

Magnetic field calculations with second order accuracy using linear finite elements

R. Albanese^{1,2}, M. Neri^{1,2,3}, P. Zumbolo^{1,2}

¹ Università degli studi di Napoli Federico II, via Claudio 21, I-80124 Napoli, Italy, marco.neri@unina.it

² Consorzio CREATE c/o Università degli studi di Napoli Federico II, via Claudio 21, I-80124 Napoli, Italy

³ Università degli Studi di Padova, Padova, Via Gradenigo 6/b, I-35131 Padova, Italy

This paper shows an innovative method to evaluate the magnetic field in the vacuum in the presence of ferromagnetic materials with second order accuracy, still using the finite element method (FEM) with linear triangular elements. The procedure is the extension of a technique applied to the solution of magnetohydrodynamic equilibria, based on Helmholtz's theorem. In this way, post-processing the standard FEM solution, continuity of the magnetic field is ensured by using nodal shape functions for the field components. The effectiveness of the method is shown in 2D cases for which analytical solutions are available. The possible extension to the calculation of 3D fields is also discussed.

Index Terms— Finite element analysis, Magnetostatics, Accuracy.

I. INTRODUCTION

THIS PAPER extends the formulation presented in [1] to a wider class of elliptic problems. The method, successfully applied in [1] for the solution of magnetohydrodynamic (MHD) equilibria, was based on the formulation originally proposed in [2] as an application of Helmholtz's theorem, which guarantees the uniqueness of a vector field of which its curl and its divergence are known.

The formulation presented in [1] ensures continuity and a second-order convergence rate for the gradient of the magnetic field in 2D axisymmetric domain using triangular elements with three degrees of freedom, hence imposing continuity of the field.

The present paper shows how the method can be extended to the magnetostatic problem in 2D or 2D axisymmetric geometries in the presence of magnetic discontinuities. Then, the method can also be used to calculate 3D magnetic fields in tokamaks with a second order accuracy.

Improving the accuracy of FEM calculation is significant for particular applications, e.g., evaluating the forces on magnetic components of electrical machines using Maxwell stress tensor, or tracing long magnetic field lines in nuclear fusion devices.

The paper is organized as follows. Section II briefly summarizes the features of the method presented in [1]. Section III deals with 2D magnetostatic calculations in the presence of magnetic materials. Finally, Section IV describes an application to the 3D case.

II. ACCURATE CALCULATION OF THE MAGNETIC FIELD IN 2D AXISYMMETRIC DOMAINS

Using a cylindrical (r, ϕ, z) , co-ordinate system, the poloidal (radial and vertical) components of the magnetic field $\mathbf{B}$ in a 2D axisymmetric problem where $\mathbf{J} = J \mathbf{i}_\phi$ can be obtained in terms of the poloidal flux per radian ψ :

$$\mathbf{B} = \nabla\psi \times \mathbf{i}_\phi / r \quad (1)$$

by solving

$$L\psi = J, \quad \text{with } L\psi = -\frac{\partial}{\partial r} \frac{1}{\mu r} \frac{\partial \psi}{\partial r} - \frac{\partial}{\partial z} \frac{1}{\mu r} \frac{\partial \psi}{\partial z} \quad (2)$$

The finite element method (FEM) using linear triangular elements provides the poloidal flux per radian with a

$O(h^2)$ convergence rate at the nodes [1,3-4]. Instead, the convergence rate of the magnetic field, linked to $\nabla\psi$, is $O(h)$. The procedure adopted in [1], based on Helmholtz's theorem, guarantees the continuity of $\mathbf{B}$ as well as a convergence rate increasing from $O(h)$ to $O(h^2)$. This is made still using the FEM method and linear triangular elements, this time with linear shape functions for both $\partial\psi/\partial r$ and $\partial\psi/\partial z$ in the second order calculation. With the technique proposed in [1], where $J = J(r, \psi)$, the poloidal magnetic field can be determined by post-processing the standard FEM flux map as the solution to the minimization problem:

$$\min \left\{ \int_{\Omega} \left[(\nabla \cdot \mathbf{B})^2 + (\nabla \times \mathbf{B} - \mu_0 J \mathbf{i}_\phi)^2 \right] r^2 d\Omega + \alpha \int_{\partial\Omega} [\mathbf{n} \cdot \mathbf{B} - (\partial\psi/\partial t)/r]^2 r d\Gamma \right\} \quad (3)$$

where α is a suitable non-dimensional factor, whereas $J(r, \psi)$ and $\partial\psi/\partial t$, which is the tangential component of $\nabla\psi$, are taken from the standard FEM solution.

