

Exact Modal Decomposition of Axial Currents in a Conducting Cylinder

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The modal decomposition of eddy current problems is commonly treated numerically via integral formulations or via the imposition of arbitrary homogeneous boundary conditions at some distance from the sources in case of differential formulations. In this paper the eigenvalues and eigenvectors for axial currents in an infinitely long conducting cylinder surrounded by vacuum are computed analytically, rigorously accounting for the solution of the outer magnetostatic problem. The analytically derived eigenvalues and eigenvectors are validated against an ad hoc numerical model based on an integral formulation, showing that our analytical solution can be used as a reference for other general-purpose numerical models. The procedure highlighted is general and could be extended to different geometries.

Index Terms—Eddy currents, Eigenvalues and eigenfunctions, Analytical models

I. INTRODUCTION

THIS PAPER deals with the modal decomposition of Magneto-Quasi-Static (MQS) problems in cylindrical geometry. The important result that MQS problems admit an infinite discrete set of real eigenvalues was illustrated rigorously in [1]. The identification of typical time constants and associated current distributions can be beneficial for different engineering purposes, e.g. for modal truncation during the design phase of cables or wires. The magneto-quasi-static eigenvalues for a conducting sphere were presented and compared to the poles of Mie coefficients in [2].

Modal approaches to the solution of eddy current problems in cables, wires, pipes, disks and other axisymmetric conducting samples are found in the literature which use a truncation of the domain of interest based on the expected decay of magnetic field at distance from the source, hence on arbitrary Dirichlet or Neumann boundary conditions, see [3] and References therein. In this paper, with the purpose of illustrating how these truncation approximations can be relaxed, the exact eigenvalues and eigenvectors for the problem of axial currents in an infinitely long cylinder are computed.

II. FORMULATION

In Coulomb gauge, neglecting for the moment the net current through the cylinder, the differential formulation for the eddy currents' eigenmode problem is:

$$\nabla^2 \mathbf{A} = -\mu_0 \sigma \lambda \mathbf{A} \quad \text{in } \Omega_{int} \quad (1)$$

$$\nabla^2 \mathbf{A} = 0 \quad \text{in } \Omega_{ext} \quad (2)$$

$$\left[\left[\frac{\partial \mathbf{A}}{\partial n} \right] \right] = 0 \quad \text{on } \partial \Omega_{int} \quad (3)$$

Condition (3) expresses the fact that the boundary conditions to impose for the Helmholtz equation in Ω_{int} are related to the solution of the Laplace problem in Ω_{ext} and vice-versa. Hence, the determination of an eigenmode of the current density within the conducting volume also depends on the solution of the vacuum magnetostatic problem outside the conductor.

In the following we consider as conductor an infinite cylinder and we study exclusively the translational-invariant modes (*i.e.* $\partial/\partial z = 0$) associated to axial currents (*i.e.* we are interested exclusively in the j_z component of the current density). Considering explicitly the fact that the problem is translational invariant, and making the hypothesis the solution is separable, we have that $A_z(r, \varphi, z) = R(r)F(\varphi)$. In particular the function $F(\varphi)$ must be a periodic function of the toroidal angle, which can be expressed as a standard Fourier series. For the m -th toroidal harmonic, the function $R(r)$ accounting for the radial dependence is constrained to satisfy the equation:

$$\frac{d^2 R(r)}{dr^2} + \frac{1}{r} \frac{dR(r)}{dr} + \left(k^2 - \frac{n^2}{r^2} \right) R(r) = 0 \quad (4)$$

where $k^2 = \mu_0 \sigma \lambda$.

A. The outer problem

The solution of the outer Laplace problem can be studied setting $k^2 = 0$ in (4), which is equivalent to set the electric conductivity to zero. Simple calculations reveal that a general solution is given by the expansion:

$$A_z(r, \varphi) = B_0 \ln(r) + \sum_{m=1}^{\infty} (B_m r^{-m}) \cdot [C_m \cos(m\varphi) + D_m \sin(m\varphi)] \quad (5)$$

For the axisymmetric part of the solution, Ampère's law immediately reveals that $B_0 = \mu_0 I_c / 2\pi$, where I_c is the net current through the cylinder, *i.e.* the Neumann boundary condition which we can later use for the axisymmetric part of the inner problem:

$$\left. \frac{\partial A_z^0}{\partial n} \right|_{r=a} = \frac{\mu_0 I_c}{2\pi a} \quad (6)$$

Further, harmonic by harmonic, we have the Dirichlet to Neumann map for the non-axisymmetric solutions:

$$\left. \frac{\partial A_z^m}{\partial n} \right|_{r=a} = -\frac{m}{a} A_z^m \Big|_{r=a} \quad (7)$$

B. The inner problem

For the inner problem, the radial dependence is given via Bessel functions:

$$A_z(r, \varphi) = B'_0 J_0(kr) + \sum_{m=1}^{\infty} (A'_m J_m(kr) + B'_m Y_m(kr)) \cdot [C_m \cos(m\varphi) + D_m \sin(m\varphi)] \quad (8)$$

C. Axisymmetric eigenmodes

It is possible to show that a net current distribution would not extinguish in a infinitely long cylinder without a return circuit. Hence we can set up the boundary condition:

$$\frac{\partial A_z^0}{\partial n} = 0 \rightarrow x J_1(x) = 0 \quad (9)$$

where we defined $x = ka$. The axisymmetric eigenvalues for the problems are hence given by the zeroes of the Bessel function $J_1(x)$, which are tabulated in the literature, via the inverse formula $\lambda_{0,n} = x_{1,n}^2 / \mu_0 \sigma a^2$. The associated eigenfunctions are clearly $A_z^{0,n} = J_0(x_{1,n}\rho)$, where $\rho = r/a$.

