

Incomplete Balancing Domain Decomposition Methods for Magnetostatic Problems Based on Discrete de Rham Sequences

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A Balancing Domain Decomposition (BDD) method is considered as the preconditioner of an iterative Domain Decomposition Method (DDM) for perturbed magnetostatic problems, where the magnetic vector potential is regarded as an unknown function. To reduce the number of the Degrees Of Freedom (DOF) of coarse spaces in BDD methods, Polytopal Element Methods (PEMs) are introduced, which are originally proposed by Di Pietro, et al. The PEMs can construct any order elements, satisfying Discrete de Rham Sequences. Owing to the introduction of PEMs and the result of BDD method originally proposed by Mandel, the condition number of coefficient matrices derived from the iterative DDM is evaluated. Therefore, the number of iterations of the iterative DDM can be kept even when the number of the subdomains becomes larger. Moreover, the approximate coarse space by PEMs can admit that each subdomain becomes more general polyhedron. The results in case of higher order PEMs are also shown.

Index Terms—magnetostatic, balancing domain decomposition, discrete de Rham sequence, perturbed magnetostatic problem

I. INTRODUCTION

A Balancing Domain Decomposition (BDD) method is originally proposed by Mandel [1], and is regarded as the preconditioner of linear iterative solvers for artificial boundary problems appearing in Domain Decomposition Methods (DDM; see [2]). There exist many literatures on the application of BDD into large-scale computational models derived from finite element methods for partial differential equations; see, for example in case of elastic problems, Ogino, et al. [3].

We have already computed ultra large-scale computational models derived from finite element methods for magnetic field problems by using iterative DDM; see, for example, Sugimoto, et al. [4]. However, in [4], we used the simplified diagonalization as the preconditioner. Therefore, to reduce computational costs, more effective preconditioner should be required, and it is valuable to apply BDD into linear iterative solvers for artificial boundary problems appearing in iterative DDM for magnetic field problems.

However, when we apply BDD methods directly into magnetostatic problems, the number of the Degrees Of Freedom (DOF) of coarse spaces in BDD methods is equal to the number of the subdomains times the number of the nodes of tetrahedron in each subdomains. Therefore, theoretically we may keep the condition number, but applicably we cannot expect to reduce the computational cost. To reduce the number of DOF of coarse spaces in BDD methods, Polytopal Element Methods (PEMs) are introduced, which are originally proposed by Di Pietro, et al. [8]. The PEMs can construct any order elements, satisfying Discrete de Rham sequences. Owing to the introduction of PEMs and the result of BDD method originally proposed by Mandel, the condition number of coefficient matrices derived from the iterative DDM is evaluated. Therefore, the number of iterations of the iterative DDM can be kept even when the number of the subdomains becomes larger. Moreover, the approximate coarse space by PEMs can admit that each subdomain becomes more general polyhedron. The results in case of higher order PEMs are also shown.

II. FORMULATION OF PERTURBED MAGNETOSTATIC PROBLEMS

Let Ω be a polytopal domain with its boundary Γ , and $\mathbf{n}$ an outward unit normal to Γ . Let $\mathbf{u}$ be a magnetic vector potential, $\mathbf{f}$ excitation current density, and μ magnetic permeability. Now, for $\varepsilon > 0$, let us consider a perturbed magnetostatic problems with unknown function $\mathbf{u}$; see [5]:

$$\begin{cases} \mathbf{rot}(\nu \mathbf{rot} \mathbf{u}) + \varepsilon \mathbf{u} = \mathbf{f} & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma. \end{cases} \quad (1a) \quad (1b)$$

By decomposing the domain Ω into tetrahedra and by approximating the magnetic vector potential by Nédélec element [6], we can obtain the finite element equation of (1)

$$(\mathbf{rot} \mathbf{u}_h, \mathbf{rot} \mathbf{v}_h) + \varepsilon(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h. \quad (2)$$

Here, V_h denotes finite dimensional space approximating the magnetic vector potential by using Nédélec element. Moreover, the finite element equation (2) is described as follows by the matrix form.

$$\mathbf{K}\mathbf{U} = \mathbf{F} \quad (3)$$

III. INTRODUCTION OF BDD

Now, we consider N -subdomains $\{\Omega^{(i)}; i = 1, 2, \dots, N\}$ and an artificial boundary γ_B satisfying

$$\bar{\Omega} = \bigcup_{i=1}^N \bar{\Omega}^{(i)}, \quad \Omega^{(i)} \cap \Omega^{(j)} = \emptyset \ (i \neq j), \quad \gamma_B := \bigcap_{i=1}^N \bar{\Omega}^{(i)}.$$

Then, the finite element equation (3) can be rewritten as follows:

$$\begin{pmatrix} \mathbf{K}_{11} & \cdots & \mathbf{O} & \mathbf{K}_{1B} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{O} & \cdots & \mathbf{K}_{NN} & \mathbf{K}_{NB} \\ \mathbf{K}_{1B}^T & \cdots & \mathbf{K}_{NB}^T & \mathbf{K}_{BB} \end{pmatrix} \begin{pmatrix} \mathbf{U}_1 \\ \vdots \\ \mathbf{U}_N \\ \mathbf{U}_B \end{pmatrix} = \begin{pmatrix} \mathbf{F}_1 \\ \vdots \\ \mathbf{F}_N \\ \mathbf{F}_B \end{pmatrix}, \quad (4)$$

where subscripts 1, 2, N , and B denote the degrees of freedoms (DOFs) corresponding to $\Omega^{(1)}$, $\Omega^{(2)}$, $\Omega^{(N)}$, and γ_B ,

respectively. Moreover, applying the static condensation into the linear system (4), the artificial boundary problem has been obtained, where U_B denotes the unknown vector:

$$SU_B = G, \quad (5)$$

where S, G denote the given vectors defined by

$$S := K_{BB} - \sum_{i=1}^N K_{iB}^T K_{ii}^\dagger K_{iB}, \quad G := F_B - \sum_{i=1}^N K_{iB}^T K_{ii}^\dagger F_i.$$