The method is here applied also to a subdomain with a uniform magnetic permeability, e.g., air or vacuum, even in the presence of magnetic materials. It is sufficient to use (3) in the subdomain by using the value of ψ given by the first order solution at the interface to prescribe the boundary condition.

III. 2D MAGNETOSTATIC CALCULATIONS IN THE PRESENCE OF MAGNETIC MATERIALS

The first application is the computation of the perturbation of the magnetic field generated by an iron sphere inside an externally applied uniform magnetic field $B_{z\infty}$. The sphere is made of linear ferromagnetic material with relative magnetic permeability $\mu_{ri} > 1$:

$$\mu_r = \begin{cases} \mu_{ri}, & 0 \leq r \leq R_s \\ 1, & r > R_s \end{cases} \quad (4)$$

Figure 1 shows the field lines and the comparison between the numerical solutions and the analytical value of the magnetic field on the equatorial plane [5].

The second example tests the method efficiency with an inhomogeneous ferromagnetic material. The aim is to determine the magnetic field produced by a co-axial indefinite coil with a ferromagnetic insert (Fig. 2a). The magnetic permeability of the iron is:

$$\mu_r = 1 + 10(x^2 + y^2)/a_2^2 \quad (5)$$

This 1D problem has a straightforward analytical solution, but it is solved as a 2D problem. It is worth noticing that this test can also be formulated as a non-linear problem where the magnetic permeability is a function of the field in the iron. Figure 2b shows the azimuthal field in the vacuum given by the standard *FEM* solution and by the second order procedure in comparison with the analytical solution. With a maximum step size of 150 mm, the relative errors obtained by the 1st and 2nd order procedures are 2.02% and 1.14%, respectively. With a maximum step size of 75 mm, the corresponding errors are reduced to 1.18% and 0.30%, showing the quadratic convergence of the 2nd order method.

The main computational cost is related to the pre-processor and the solution of the non-linear system (2) with first order triangles. The number of floating point operations (FLOPS) for a single non-linear iteration would scale with $7N^2$, where N is the number of nodes. Instead, the second order calculation requires a total number of about $54N^2$ FLOPS. Using an alternative method [8], to get the same level of accuracy with $\psi \in C^1$ using reduced quintic order finite elements, the number of unknowns would rise to $6N$. Therefore, the number of FLOPS for a single non-linear iteration would scale with $1512N^2$ [7-8].

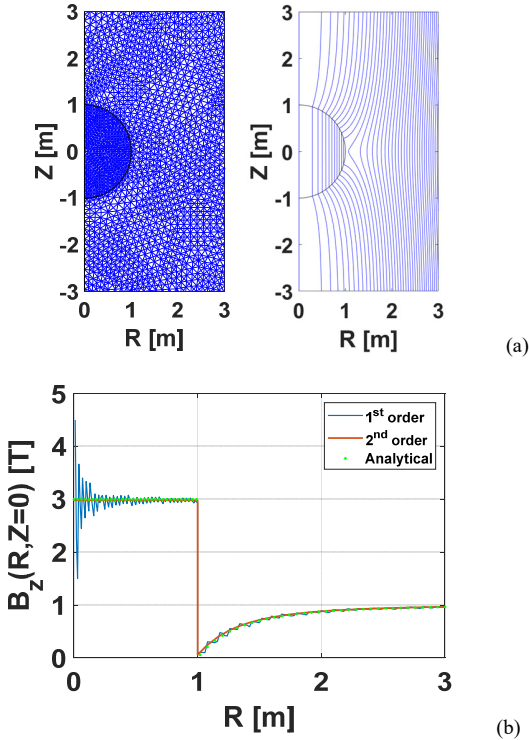


Fig. 1. Magnetized sphere in a uniform field: a) field lines given the standard *FEM* solver; b) vertical field at $Z=0$ given by the standard *FEM* solution and by the second order procedure compared to the analytical solution.

