

Virtual Casing Principle in presence of sink/source currents

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The possibility of describing the magnetic field due to currents within a finite control volume via a set of equivalent surface current on the boundary of such volume is given by the well-known Virtual Casing Principle. This was illustrated and implemented in the past for the case of solenoidal currents in the volume of interest, *i.e.* when the currents to represent have a null normal component to the boundary of the control volume. Here we illustrate how to apply and implement the Virtual Casing Principle for the more general case where the currents to represent can cross the boundary of the control volume.

Index Terms—Boundary conditions, Integral equations, Eddy currents.

I. INTRODUCTION

WHEN studying magnetostatic or magneto-quasi-static problems via differential formulations, the imposition of boundary conditions may go through the coupling of a Finite Element Method (FEM) within the computational domain with a Boundary Element Method (BEM) on the boundary of the computational domain, which solves for the vacuum magneto-static problem outside the domain[1].

In applications related to magnetic confinement fusion devices, among the various possible BEM methods, it is quite popular to use indirect approaches based on the so-called *Virtual Casing Principle* (VCP) [2], [3], [4], [5]. The magnetic vector potential due to plasma currents within the computational domain is associated to an equivalent current density distribution on its boundary. This current perfectly describes the magnetic field due to internal currents in the outer domain, hence a Biot-Savart integral can be calculated to provide the required $\mathbf{A} \mapsto \mathbf{B}$ map.

The presence of currents shared between plasmas and surrounding conductors (the so-called *halo currents*) calls for the development of FEM-BEM methods which are capable to handle current distributions which can cross the boundary of the domain[6]. In this paper, we describe how a standard VCP-based Boundary Element Method can handle this situation. The adaptation of other FEM-BEM methods to this situation will be subject of future work.

II. FORMULATION

The first step of the virtual casing principle consists of identifying a certain domain V_{in} which contains a current source (IN) which we want to represent via a set of equivalent currents at the boundary of the domain itself ∂V_{in} . We imagine a Perfectly Conducting Shell (PCS) on the domain boundary. Turning on the current source within the control domain V_{in} , a surface current immediately appears on the PCS which immediately cancels any induced electric field in the outer domain. In the general case where the current density is crossing the boundary

of the control volume, the electric field in the outer conductors is associated both to the contact potentials and to the magnetic flux variations. Hence, the condition of zero electric field on the PCS shall read: $\mathbf{E} = -\partial\mathbf{a}/\partial t - \nabla\phi = 0$. Using the Coulomb gauge, the magnetic vector potential due to different current source contributions is easily set up via the Biot-Savart law, *e.g.* for the PCS we set the notation $\mathbf{a}_{PC} = \mathcal{A}\mathbf{j}_{PC}$. The resulting system of integro-differential equations is

$$\mathcal{A}\mathbf{j}_{PC} + \nabla\chi = -\mathcal{A}\mathbf{j}_{IN} \quad \text{on } \partial V_{IN} \quad (1)$$

$$\nabla_S \cdot \mathbf{j}_{PC} = J_n \quad \text{on } \partial V_{IN} \quad (2)$$

Here the unknown terms are the PCS surface current density $\mathbf{j}_{PC}$ and the time integral of the electric potentials $\chi(x, t) = \int \phi(x, \tau) d\tau$, whereas the assigned quantities are the volume current density within the control volume $\mathbf{j}_{IN}$ and the injected currents in the superconducting shell J_n . Considering the curl of (1), it is evident that the current distribution $\mathbf{j}_{PC} + \mathbf{j}_{IN}$ does not produce any magnetic field in the outer domain V_{ext} . Hence the equivalent current on ∂V_{IN} which reproduces exactly the magnetic field due to internal currents in the outer world is clearly $\mathbf{j}_{eq} = -\mathbf{j}_{PC}$.

The electric current density needs to be divergent free in the magneto-quasi-static assumption, hence it is convenient to introduce an approximation of $\mathbf{j}_{PC}$ via a finite subspace of divergence-free basis functions, *i.e.* $\mathbf{j}_{PC} \simeq \sum_k I_k \mathbf{w}_k$, where $\nabla \cdot \mathbf{w}_k = 0$ [7]. Furthermore we can introduce a piecewise constant basis function to each electrode used for current injection, unitary on the electrode and null elsewhere. The weak form of equation (1) against the div-free basis functions $\mathbf{w}_k$, together with the weak form of equation (2) against piecewise constant basis functions λ_k yields the numerical approximation of the problem:

$$\mathbf{L} \mathbf{i}_{eq} + \mathbf{F} \chi_{eq} = \mathbf{V} \quad (3)$$

$$\mathbf{F}^T \mathbf{i}_{eq} = \mathbf{J} \quad (4)$$

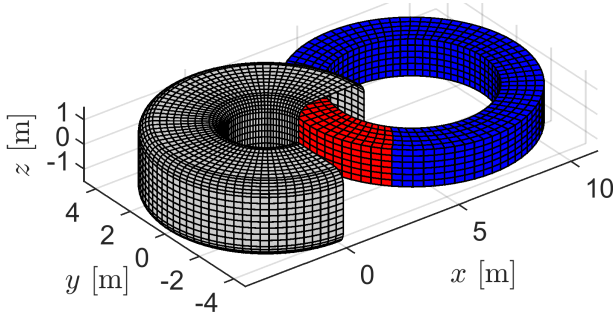


Fig. 1. Geometry for the low resolution test case. A portion of the axisymmetric toroidal surface used for the equivalent current computation is depicted in grey. The portion of the bulk conducting coil contained within the toroidal surface is depicted in red.

