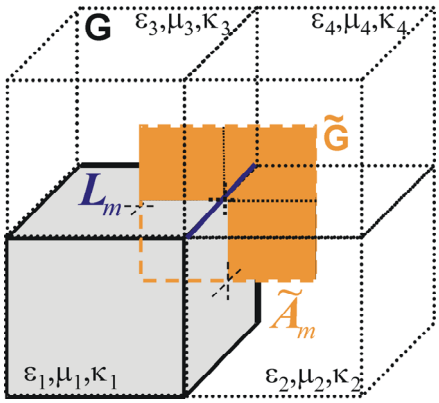

Fig.4 The coupling of the degrees of freedom on G and $\tilde{G}$ is performed in the constitutive material equations. The electric grid voltages $\hat{e}_m$ allocated onto the edge $L_m \in G$ is coupled to a facet flux $\hat{\hat{j}}_m$ allocated on a dual facet $\tilde{A}_m \in \tilde{G}$. This involves an averaging process of the four cell conductivities $\kappa_1,\ldots,\kappa_4$ to a value $\bar{\kappa}$. The coupling constitutive equation here is $\hat{\hat{j}}_m / |\tilde{A}_m| = \bar{\kappa} \cdot \hat{e}_m / |L_m|$,

In [19] this principle was proposed for the definition of the constitutive material equations under the assumption of local uniformity of the fields involved, thus eliminating the need to resort to the state variables of the differential formulation of Maxwell's equations for a point-wise definition. In this presentation, though, no averaging for the material parameters was introduced. It should be noted, however, that the uniformity assumption always requires a homogeneous material distribution. At interfaces of different materials, as e.g. depicted in Fig. 4 with a situation of different cell conductivity values $\kappa_1 \neq \kappa_2 = \kappa_3 = \kappa_4$ for the intersection point of L_m and $\tilde{A}_m$, always a non-uniform situation will occur due to the discontinuity of the flux and current density vectors. This argument requires averaging of the material distribution of the cells in G to keep up the argument of uniformity.

Nonetheless, keeping in mind the character of the constitutive material relations as a result of an approximative physical modeling of the actual material behavior, this approach can be considered to be as valid as the well-known point-wise introduction of the constitutive material equations.

When using the approach in (8) within a computational scheme, where the uniformity assumption is supposedly false, the availability of continuous integrand fields, however, allows to use Taylor's expansion for estimating the order of the truncation error convergence of this approach.

With the definition of a maximum length $h = \max\{|L|, |\tilde{L}| \;\big|\; L \in G, \tilde{L} \in \tilde{G}\}$ of the grid cell edges for e.g. a Cartesian grid doublet $\{G, \tilde{G}\}$, an entry of the diagonal material matrix of conductivities is derived by

$$\frac{\hat{\hat{j}}_m}{\hat{e}_m} = \frac{\int_{\tilde{A}_m} \vec{J} \cdot d\vec{A}}{\int_{L_m} \vec{E} \cdot d\vec{s}} = \frac{\int_{\tilde{A}_m} \kappa \cdot \mathrm{mean}(\vec{E}) \cdot d\vec{A} + O(h^{l_1})}{\int_{L_m} \mathrm{mean}(\vec{E}) \cdot d\vec{s} + O(h^{l_2})} = \frac{\int_{\tilde{A}_m} \kappa \cdot d\vec{A}}{\int_{L_m} d\vec{s}} + O(h^{l_1 - l_1}) \quad \approx \quad \{\mathbf{M}_\kappa\}_{m,m} := \bar{\kappa} \cdot \frac{\int_{\tilde{A}_m} d\vec{A}}{\int_{L_m} d\vec{s}}$$

$$\Rightarrow \hat{\hat{j}}_m - \{\mathbf{M}_\kappa\}_{m,m} \hat{e}_m = O(h^{l_1}) \tag{9}$$

for the corresponding coupling of a grid voltage $\hat{e}_m$ along the edge $L_m \in G$ and a facet flux $\hat{\tilde{j}}_m$ through the facet $\tilde{A}_m \in \tilde{G}$. The specific construction of the constitutive relation in (9) takes into account the continuity of the electric field intensity vector along the edge L_m which holds even for different material distributions within the adjacent cells. The truncation error exponents l_1, l_2 have values $l_1 = 3, l_2 = 3$ in the case of non-uniform grid spacing or if the cell conductivities κ_i differ in their value, otherwise $l_1 = 4, l_2 = 3$ holds. The material relations for the permittivities are derived analogously to (8) and (9), whereas the coupling of the magnetic fluxes and the magnetic voltages involves the consideration of the averaged reluctivities ν_i along the dual edges $\tilde{L}_m \in \tilde{G}$ [24]. This coupling approach differs from the facet oriented coupling in (9) for the electric grid voltages and the facet currents, since it takes into account the normal continuity of the magnetic flux density vectors.