D. Non-axisymmetric eigenmodes

For any toroidal harmonic index, we can use the Dirichlet to Neumann boundary condition (7), which via Bessel function properties reduces to:

$$x J_{m-1}(x) = 0 \quad (10)$$

Again solutions of (10) reveal the eigenvalues for the problem, which are given by the zeroes of the corresponding Bessel functions. The null eigenvalue given by $x = 0$ always corresponds to a null eigenvector by properties of Bessel functions ($J_\nu(0) = 0 \forall \nu > 0$). The associated eigenfunctions are clearly $A_z^{m,n} = J_m(x_{m-1,n}\rho)$.

III. NUMERICAL MODEL VIA INTEGRAL FORMULATION

The integral formulation of the problem, considering directly only the axial currents and a generic axial cross section of the cylinder Ω_c , consists in finding the function $j_z(x) \in L_2(\Omega_c)$ such that, $\forall w \in L_2(\Omega_c)$,

$$\begin{aligned} (a) \quad & -\lambda \langle \mathcal{A} j_z, w \rangle - \Delta_z \phi \langle 1, w \rangle = \langle \sigma j_z, w \rangle, \\ (b) \quad & \langle j_z, 1 \rangle = I_c. \end{aligned} \quad (11)$$

where we have introduced the 2D Biot-Savart operator $\mathcal{A}$ and the voltage drop per unit length, which is homogeneous here *i.e.* $\Delta_z \phi(r, z) = v$. This is the voltage associated to two virtual electrodes placed at the extremities of the cylinder ($z \rightarrow +\infty$ and $z \rightarrow -\infty$). Notice that the classical divergence-free requirement for the current density is here automatically satisfied since the current is z -directed and by assumption the physical system is invariant for translations in that direction.

We solve the above problem introducing a triangularization $\mathcal{T}_h$ of the cylinder cross-section. The triangles are enumerated by the symbol T_k . The Whitney 2-form defined for each triangle facet can be associated to a constant (unitary) basis function on the facet itself, which is null on other facets. The

triangulation induces the definition of the finite-dimensional sub-space of $L_2(\Omega_c)$:

$$V_h = \{w_k = 1 \forall x \in T_k, \quad w_k = 0 \forall x \notin T_k\} \quad (12)$$

Finally using a Galerkin formulation based on formulation (11) and the finite element space (12), problem (11) is reduced to

$$\begin{aligned} (a) \quad & -\lambda \mathbf{L} \mathbf{i} - \mathbf{1} v = \mathbf{R} \mathbf{i} \\ (b) \quad & \sum_k I_k = I_c \end{aligned} \quad (13)$$

Further details on matrix computations and how to transform problem (13) effectively in a generalized eigenvalue problem will be reported in the extended version of the paper, see [4] for a general treatment.

IV. DISCUSSION

The eigenvalues for axisymmetric modes and $m = 2$ modes are the same, hence the axisymmetric eigenvalues will have multiplicity equal to three (*i.e.* $J_0(x_{1,m}\rho)$, $J_2(x_{1,m}\rho) \cos m\varphi$, and $J_2(x_{1,m}\rho) \sin m\varphi$). Clearly, the slowest eigenmodes of the system are associated with the smallest zero of $J_0(x)$, *i.e.* to an $m = 1$ toroidal harmonic. Setting as reference our analytical estimates, in Figure 1 the error of the numerical model in the estimation of the six slowest time constants is illustrated. Each time constant has multiplicity 2 or 3, corresponding respectively to 2 or 3 different numerical estimations for the same time constant. At given resolution, the error is smaller for larger time constants. Clearly, the relative error decreases for smaller maximum accepted mesh edge length h , *i.e.* for finer mesh discretization and increased computational cost.

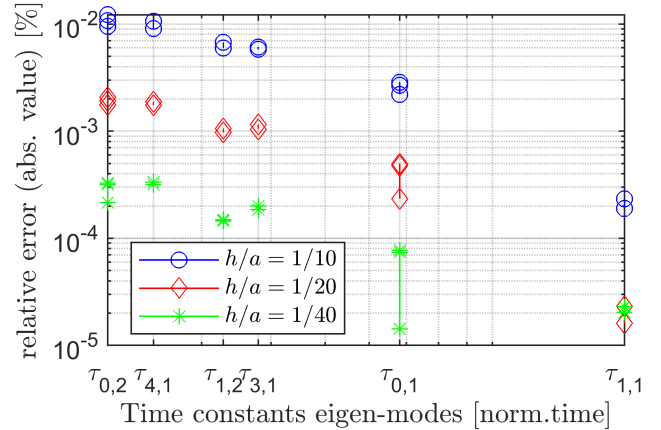


Fig. 1. Relative error in the computation of time constants via the numerical model, for different discretization levels. h is the maximum length of a mesh edge, and a is the radius of the cylinder.

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