

Easy Computation of the Full Helmholtz Decomposition of Tangential Fields on General Multiply Connected Surfaces using Deterministic and Randomized Algorithms

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In this work, we construct a full Helmholtz-Hodge decomposition for tangential H_{div} -conforming fields on general multiply connected, eventually open, surfaces using either a singular value decomposition or a randomized algorithm for the computation of the global loop functions. In particular, our method yields explicit coefficients for the global loop functions spanning the harmonic subspace in the Rao-Wilton-Glisson (RWG) basis in the same fashion as for the *loop-tree* or *loop-star* decompositions. This is particularly interesting for users who already have a code based on such decompositions as the implementation of our two distinct algorithms can be done in a few lines of code. We give a numerical example in the context of the simulation of the eddy current-based non-destructive testing of metallic structures using Boundary Integral Equations.

Index Terms—integral equations, computational electromagnetics, eddy currents, numerical analysis

I. INTRODUCTION

THIS WORK offers a new and innovative approach for the construction of the harmonic subspace when computing the Helmholtz-Hodge decomposition (HHD) of tangential fields in the context of low-frequency electromagnetics on general multiply connected surfaces. It takes its source in the context of the simulation of the non-destructive testing of metallic structures using eddy current probes. It appears that Boundary Integral Equations (BIE) are particularly well-suited for such simulations as they only require a surface mesh on the contrary to more classical Finite Element approaches which would require a volume mesh adapted to the skin depth. A well-known BIE is the Poggio-Miller-Chan-Harrington-Wu-Tsai (PMCHWT) formulation [1]. While theoretically well-posed for any frequency, it suffers from a so-called *low-frequency breakdown* because the computation of the *single layer* operator becomes numerically unstable as the frequency goes to zero. In order to circumvent this issue, the HHD of the unknowns is computed: they are split into a curl-free component and its complementary. Formally, it allows some operator splitting and numerically, stability is recovered (see for example [3]). In this work, we assume that the approximation space for the PMCHWT formulation is the RWG space of H_{div} -conforming functions and, without loss of generality, that the components of the HHD are spanned by the *loop-star* (LS) subspaces (see Section II, eq. (24)-(25) in [2]). If the surface is simply connected, the LS space spans the whole discrete subspace and there are no other “difficulties” than computing the basis functions. On multiply connected surfaces, there exists an additional “harmonic subspace”, orthogonal to the others, and spanned by the so-called *global loop functions*. Assuming that the surface is closed, it can be shown that there are $2g$ global loops where g is the genus of the underlying surface. We note that the authors in [2] combine quasi-Helmholtz projectors on the *loop* and *star* subspaces with a variant of the PMCHWT formulation to avoid the computation

of the global loops functions. We will not dwell precisely into the computations of these functions but we would like to mention this recent work [4] based on polynomials as we wish to stick to the original RWG-based discretisation.

We propose a non-intrusive method which finds the coefficients of the global loops on the RWG basis by numerically evaluating the range of the quasi-Helmholtz projector on the harmonic subspace. The first algorithm is purely deterministic and consists in computing a low-rank Singular Value Decomposition (SVD) of the matrix of the projector while the second algorithm is based on the Randomized Range Finder (RRF) [5]. Both algorithms only require very few lines of code (less than five when using our Matlab implementation) and are easily integrated in an existing code basis. Moreover, they are computationally extremely efficient compared to the traditional cost of a BIE computation and can also be used on *open* surfaces. We give practical comparisons with a previous method implemented in a commercial software.

II. BASIS FUNCTIONS FOR THE HARMONIC SUBSPACE

The purpose of this section is to compute the coefficients of the expansion of the harmonic subspace basis functions in the RWG basis in order to obtain the corresponding transition matrix from the discrete harmonic subspace to the RWG subspace. We propose two algorithms. To that purpose, we make use of the quasi-Helmholtz projector on the harmonic subspace. Following [2] and reusing the notations

- Λ : transition matrix “*loop* to RWG space”,
- Σ : transition matrix “*star* to RWG space”,

we have closed-form expressions for the projection matrix on the corresponding subspaces

$$\Pi^\Lambda = \Lambda (\Lambda^T \Lambda)^{-1} \Lambda^T \quad \text{and} \quad \Pi^\Sigma = \Sigma (\Sigma^T \Sigma)^{-1} \Sigma^T. \quad (1)$$

Therefore, the projector matrix on the *harmonic subspace* is

$$\Pi^H = \mathbf{I} - \Pi^\Lambda - \Pi^\Sigma, \quad (2)$$

where $\mathbf{I}$ is the identity matrix. We emphasize the fact that Π^H projects the whole RWG space onto the harmonic subspace. In particular, it means that the rank k of Π^H is *exactly* the number of sought basis functions. It is known *a priori* or it can be computed at runtime. At this point, all we have to do is to find the range of Π^H .