With the one-to-one correspondence of the dual facets and their orthogonally penetrating edges the construction principle in (8) results in the discrete material matrix relations

$$\hat{\tilde{\mathbf{d}}} = \mathbf{M}_\varepsilon \hat{\mathbf{e}} + \hat{\tilde{\mathbf{p}}}_E, \quad \hat{\tilde{\mathbf{j}}} = \mathbf{M}_\kappa \hat{\mathbf{e}}, \quad \hat{\mathbf{h}} = \mathbf{M}_{\nu_0} \hat{\tilde{\mathbf{b}}} - \hat{\mathbf{p}}_m \tag{10}$$

featuring only diagonal matrices for materials modeled with diagonal or isotropic material tensors [24]. Here $\mathbf{M}_\varepsilon$ is the permittivity matrix, $\mathbf{M}_\kappa$ is the (usually singular) matrix of electric conductivities, $\mathbf{M}_{\nu_0}$ the matrix of magnetic reluctivities and $\hat{\tilde{\mathbf{p}}}_E$ and $\hat{\mathbf{p}}_M$ arise from electric and magnetic polarizations. Due to the specific requirements on the orientations of the edges and their dual facet these matrices are diagonal, symmetric and positive-definite (or semi-definite in the case of $\mathbf{M}_\kappa$, since electromagnetic problems typically feature also nonconductive regions.) Whereas the integral state variables of the FIT arise from the exact reformulation of the Maxwell equations (1), the constitutive material matrices of the FIT contain the averaged information of the material and the grid dimensions. Any discretization error of the method is found to be located in the discrete constitutive material equations [24].

A critical issue for practical calculations is the generation of the dual-orthogonal cell complex pair $\{G, \tilde{G}\}$ for a given material distribution of an electromagnetics problem. Coordinate axis parallel orthogonal grids, where each cell is filled with only one material as shown in Fig. 4 will result in the problem of staircase approximations of curved boundary surfaces. To overcome this problem, sophisticated *partially filled cell* techniques have been developed in the context of the FIT for an improved geometry approximation and material averaging inside the cells such as the *triangular subvolume* technique [22] and the computationally more expensive *tetrahedral subvolume* technique, which goes back to nodal-oriented Finite Difference formulations in [11].

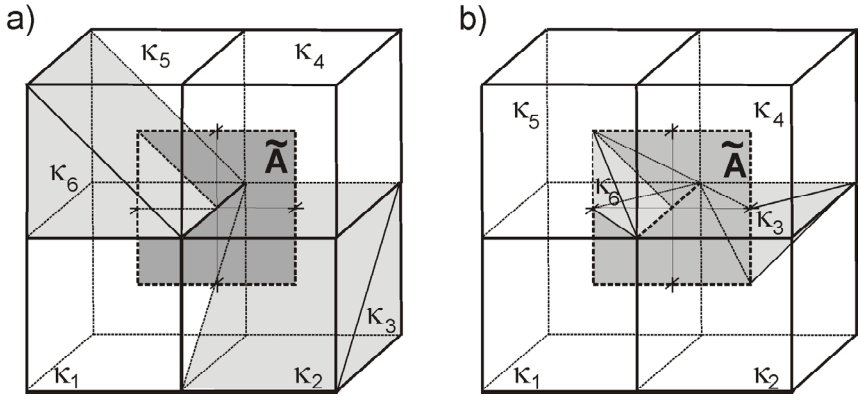

Fig.5 Both figures show the averaging process of the classical FIT for the cell material properties for the dual cell facet $\tilde{A} \in \tilde{G}$ in the presence of cell subvolumes, which are filled with materials of different electrical conductivities $\kappa_1,\ldots,\kappa_6$. Fig. 5a) depicts the situation for triangularly filled cells, whereas Fig. 5b) features tetrahedral cell subvolumes. In both cases the averaging process for the conductivity on $\tilde{A}$ yields $\int_{\tilde{A}} \kappa \cdot d\vec{A} / |\tilde{A}| \approx \bar{\kappa} := 1/|\tilde{A}| \cdot \sum_{i=1,i\neq 3}^{6} \kappa_i \cdot |\tilde{A}_i|$, where $|\tilde{A}_i|$ is the area of $\tilde{A}$ cutting the subcell with conductivity κ_i.