Here we indicated by $\mathbf{i}_{eq}$ the unknown degrees of freedom associated with the equivalent currents and by χ_{eq} the corresponding time integrated potentials on the electrodes of the PCS. The forcing term $\mathbf{V}$ in equation (3) is given by projections of the magnetic vector potentials due to currents within the control volume on the basis functions on the PCS, i.e. $V_i = \langle \mathcal{A} \mathbf{j}_{IN}, \mathbf{w}_i \rangle_{SS}$. The currents injected into the equipotential electrodes are stored in the array $\mathbf{J}$ of equation (4), i.e. $J_i = \langle J_n, \lambda_i \rangle_{SS}$. Clearly the inductance matrix coefficients are defined by $L_{ij} = \langle \mathcal{A} \mathbf{w}_i, \mathbf{w}_j \rangle_{SS}$ and the Kirchoff incidence matrix for electrodes is calculated as $F_{ij} = \langle \mathbf{w}_i \cdot \mathbf{n}, 1 \rangle_{\Sigma_j}$.

III. RESULTS

A test case for the system of equations (3)-(4) is set up: the geometry and nominal discretization for this toy model is depicted in Figure 1. A bulk toroidal conductor of rectangular cross section ($R = 3$ m, $\Delta Z = 1.5$ m, $\Delta R = 1.5$ m) is crossing a toroidal surface of rectangular cross section ($R = 3$ m, $\Delta Z = 3$ m, $\Delta R = 3$ m), which can represent the boundary of some computational domain in the future. The centers of the two toroids are distant 6.5 m along the x -direction.

In the bottom panel of Figure 2 we show the equivalent current distribution to the one inside the toroidal surface, shown in the top panel. This is obtained inverting the system of equations (3)-(4). Different strategies can be adopted for efficient inversion exploiting the block-structure of the resulting algebraic system, which will be described in the extended version of the paper.

We compare the magnetic field contribution on the outer page of the toroidal shell generated by our original current distribution and by our equivalent surface current, for different discretization levels. Figure 3 summarizes the result of the scan, indicating that decreasing the typical linear dimension of the mesh size, the error decreases proportionally.

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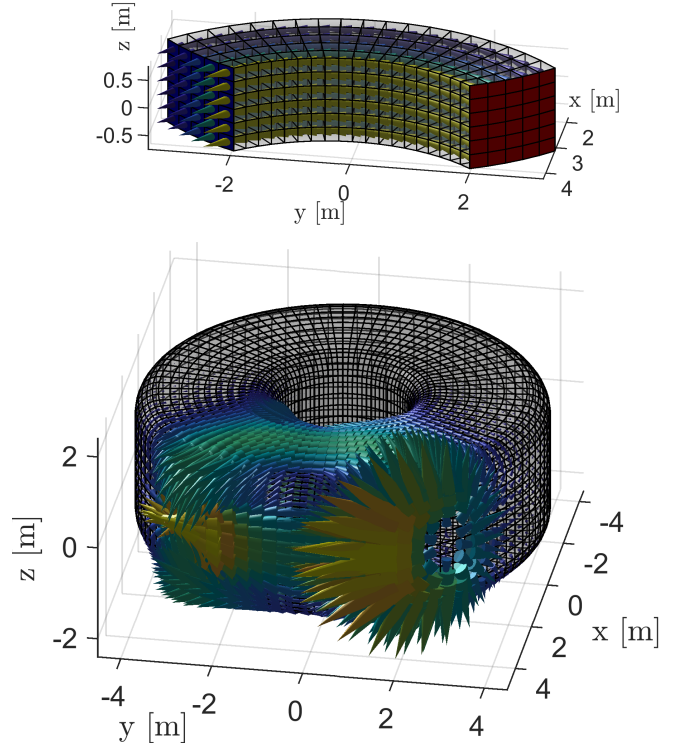


Fig. 2. Top panel: assigned current density distribution within the toroidal surface. Bottom panel: corresponding equivalent current distribution.

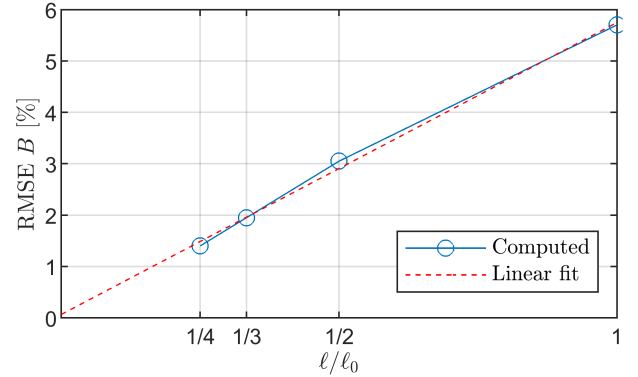


Fig. 3. Scan of Magnetic Field RMSE with typical linear size of boundary element. The coarsest case $l = l_0 = 23$ cm corresponds to the equivalent current in Figure 2.

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