In the magnetostatic case, since the coefficient matrix S in (5) is symmetric and real valued, Conjugate Gradient (CG) method are used to solve (5). However, in the iterative DDM, S and G are not explicitly constructed, and the matrix-vector product appearing in the CG method is computed by solving the corresponding finite element equations in the subdomains. Therefore, we cannot use preconditioners that requires concrete values of the coefficient matrix, such as the incomplete Cholesky decomposition. In this case, the BDD method proposed by Mandel [1] is used.

When constructing the preconditioner matrix M for the BDD method, the key is to select the matrix Z_i satisfying

$$\text{Ker}(S_i) \subset \text{Range}(Z_i), \quad (7)$$

where the subscript i denotes the matrix corresponding to the subdomain $\Omega^{(i)}$, and S_i denotes the matrix S corresponding to the subdomain $\Omega^{(i)}$. Let Q_h be the finite element space by using $P1$ elements, and the function $q_h \in Q_h$ is satisfying $q_h = 0$ on Γ . The space Q_h is approximating the Lagrange multiplier appearing in the original mixed formulation; see [5] and [7]. In case of (3), the inclusion (7) become as follows:

$$\text{grad } Q_h = \text{Ker}(\text{rot } V_h) \approx \text{Ker}(S_i). \quad (8)$$

Therefore, as the same analogy in [1], theoretically we can evaluate the condition number as follow:

$$\text{Cond}(M^{-1}S) \leq C \left(1 + \log^2 \frac{H}{h} \right), \quad (9)$$

where H denotes the maximum value of the diameter of the subdomains, h denotes the maximum value of the diameter of the elements. The inequality (9) suggests that if a computational model is constructed to keep the ratio of H to h constant, the condition number of the preconditioned coefficient matrix is bounded, and this will not lead to an increase in computational costs.

IV. DISCRETE DE RHAM SEQUENCES

Now, we recall the de Rham sequences

$$\begin{aligned} \mathbb{R} &\xrightarrow{i_T} H^1(T) \xrightarrow{\text{grad}} \mathbf{H}(\text{rot}; T) \\ &\xrightarrow{\text{rot}} \mathbf{H}(\text{div}; T) \xrightarrow{\text{div}} L^2(T) \xrightarrow{0} \{0\}, \end{aligned} \quad (10)$$

where T denotes a polyhedron, i_T denotes a map from $\mathbb{R}$ to the constant function defined in T , $\mathbf{H}(\text{rot}; T)$ and $\mathbf{H}(\text{div}; T)$ denote functional spaces defined by

$$\mathbf{H}(\text{rot}; T) := \left\{ \mathbf{v} \in (L^2(\Omega))^3; \text{rot } \mathbf{v} \in (L^2(\Omega))^3 \right\}, \quad (11a)$$

$$\mathbf{H}(\text{div}; T) := \left\{ \mathbf{v} \in (L^2(\Omega))^3; \text{div } \mathbf{v} \in L^2(\Omega) \right\}. \quad (11b)$$

The inclusion (8) denotes that the corresponding finite element spaces satisfy the de Rham sequences with the finite dimension.

The Polynomial Element Method (PEM) proposed by Di Pietro, et al. [8] satisfies the de Rham sequences. Especially, arbitrary order elements based on PEM satisfy the de Rham sequences. Therefore, when the approximate spaces for the magnetic vector potential and the Lagrange multiplier appearing in the original mixed formulation replace with PEM corresponding to $\mathbf{H}(\text{rot}; T)$ and $\mathbf{H}(\text{div}; T)$, the condition number of the preconditioned coefficient matrix may be evaluated as (9).

V. INCOMPLETE BDD METHOD

Even if approximate spaces of the magnetic vector potential and the Lagrange multiplier are replaced with PEM, the fact remains that the dimension of the coarse space problem required by the BDD method is the product of DOF of $\text{grad } Q_h$ and the number of subdomains. Therefore, we introduce Q_H that consists of functions defined by PEM in each subdomain $\Omega^{(i)}$, and we approximate $\text{Range}(Z_i)$ with the space constructed by Q_H .

For example, in case of PEM, for each subdomain $\Omega^{(i)}$, $\underline{X}_{\text{grad } \Omega^{(i)}}^k$ is used for the approximation of $\text{Range}(Z_i)$, where the natural number k denotes the order of PEM (for details of the definition of $\underline{X}_{\text{grad } \Omega^{(i)}}^k$, see Di Pietro–Droniou [8]). In this case, by replacing the sparse space $\mathbf{W}$ with its approximate sparse space $\mathbf{W}_H$ and following the algorithm of BDD, an approximate preconditioner matrix M_H can be constructed. The author calls this method the incomplete BDD method.

As the same analogy in case of standard BDD method, the condition number of the preconditioned coefficient matrix in case of incomplete BDD methods may be evaluated as in (9). In addition, since the dimension of the approximate coarse space $\mathbf{W}_H$ become low rather than one of $\mathbf{W}$, it is expected that the computational cost of the incomplete BDD method will be reduced.

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