Of course, we do not want to assemble Π^H which is dense. However, we can use iterative procedures because Λ , Σ are *sparse matrices* and $\Lambda^T \Lambda$ and $\Sigma^T \Sigma$ can be factorized beforehand. The first algorithm is the low-rank SVD (accessed through the `svds` function in Matlab) computing $\Pi^H \approx \mathbf{U} \cdot \mathbf{S} \cdot \mathbf{V}^T$. The coefficients are given by the first k columns of the $\mathbf{U}$ matrix (or k columns of $\mathbf{V}$), see [6] for insights and notations. Theoretical results guarantee the convergence of the algorithm, but it may be *very slow*. The second algorithm is simply the Randomized Range Finder (ALGORITHM 4.1 in [5]). While the convergence result is only obtained up to some probability, it can be shown that the expectancy of the error is equal to 0 because the first singular values are equal to 1 and the remaining ones are 0. The Matlab code for the SVD-based algorithm is provided below (the RRF-based algorithm is implemented pretty much in the same fashion...).

```
% SVD-based global loops in Matlab
% L/S: loop/star to RWG transfer matrix
% k : number of global loops/oversampling
dL=decomposition(L'*L); dS=decomposition(S'*S);
PiH = @(V,~) V-L*(dL\((L'*V))-S*(dS\((S'*V)));
[H,~,~] = svds(PiH,k,'largest', ...
    'Tolerance',1e-14);
% H: global loops coef. in the RWG basis
```

III. NUMERICAL EXAMPLE

We illustrate the use of the algorithms above with the inspection of steam generator tubes using an eddy current probe. The simulations are performed using the dedicated module of the commercial CIVA software[7] which we have modified to use any of the proposed algorithms. In the following, the *legacy* label means that the global loops are computed by first identifying loops of mesh cells in a structured mesh of the object surface, then by computing a linear combination (not described here) of the relevant RWG basis functions. The BIE formulation used to produce the results is described in [3].

In this digest, we consider a simple test case where a steam generator tube (inner radius 8.435 mm, thickness 1.09 mm, conductivity $\sigma = 0.87 \text{ MS.m}^{-1}$, relative permeability $\mu_r = 1$) features a flaw which is a single traversing hole (diameter 1 mm). The length of the tube is sufficiently big with respect to the inspection area so that any perturbation caused by the finite size of the numerical model is neglectable. The probe consists in two coils (inner radius 7 mm, thickness 0.5 mm, height 2 mm, 45 turns, and a 1 mm space inbetween) working at 600 kHz with a current of 1 mA. The output is the *difference of the impedances between the first and second coil* displayed in the complex plane, yielding a so-called Lissajous curve. Because of the symmetries of the setup, the Lissajous *must* feature some

symmetry with respect to the origin: the forward coil "sees" first the flaw, then the sensibility shifts as the probe moves forward.

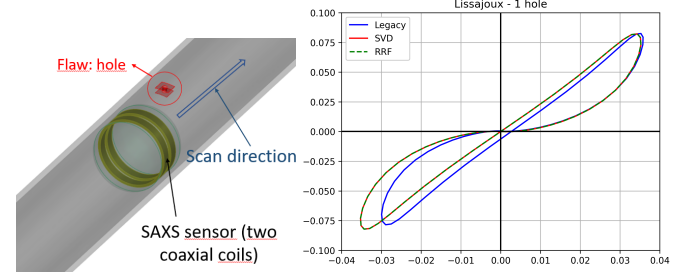


Fig. 1. Tube with a circular traversing flaw and SAXS axial probe - Setup (left) et impedance measurement (right)

The outputs are displayed on Fig. 1. Using the "legacy global loops", the Lissajous is clearly not symmetrical (for example, the two lobes do not have the same extent). Using the SVD- or RRF-based global loops, we recover the expected result.

IV. CONCLUSION

With this work, we propose two algorithms for the computation of the harmonic subspace whose implementation is straightforward. In practice, their cost is totally neglectable compared to the full simulation (seconds or less versus minutes or hours). The low-rank SVD will not fail but its convergence can be slow. On the other side, the RRF is much cheaper but may theoretically fail, even if it was never achieved in our tests. By construction these algorithms *will succeed* on open surfaces (tested but not presented here). More detail will be given in the extended version of this paper, including error indicators. Moreover, if higher order *loop* and *star* functions are available, both algorithms should yield the corresponding higher order *global loop functions*. Finally, it is worth mentioning that the coherence with the *loop-star* decomposition of the dual Buffa-Christiansen space is ensured and may be used to preconditioning ends.

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