

Radial cylindrical multipole expansion of the free-space Green function for the three-dimensional Laplace equation

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Free-space Green function in cylindrical coordinates has multipole expansion containing improper integral of modified Bessel functions. In order to make multipole expansion convenient for analytical calculations, it should contain multiple sums instead of integrals. In this paper, we prove radial multipole expansion of free-space Green function in cylindrical coordinates, which does not contain any integrals. Presented expansion contains triple sum of Gaussian hypergeometric functions, powers and trigonometric functions. Radial multipole expansion can be employed to analytically calculate electrostatic and magnetostatic potentials, fields, forces and inductances in cylindrically symmetric systems.

Index Terms—free-space Green function, hypergeometric functions, modified Bessel functions, multipole expansion

I. INTRODUCTION

FREE-SPACE Green function for the three-dimensional Laplace equation can be expanded in the complete basis of some other real functions defined on $\mathbb{R}^3$. In cylindrical coordinates, Bessel functions form uncountable complete basis in radial direction and free-space Green function has following expansion in the form of an integral representation involving modified Bessel functions of the first and second kind and m -th order denoted by $I_m(x)$ and $K_m(x)$, respectively [1]

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi - \phi')} \int_0^{\infty} I_m(k\rho_{<}) K_m(k\rho_{>}) \times \cos[k(z - z')] dk. \quad (1)$$

Here $\mathbf{r}$ denotes position vector in $\mathbb{R}^3$, (ρ, ϕ, z) are cylindrical coordinates associated with position vector $\mathbf{r}$ and $\rho_{>} = \max\{\rho, \rho'\}$, $\rho_{<} = \min\{\rho, \rho'\}$.

In order to efficiently use multipole expansion in analytical calculations, it is convenient to have multiple sums instead of integrals, since those integrals could be complicated to evaluate. One way to eliminate integral from (1) is to use the fact that integral with respect to k has analytical solution in terms of toroidal functions [2]. This leads to expansion of Green function in toroidal functions with single sum over m which has been extensively studied in [3]-[6]. Although toroidal multipole expansion does not contain integrals, it is impossible to separate coordinates of $\mathbf{r}$ from coordinates of $\mathbf{r}'$ making computation of multipole moments cumbersome.

Cylindrical multipole expansion convenient for application should not contain any integrals, and coordinates of $\mathbf{r}$ need to be separated from coordinates of $\mathbf{r}'$. One such expansion

of the free-space Green function has been derived from the integral representation of the free-space Green function involving Bessel functions [7]. We call expansion derived in [7] axial multipole expansion. In this paper, we derive another multipole expansion satisfying mentioned conditions called radial multipole expansion. Radial multipole expansion can be interpreted as dual result of the axial multipole expansion in sense that analogous techniques are used in the proofs of both expansions, and every result that holds for axial multipole expansion involving axial coordinates has its dual result that holds for radial expansion involving radial coordinates. For example, condition of validity for the radial multipole expansion is $\rho > \rho'$, while the validity of the axial multipole expansion is determined by the condition $z > z'$.

II. RADIAL MULTIPOLE EXPANSION OF FREE-SPACE GREEN FUNCTION

In this section, we formally state and prove radial multipole expansion of free-space Green function.

Theorem 1. *Let $\mathbf{r} = (\rho, \phi, z) \in (\mathbb{R}^+ \cup \{0\}) \times [0, 2\pi) \times \mathbb{R}$ and $\mathbf{r}' = (\rho', \phi', z') \in (\mathbb{R}^+ \cup \{0\}) \times [0, 2\pi) \times \mathbb{R}$ be arbitrary position vectors of two points in $\mathbb{R}^3$ such that $\rho > \rho'$ and $\rho > |z'|$. Then, radial multipole expansion (3) holds true.*

Proof. We are starting the proof from (1). Since $\rho > \rho'$ and $I_{-m}(x) = I_m(x)$, $K_{-m}(x) = K_m(x)$ for all $m \in \mathbb{N}_0$, $x \in \mathbb{R}$, we can rewrite expression (1) as follows

