

Finite Element Method with Unbounded Approach

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Abstract—This paper aims to describe the Finite Element Method (FEM) applied to boundary value problems governed by partial differential equations. However, domain truncation can lead to inaccuracies. To address this issue, a mapping method (Unbounded Approach) is applied to the Laplace's Equation in electrokinetic problems.

Index Terms—Boundary value problems, Electrokinetics, Electromagnetic fields, Finite element analysis, Laplace equations.

I. INTRODUCTION

For infinite domain problems, the truncation used in the FEM can distort the results. Using a larger mesh improves accuracy, but at the cost of significantly more processing time [1] and reduced visibility of the region of interest due to the smaller scale. To solve it, a mapping method for simulating a larger domain will be presented. The Unbounded Approach [2] allows each triangular element in the mapped regions of the finite element mesh to assume larger values for its vertex coordinates without increasing the size of the mesh or computational effort.

The method will be applied in two electrokinetic problems: two buried pipelines and a single grounding rod. These problems can be solved using two different mappings: for the double pipeline case, a simple 2D cartesian mapping will be used, while for the single grounding rod, an axisymmetric mapping will be applied. However, the equation to be solved is the same for both cases, the Laplace's Equation [3]:

$$\nabla \cdot (\sigma \nabla V) = 0 \quad (1)$$

In equation (1), σ represents the conductivity of the soil and V is the electric potential.

II. FINITE ELEMENT METHOD

Applying the Weighted Residual Method [4] in equation (1) leads to the following equation:

$$\int D_{xy}^T W_0^e \sigma^e D_{xy} V^e dx dy = 0 \quad (2)$$

In equation (2), W_0^e is the weight function and D_{xy}^T is the transpose of the differential operator ∇ , which is represented as $[\partial/\partial x \ \partial/\partial y]$. So, D_{xy} is a 2×1 matrix, and the conductivity σ^e is a 2×2 matrix in this formulation.

Solving equation (2) for each vertex of a particular element in the mesh results in a set of equations that can be expressed in matrix form. Subsequently, through an assembly process, the global matrix of the finite element mesh is constructed [5].

III. UNBOUNDED APPROACH AND TEST CASES

In order to explain the mapping method, it is necessary to show the mapping regions for both test cases. For the double pipeline case, this is illustrated in Fig. 1. The metallic structures are completely buried in Region I. Moreover, the pipeline on the left is set to an electric potential of 100 V, while the pipeline on the right is at -100 V.

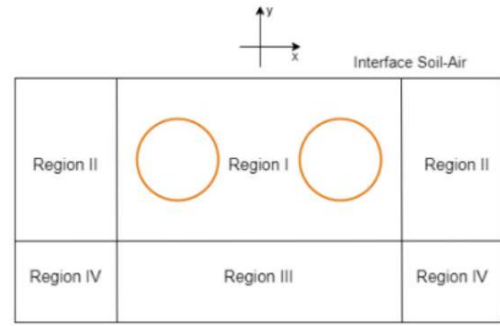


Fig. 1. Double pipeline scheme for the mapping method.

The mapped variables are (u, v) for both mappings. Table I shows the coordinate transformations for the 2D Cartesian mapping.

TABLE I
MAPPED VARIABLES FOR EACH REGION

Region	u	v
Region I	u	v
Region II	$\frac{x}{u}(x_u - x_m)$	y
Region III	x	$\frac{y_u(y_u - y_m)}{y - y_m}$
Region IV	$\frac{x_u(x_u - x_m)}{x - x_m}$	$\frac{y_u(y_u - y_m)}{y - y_m}$

The subscripts u and m indicate the unmapped and mapped limits, respectively. The single grounding rod scheme is presented in Fig. 2. The electric potential of the rod was fixed at 100 V.

Table I also applies to the axisymmetric mapping, with the substitution of (x, y) by (r, z) . The region of greatest interest in both mappings is Region I, where the double pipeline and the single grounding rod are located.

Applying the mapping method, equation (2) is modified to:

$$\int D_{uv}^T W_0^{e'} \sigma^{e'} D_{uv} V^{e'} du dv = 0 \quad (3)$$

In equation (3), $\sigma^{e'} = J^T \sigma J |J^{-1}|$, and J is the 2×2 Jacobian matrix, which varies depending on the mapping region [2].