For the more sophisticated *Conformal Finite Integration Technique* (CFIT) [6] arbitrarily filled cells allow an improved approximation of curved surface material interfaces independent of the grid topology (see Fig. 6), using a technique similar to the one proposed in [26] for so-called *Conformal FDTD* schemes. This approach allows to use computationally efficient and easy to generate structured Cartesian, i.e., orthogonal, grid complexes, while at the same time significantly reducing the geometry approximation error of the method when compared to the classical approach.

In the past, the construction of material relations using dual-orthogonal grid complexes for the FIT has been performed for tensor-product-type grids in 2D and 3D [24], as well as for general 2D triangular dual orthogonal cells complexes [20]. While this common choice was mainly motivated by reasons of the resulting higher computational efficiency, it might have been cause for an often-encountered misconception, that the FIT might be restricted to these grid types only.
For topologically regular, non-orthogonal cell complexes some of these computational efficiency advantages of coordinate parallel grid types may be maintained, but they also allow for a better approximation of curved material interfaces within computational schemes (See Fig. 6).
For these non-orthogonal cell complexes, however, the one-to-one correspondence of cell edges and dual facets will not necessarily coincide with a one-to-one relation of the corresponding voltage and flux degrees of freedom, since an additional interpolation process is required for the coupling of G and $\tilde{G}$, which involves additional flux degrees of freedom. The resulting material matrices of the *Non-orthogonal Finite Integration Technique* (NFIT) are symmetric, but no longer diagonal [13], [14], [15].

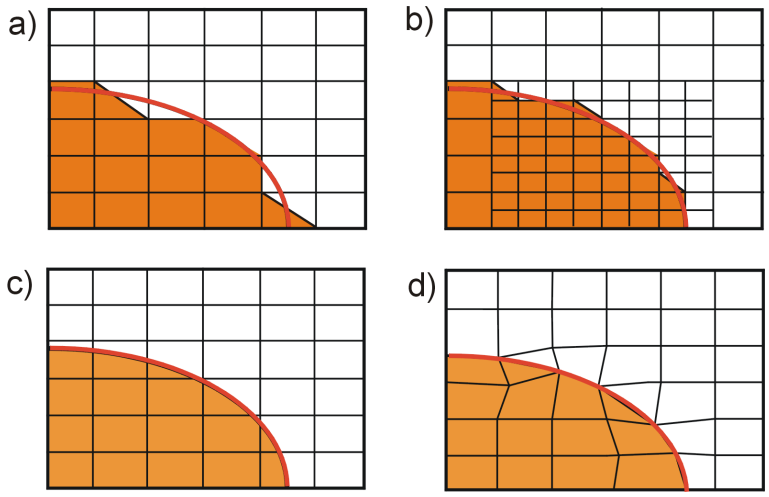

Fig.6 Techniques to geometrically model curved material surfaces within the framework of the FIT: Fig. a) depicts the use of triangularly filled cells used in the standard FIT. Fig. 6b) introduces a topologically non-regular local grid refinement with a *subgridding technique*. In 6c) the refined partially filled cell technique of the *Conformal FIT* yields a smaller geometric volume error than the standard FIT of Fig. 6a), while maintaining the same orthogonal grid. Fig. 6d) shows an approximation of the surface for the Non-orthogonal FIT (NFIT) using a topologically regular grid.

A treatment of dispersive, gyrotropic or nonlinear material properties within the FI Technique requires numerical schemes which concentrate on suitable modifications of the material matrices. Typically these numerical schemes, however, require to consider averaged components of field intensity or flux density vectors, which contradicts the proposed idea in [19] to completely discard these state-variables of the differential formulation in the electromagnetics equations.