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi - \phi')} \int_0^{\infty} I_{|m|}(k\rho') K_{|m|}(k\rho) \times \cos[k(z - z')] dk. \quad (2)$$

$$\begin{aligned} \frac{1}{|\mathbf{r} - \mathbf{r}'|} = & \sum_{m=-\infty}^{\infty} \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (2l + 2|m| + 2n)! (2l + 2n)! e^{im(\phi - \phi')}}{(2n)! l! (l + |m|)! (l + |m| + n)! (l + n)!} \left\{ F\left(\frac{2l + 2|m| + 2n + 1}{2}, \frac{2l + 2n + 1}{2}; \frac{1}{2}; -\frac{z^2}{\rho^2}\right) \right. \\ & \left. + \frac{zz'(2l + 2|m| + 2n + 1)(2l + 2n + 1)}{(2n + 1)\rho^2} F\left(\frac{2l + 2|m| + 2n + 3}{2}, \frac{2l + 2n + 3}{2}; \frac{3}{2}; -\frac{z^2}{\rho^2}\right) \right\} \frac{z'^{2n} \rho'^{2l + |m|}}{2^{4l + 2|m| + 2n} \rho^{2l + |m| + 2n + 1}}. \end{aligned} \quad (3)$$

Modified Bessel function of the first kind $I_{|m|}(x)$ has convergent Taylor series expansion for all $x \in \mathbb{R}$. By using Taylor series expansion of $I_{|m|}(x)$ at $x = 0$ which converges for all $x \in \mathbb{R}$, we have

$$\begin{aligned} & \int_0^\infty K_{|m|}(k\rho) I_{|m|}(k\rho') \cos[k(z - z')] dk \\ &= \int_0^\infty K_{|m|}(k\rho) \cos[k(z - z')] \sum_{l=0}^\infty \frac{\left(\frac{k\rho'}{2}\right)^{2l+|m|}}{l!(l+|m|)!} dk. \end{aligned}$$

By using dominated convergence theorem, it can be shown that summation over l and integration order can be interchanged if $\rho > \rho'$. Since this is assumption of the theorem, we indeed can interchange order of summation over l and integration to obtain

$$\begin{aligned} & \int_0^\infty K_{|m|}(k\rho) I_{|m|}(k\rho') \cos[k(z - z')] dk \\ &= \sum_{l=0}^\infty \int_0^\infty K_{|m|}(k\rho) \cos[k(z - z')] \frac{\left(\frac{k\rho'}{2}\right)^{2l+|m|}}{l!(l+|m|)!} dk. \end{aligned} \quad (4)$$

By using addition formula for cosine and Taylor series expansions for $\sin x$ and $\cos x$ converging for all $x \in \mathbb{R}$, we have

$$\begin{aligned} \cos[k(z - z')] &= \cos(kz) \sum_{n=0}^\infty \frac{(-1)^n (kz')^{2n}}{(2n)!} \\ &+ \sin(kz) \sum_{n=0}^\infty \frac{(-1)^n (kz')^{2n+1}}{(2n+1)!}. \end{aligned} \quad (5)$$

Substitution of (5) in (4) yields

$$\begin{aligned} & \int_0^\infty K_{|m|}(k\rho) I_{|m|}(k\rho') \cos[k(z - z')] dk = \\ & \sum_{l=0}^\infty \int_0^\infty K_{|m|}(k\rho) \frac{\left(\frac{k\rho'}{2}\right)^{2l+|m|}}{l!(l+|m|)!} \left\{ \cos(kz) \sum_{n=0}^\infty \frac{(-1)^n (kz')^{2n}}{(2n)!} \right. \\ & \left. + \sin(kz) \sum_{n=0}^\infty \frac{(-1)^n (kz')^{2n+1}}{(2n+1)!} \right\} dk. \end{aligned}$$