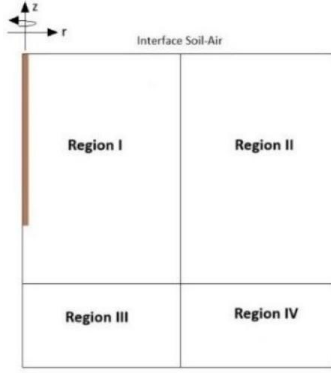


Fig. 2. Single grounding rod scheme for the mapping method.

IV. RESULTS

The results of the electric potential calculations for both test cases, with and without the mapping technique, are presented in Figs. 3–6.

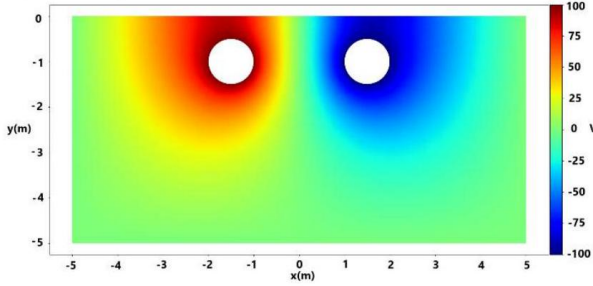


Fig. 3. Electric potential for the double pipeline case without the mapping method.

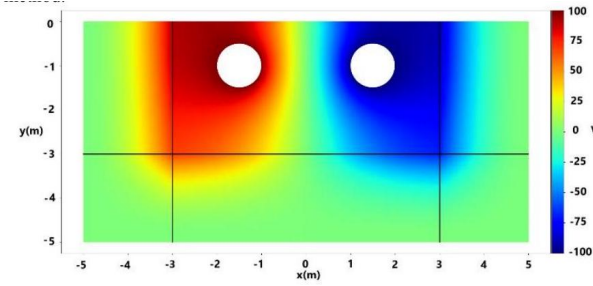


Fig. 4. Electric potential for the double pipeline case with the mapping method.

From the results shown, it is evident that the mapping technique increases the electric potential in Region I, addressing distortions caused by the proximity of boundary conditions simulating infinity to the region of interest.

Furthermore, the total simulation time remained nearly the same. For the double pipeline case, the mapped domain required only about 4% more time than the unmapped one. In the single grounding rod case, the mapped version was slightly faster, with a simulation time approximately 1

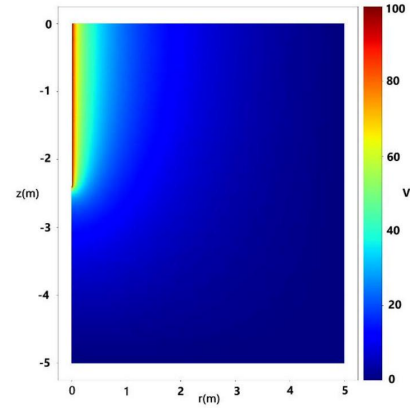


Fig. 5. Electric potential for the single grounding rod case without the mapping method.

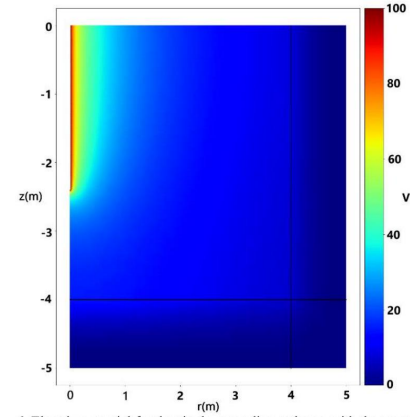


Fig. 6. Electric potential for the single grounding rod case with the mapping method.

To validate the approach, results from the mapped domain simulation of the grounding rod were compared to analytical solutions. The mapped domain showed better agreement in Region I than the unmapped counterpart.

V. CONCLUSION

The mapping method improves FEM accuracy in open domains without added computational cost. This work isolates its impact via quantitative comparisons on identical meshes and will include L2-norm error metrics and validation against an analytical solution.

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