

A vector potential formulation with vector hysteresis and its systematic numerical solution

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The modelling and simulation of ferromagnetic materials requires the consideration of magnetic hysteresis. We discuss the systematic incorporation of the energy-based vector hysteresis model of Henrotte et al., based on previous work of Bergqvist, into a vector potential formulation for the magnetic field equations. The field model describing a single step in a load cycle is phrased as a convex minimization problem which allows us to establish existence and uniqueness of solutions and to obtain accurate approximations by appropriate finite element discretization. Consistency of the model with the governing field equations is deduced from the first order optimality conditions. In addition, two globally convergent iterative methods are presented for the solution of the underlying minimization problems. The efficiency of the approach is illustrated by numerical tests for a typical benchmark problem.

Index Terms—Ferromagnetic materials, vector potential formulation, finite element analysis, minimization algorithms.

I. INTRODUCTION

HYSTERESIS is a key property of ferromagnetic materials and responsible for core losses in electric motors and transformers as well as the appearance of permanent magnets. A vector hysteresis model consistent with the principles of thermodynamics has been introduced by Bergqvist [1] and later refined by Henrotte et al. [2], [3]. This model describes the evolution of certain internal states of the system in response to external excitations. In its simplest form, it can be phrased as

$$\mathbf{J} = \arg \min_{\mathbf{J}'} U(\mathbf{J}') - \langle \mathbf{H}, \mathbf{J}' \rangle + \chi |\mathbf{J}' - \mathbf{J}_p|. \quad (1)$$

Here $\mathbf{J}_p$ and $\mathbf{J}$ denote the magnetic polarizations at the previous and current load step and $\mathbf{H}$ the exciting magnetic field intensity. The specific material properties are encoded in the internal energy density U and the parameter χ , which is related to the strength of pinning forces. Together with

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{J} \quad (2)$$

one obtains a constitutive relation $\mathbf{B} = \mathbf{B}(\mathbf{H}; \mathbf{J}_p)$ between the magnetic field intensity $\mathbf{H}$ and the flux density $\mathbf{B}$ which is influenced by the internal variable $\mathbf{J}_p$. The model (1)–(2) describes the material behaviour for one single load step. By solving (1), the internal variable, which encodes the history of the material, is continuously updated during a load cycle. In order to increase the accuracy of the model [3], one can split $\mathbf{J} = \sum_k \mathbf{J}_k$ into partial polarizations $\mathbf{J}_k$, each of which can be updated separately by an equation similar to (1). For ease of notation, we stay with one polarization for the moment.

The model (1)–(2) is well-suited for a scalar potential formulation of the magnetic field equations [4]. The incorporation into a vector potential formulation, on the other hand, requires the inversion of the material law $\mathbf{B} = \mathbf{B}(\mathbf{H}; \mathbf{J}_p)$, which may cause significant numerical overhead [5]. A systematic way to derive an inverse hysteresis operator $\mathbf{H} = \mathbf{H}(\mathbf{B}; \mathbf{J}_p)$ has been presented recently in [6]. In this paper, we study a more direct approach for incorporating the model (1)–(2) into a vector potential formulation of the governing field equations.

II. VECTOR POTENTIAL FORMULATION

Our mathematical model describing the coupling of the material law (1)–(2) to the field equations of magnetostatics is the following: We express $\mathbf{B} = \text{curl } \mathbf{A}$ via a vector potential $\mathbf{A}$ and consider the minimization problem

$$\min_{\mathbf{A} \in V} \min_{\mathbf{J} \in Q} \int_{\Omega} \frac{\nu_0}{2} |\text{curl } \mathbf{A} - \mathbf{J}|^2 - \langle \mathbf{j}_s, \mathbf{A} \rangle + U(\mathbf{J}) + \chi |\mathbf{J} - \mathbf{J}_p|_{\varepsilon} dx. \quad (3)$$

Here $\nu_0 = 1/\mu_0$ is the vacuum reluctivity, $\mathbf{j}_s$ a square-integrable current density, U is proper, smooth, and strongly convex, see [3, eq. (18)] for example, and $|\mathbf{J}|_{\varepsilon} = \sqrt{|\mathbf{J}|^2 + \varepsilon}$ is a regularization of the norm $|x|$. The domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is assumed sufficiently regular. The function space $V \subset H(\text{curl}; \Omega)$ in the minimization problem incorporates appropriate boundary and gauging conditions for the vector potential $\mathbf{A}$, and the space $Q = L^2(\Omega)^d$ contains the admissible polarizations. We can then show the following results:

Well-posedness. For any $\varepsilon, \chi \geq 0$ and $\mathbf{J}_p \in Q$, the functional to be minimized in problem (3) is proper, strictly convex, and coercive. Hence there exists a unique minimizer. Moreover, the minimizers depend continuously on the parameter ε .