As in previous case, here we can interchange summation over n and integration order if $\rho > |z'|$, which can be justified by means of dominated convergence theorem. Since $\rho > |z'|$ is assumption of the theorem, we can write

$$\begin{aligned} & \int_0^\infty K_{|m|}(k\rho) I_{|m|}(k\rho') \cos[k(z - z')] dk = \\ & \sum_{l=0}^\infty \sum_{n=0}^\infty \frac{(-1)^n z'^{2n}}{(2n)! l! (l+|m|)!} \left(\frac{\rho'}{2}\right)^{2l+|m|} \int_0^\infty K_{|m|}(k\rho) \\ & \times k^{2l+|m|} \left\{ \cos(kz) k^{2n} + \frac{z'}{2n+1} \sin(kz) k^{2n+1} \right\} dk. \end{aligned} \quad (6)$$

Let us introduce following symbol defined for all $(l, m, n, p) \in \mathbb{N}_0 \times \mathbb{Z} \times \mathbb{N}_0 \times \{1, 3\}$

$$\begin{aligned} g_{(l,m,n,p)} &:= \frac{2^{2n-1} \Gamma\left(\frac{2l+2|m|+2n+1}{2}\right) \Gamma\left(\frac{2l+2n+1}{2}\right)}{\rho^{2l+|m|+2n+1}} \\ & \times F\left(\frac{2l+2|m|+2n+1}{2}, \frac{2l+2n+1}{2}; \frac{p}{2}; -\frac{z^2}{\rho^2}\right). \end{aligned}$$

For $\rho > 0$, $z \in \mathbb{R}$, integrals with respect to k from expression (6) can be found in [2]

$$\begin{aligned} & \int_0^\infty k^{2l+|m|+2n+1} K_{|m|}(k\rho) \sin(kz) dk = 2^{2l+|m|} z g_{(l,m,n+1,3)} \\ & \int_0^\infty k^{2l+|m|+2n} K_{|m|}(k\rho) \cos(kz) dk = 2^{2l+|m|} g_{(l,m,n,1)}. \end{aligned}$$

By substituting these integrals in (6), we have

$$\begin{aligned} & \int_0^\infty K_{|m|}(k\rho) I_{|m|}(k\rho') \cos[k(z - z')] dk = \\ & \sum_{l=0}^\infty \sum_{n=0}^\infty \frac{(-1)^n \rho'^{2l+|m|} z'^{2n}}{(2n)! l! (l+|m|)!} \left(g_{(l,m,n,1)} + \frac{z z'}{2n+1} g_{(l,m,n+1,3)} \right). \end{aligned} \quad (7)$$

Note that $g_{(l,m,n,p)}$ has following equivalent form obtained by using formula for gamma function of half integer

$$\begin{aligned} g_{(l,m,n,p)} &= \frac{\pi(2l+2|m|+2n)!(2l+2n)!}{2^{4l+2|m|+2n+1} \rho^{2l+|m|+2n+1} (l+|m|+n)! (l+n)!} \\ & \times F\left(\frac{2l+2|m|+2n+1}{2}, \frac{2l+2n+1}{2}; \frac{p}{2}; -\frac{z^2}{\rho^2}\right). \end{aligned}$$

Substitution of previous expression for $g_{(l,m,n,p)}$ in (7), and then (7) in (2) yields expansion (3). $\square$

III. CONCLUSION

This paper contains the mathematical proof of radial multipole expansion of free-space Green function with all relevant steps included. However, in order to make proof mathematically complete, interchange of summation and integration in (4) and (5) should be formally justified. This can be done by using dominated convergence theorem and it would be included in extended version of the paper.

Additionally, in order to fully investigate mathematical properties of obtained multipole expansion, sufficient conditions for convergence and divergence of triple sum (3) would be investigated in extended version of the paper.

Finally, presented radial multipole expansion of free-space Green function will be applied for analytical calculation of mutual inductance of two coaxial thick coils with rectangular cross section. Since axial multipole expansion has already been effectively applied to calculate mutual inductance of such coils, this can be viewed as dual approach for calculation of mutual inductance with respect to duality described in introduction.

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