The basic idea for the derivation of the material matrices in FIT is originally motivated by physical considerations, including the consideration of the continuity requirements of the fields involved. The derivation of the material matrices for Whitney-Finite-Element schemes originates from a variational formulation. It involves the element-by-element quadrature of the Whitney edge shape functions $\mathbf{w}_{ei}$ [2], given here e.g. for the *mass matrix* of conductivities on a tetrahedral element grid with

$$\left\{\mathbf{M}_{\kappa}\right\}_{i,j} := \int_{A} \kappa \mathbf{w}_{ei} \cdot \mathbf{w}_{e_j} d\vec{A} \,. \tag{11}$$

As in the NFIT case, the resultant material matrices of the discrete constitutive equations will be symmetric (and eventually badly conditioned for obtuse edge angles) and non-diagonal, which prohibits the construction of an efficient leapfrog time integration scheme, since the inverse of $\mathbf{M}_{\varepsilon}$ is not available as sparse matrix.

Algebraic Properties

Essential properties of the discrete representation of Maxwell's equations arise from the relations

$$\mathbf{SC} = 0, \quad \tilde{\mathbf{S}}\tilde{\mathbf{C}} = 0, \tag{12}$$

which can be directly derived from the grid topology [3], [5].
It allows to prove charge conservation in the MGE and the derivation of the continuity equation directly from the original MGE with

$$\tilde{\mathbf{S}} \cdot \tilde{\mathbf{C}} \hat{\mathbf{h}} = \tilde{\mathbf{S}} \cdot \left(\frac{\mathrm{d}}{\mathrm{dt}} \hat{\tilde{\mathbf{d}}} + \hat{\tilde{\mathbf{j}}} \right) \Rightarrow \frac{\mathrm{d}}{\mathrm{dt}} \mathbf{q}_E + \tilde{\mathbf{S}} \hat{\tilde{\mathbf{j}}} = 0 \tag{13}$$

Due to the additional constraints on the orientation and the numbering on the cell edges of G and its dual cell facets $\tilde{G}$, we receive the *duality relation* of the two curl-operators of G and $\tilde{G}$ with

$$\mathbf{C} = \tilde{\mathbf{C}}^T . \tag{14}$$

This property is visualized in Fig. 3 for a local situation.

The generalized symmetry in (14) and the symmetry of the material matrices in (10) results in a real-valued, non-negative eigenvalue spectrum of the curlcurl-operator $\mathbf{M}_\varepsilon^{-1} \tilde{\mathbf{C}} \mathbf{M}_\nu \mathbf{C} \mathbf{M}_\varepsilon^{-1}$. With the identities (12), this allows to prove that all static and dynamic eigensolutions represent a complete, orthogonal set of basis vectors for all grid vectors of G representing the solutions of the discrete MGEs [23], [5]. This result is a sufficient prerequisite for the stability of numerical time integration schemes. In addition, the duality relation (14) allows to prove energy conservation for the time continuous MGE [18].

The duality relation (14), the relations (12) and the identities $\mathbf{G} = -\tilde{\mathbf{S}}^T$, $\tilde{\mathbf{G}} = -\mathbf{S}^T$, which can be shown to hold for the discrete gradient operator, yield the matrix operator identities

$$\mathbf{C}\mathbf{G} = \mathbf{0}, \quad \tilde{\mathbf{C}}\tilde{\mathbf{G}} = \mathbf{0}, \tag{15}$$

which both state that discrete gradient fields on G and $\tilde{G}$ are irrotational.
The discrete algebraic identities (12) and (15) correspond to the continuous relations $\operatorname{div} \operatorname{curl} = 0$ and $\operatorname{curl} \operatorname{grad} = 0$, respectively.

Time Continuous or Time Discrete?

In [19] Tonti suggested to consider a time integrated reformulation of (1), thus getting rid of the time derivatives in (1). This approach results in a dual staggered cell complex in time for the structure of his *global* (=integral) electrodynamic state-variables. In the FIT approach this duality in time is typically not considered, rendering the Maxwell Grid Equations as a set of time continuous ordinary differential equations. The FIT qualifies as a *method of lines* [8] following a *first-space-then-time* semi-discretization approach, which was specifically criticized in [10] as a shortcoming of the FIT. Following the idea of a staggered discrete structure in time as proposed in [19], Yee-like leapfrog schemes [25], such as e.g. the *Finite Integration Time Domain* scheme [24] given with