IV. APPLICATION TO A 3D CASE

Accurate tracing of 3D magnetic field lines in toroidal nuclear fusion devices is crucial for the determination of the gap between the very hot plasma and the solid walls inside the

vacuum vessel. In the vacuum region between the plasma boundary and the solid walls, the magnetic field is both solenoidal and irrotational. This makes it possible to apply the minimization procedure (3) for improving the accuracy of the magnetic field reconstruction with the procedure proposed in [6], based on FEM discretization in (r, z) and Fourier expansion along φ in a cylindrical coordinate system (r, φ, z) .

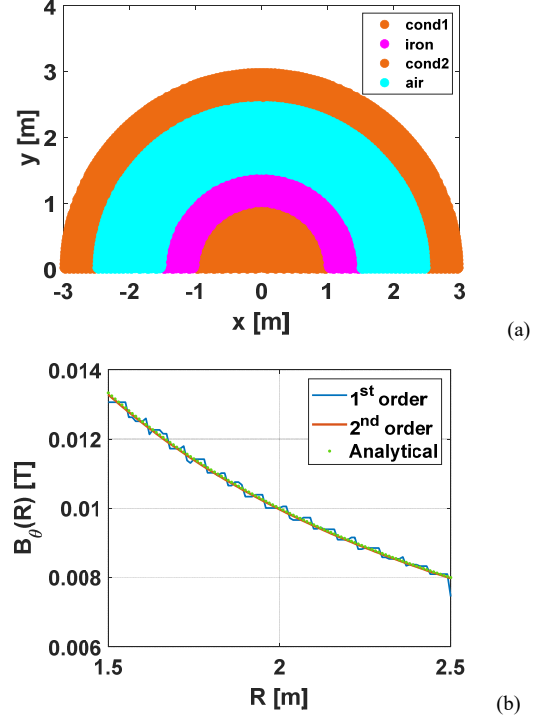


Fig. 2. Coaxial cable with a ferromagnetic insert: a) 2D computational domain, with 100 kA uniformly distributed in $R < a_1 = 1$ m, iron with the magnetic permeability given in (5) for $a_1 < R < a_2 = 1.5$ m, vacuum for $a_2 < R < a_3 = 2.5$ m, external conductor with opposite current uniformly distributed for $a_3 < R < a_4 = 3$ m; b) azimuthal field in the vacuum given by the standard *FEM* solution and by the second order procedure compared to the analytical solution.

REFERENCES

- [1] R. Albanese, A. Iaiunese, P. Zumbolo, "Solution of Grad-Shafranov equation with linear triangular finite elements providing magnetic field continuity with a second order accuracy", (2023) Computer Physics Communications, 291, art. no. 108804, doi: 10.1016/j.cpc.2023.108804
- [2] P. Neittaanmäki, J. Saranen, On finite element approximation of the gradient for solution of Poisson equation, (1981) Numerische Mathematik, 37 (3), pp. 333–337, doi: 10.1007/BF01400312
- [3] R. Albanese, R. Ambrosino, M. Mattei, "CREATE-NL+: A robust control-oriented free boundary dynamic plasma equilibrium solver", (2015) Fusion Engineering and Design, 96-97, pp. 664-667, doi: 10.1016/j.fusengdes.2015.06.162
- [4] P. Barabaschi, "The MAXFEA code", (1993) Plasma Control Technical Meeting. Naka, Japan
- [5] J. Jackson, "Classical electrodynamics", (1999) Wiley, New York, NY, 3rd edition
- [6] R. Albanese, M. de Magistris, V.P. Loschiavo, S. Minucci, "Numerical assessment of a novel technique for the reconstruction of 3D magnetic fields in tokamaks", (2017) Int. J. Appl. Electromagn. Mech., 56(7), doi: 10.3233/JAE-172284
- [7] M. Neri, P. Zumbolo, R. Albanese, "Accurate plasma boundary calculation using linear triangular finite elements", (2025) Computer Physics Communications, 109574, doi: 10.1016/j.cpc.2025.109574
- [8] S.C. Jardin, "A triangular finite element with first-derivative continuity applied to fusion MHD applications", (2004) Journal of Computational Physics, 200 (1), pp. 133-152, doi: 10.1016/j.jcp.2004.04.004