$$\begin{aligned} \hat{\tilde{\mathbf{b}}}^{(n+1)} &:= \hat{\tilde{\mathbf{b}}}^{(n)} + \Delta t \cdot \mathbf{C} \mathbf{M}_\varepsilon^{-1} \hat{\tilde{\mathbf{d}}}^{(n+1/2)}, \\ \hat{\tilde{\mathbf{d}}}^{(n+3/2)} &:= \hat{\tilde{\mathbf{d}}}^{(n+1/2)} + \Delta t \cdot \left(\tilde{\mathbf{C}} \mathbf{M}_\nu \hat{\tilde{\mathbf{b}}}^{(n+1)} - \hat{\tilde{\mathbf{j}}}^{(n+1)} \right), \end{aligned} \tag{16}$$

seem to become the natural algorithm of choice to treat electrodynamic problems. Contemporary leapfrog FDTD/FITD schemes with the extensions such *conformal path techniques* (similar to CFIT) and *perfectly matched layer (PML) absorbing boundary* conditions [1] indeed are powerful tools for numerical electromagnetic field simulations. The

leapfrog time integration scheme, however, as a mere numerical time integration scheme among others, has severe disadvantages in the restrictive Courant-Friedrich-Levi (CFL) stability limit on the length of the time steps [18], in the existence of a dispersion error and in simulations of highly resonant structures. The first restriction is especially a problem for quasistatic field simulations, where often an unfeasible high amount of time steps results from the stability condition. Here some ingenious workarounds have been devised, such as e.g. a deliberate down-scaling the speed of light suggested in [9], allowing for larger stable FDTD time steps, or a scaling of the excitation frequencies and material properties for quasistatic field simulations in biological tissue [7]. These ideas seem to successfully follow the reasoning: If your favorite tool is a hammer, make sure your problem looks like a nail.

A similar situation may occur, if grid cells of very small diameter are local part of the cell complex, where the CFL condition will restrict the maximum time step length for the whole grid. This situation, however, may be overcome with the application of *local time stepping* techniques within the leapfrog schemes or with more recent FDTD variants using the alternating direction implicit method [12], [27]. Transient leapfrog simulations of highly resonant structures typically will also result in very long simulation times. For such applications frequency domain simulations should be preferred.

The dispersion error of the standard leapfrog scheme limits the number of wave lengths that can be calculated and thus it limits the size of possible calculation domains to only a few wavelengths ($<30\ \lambda$) in each space direction. This actually prohibits to use this scheme for HF-simulations, where the computational domain is large, e.g. for a radar antenna radiation problem including the whole airport.

Hence, the discrete-in-space-continuous-in-time MGEs of FIT are more versatile with respect to varying computational needs: They allow to treat an originally physical problem as a mere set of ordinary differential equations, differential-algebraic equations for transient quasistatic problems and merely algebraic equations for stationary problems [4], [24]. Thus decoupled from a physical background, a well-stocked repository of numerical methods is available for these matrix equations. As a result, computational simulations can be performed for the whole frequency range of electrodynamic applications [24]. Even for HF-field simulations, where leapfrog schemes excel for computational domains with small dimensions, alternative time integration schemes of higher order [16] or absolutely stable and conservative, implicit formulations such as Newmark's Averaged Acceleration method are available [4].

For the solution of Maxwell's equations (1), which in the end ultimately motivates their reformulation with the FIT into the Maxwell-Grid-Equations, the time continuous formulation of both (1) and (6) yields higher flexibility with the numerical solution methods. At the same time, a discrete formulation of (1) which is also of integral nature in time identical to the one proposed in [19] is always just one additional time integration of the equations (6) away.

Conclusion

The Finite Integration Technique reformulates the original continuous Maxwell equations into a set of time continuous algebro-differential equations, the Maxwell-Grid-Equations. With integral quantities such as magnetic and electric fluxes and voltages as the state variables of the FIT, possible errors of the discrete formulation are located only in the discrete constitutive material equations. The equations of the contemporary FIT, with its various extensions not only allow to devise efficient numerical schemes for the whole range of electromagnetic problems, but they also represent the first discrete field theory in the sense of [19]. The FI Technique is explained to deliberately omit the introduction of an additional integral structure in time for its state-variables for reasons of greater flexibility with the numerical schemes to be applied.

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