Optimality conditions. The unique solution of (3) can be characterized equivalently by the Euler-Lagrange equations

$$\text{curl}(\nu_0(\text{curl } \mathbf{A} - \mathbf{J})) = \mathbf{j}_s \quad (4)$$

$$\nabla_{\mathbf{J}} U(\mathbf{J}) + \chi \frac{\mathbf{J} - \mathbf{J}_p}{|\mathbf{J} - \mathbf{J}_p|_{\varepsilon}} = \nu_0(\text{curl } \mathbf{A} - \mathbf{J}) \quad (5)$$

together with the appropriate boundary and gauging conditions.

Model consistency. From (2) we get $\nu_0(\text{curl } \mathbf{A} - \mathbf{J}) = \mathbf{H}$. Hence (4) amounts to Ampere's law $\text{curl } \mathbf{H} = \mathbf{j}_s$ and (5) can be stated as $\nabla_{\mathbf{J}} U(\mathbf{J}) + \chi \frac{\mathbf{J} - \mathbf{J}_p}{|\mathbf{J} - \mathbf{J}_p|_{\varepsilon}} = \mathbf{H}$. This is the optimality condition for $\min_{\mathbf{J}} U(\mathbf{J}) - \langle \mathbf{H}, \mathbf{J} \rangle + \chi |\mathbf{J} - \mathbf{J}_p|_{\varepsilon}$, which is the regularized version of (1). Moreover, $\text{div } \mathbf{B} = 0$ follows readily from $\mathbf{B} = \text{curl } \mathbf{A}$. Hence the solution of (3) satisfies the equations of magnetostatics and the material law (1)–(2).

III. NUMERICAL SOLUTION

We now discuss the systematic discretization of our model and present two iterative methods for the minimization.

Finite element approximation. A discretization of (3) can be obtained by simply restricting the minimization to certain subspaces V_h and Q_h . One can use standard $H(\text{curl})$ -conforming finite elements to approximate the vector potential $\mathbf{A}$ and discontinuous piecewise polynomial vector functions for the magnetic polarization $\mathbf{J}$. This turns (3) into a finite dimensional minimization problem with the same structure as on the continuous level. Existence of a unique finite element solution thus again follows by convexity arguments.

Block coordinate descent. A well-known strategy for minimizing a function $f(\mathbf{A}, \mathbf{J})$ of two variables is to minimize with respect to $\mathbf{A}$ and $\mathbf{J}$ in alternating order. One thus determines iteratively $\mathbf{A}^{n+1} = \arg \min_{\mathbf{A}} f(\mathbf{A}, \mathbf{J}^n)$ and then $\mathbf{J}^{n+1} = \arg \min_{\mathbf{J}} f(\mathbf{A}^{n+1}, \mathbf{J})$. Global convergence of such methods can be established under very general assumptions [7]. For the problem under consideration, the two steps amount to

$$\text{curl}(\nu_0 \text{curl} \mathbf{A}^{n+1}) = \mathbf{j}_s + \text{curl}(\nu_0 \mathbf{J}^n), \quad (6)$$

$$\min_{\mathbf{J}} U(\mathbf{J}) + \frac{\nu_0}{2} |\text{curl} \mathbf{A}^{n+1} - \mathbf{J}|^2 + \chi |\mathbf{J} - \mathbf{J}_p|_\varepsilon. \quad (7)$$

A similar iteration has been proposed in [3] which however uses (1) with $\mathbf{H} = \nu_0(\text{curl} \mathbf{A}^{n+1} - \mathbf{J}^n)$ instead of (7). While convergence of this method is unclear, the iteration (6)–(7) can be shown to converge globally and to yield decay of the functional to be minimized in every step of the iteration.

Newton's method. For $\varepsilon > 0$, problem (3) is convex and smooth, and global convergence of Newton's method can be established together with appropriate line search [8]. The Newton update direction $(\delta \mathbf{A}^n, \delta \mathbf{J}^n)$ is here determined by

$$\text{curl}(\nu_0(\text{curl} \delta \mathbf{A}^n - \delta \mathbf{J}^n)) = \mathbf{f}^n \quad (8)$$

$$-\nu_0 \text{curl} \delta \mathbf{A}^n + \nu^n \delta \mathbf{J}^n = \mathbf{g}^n \quad (9)$$

with the right hand sides $\mathbf{f}^n = \mathbf{j}_s - \text{curl}(\nu_0(\text{curl} \mathbf{A}^n - \mathbf{J}^n))$ and $\mathbf{g}^n = \nu_0(\text{curl} \mathbf{A}^n - \mathbf{J}^n) - \nabla_{\mathbf{J}} U(\mathbf{J}) - \chi \frac{\mathbf{J}^n - \mathbf{J}_p}{|\mathbf{J}^n - \mathbf{J}_p|}$, and reluctivity tensor $\nu^n = \nabla_{\mathbf{J}\mathbf{J}}(U(\mathbf{J}) + \chi |\mathbf{J} - \mathbf{J}_p|_\varepsilon)|_{\mathbf{J}=\mathbf{J}^n} + \nu_0 \mathbf{I}$. Together with appropriate boundary and gauging conditions, existence of a unique solution to this linearized problems can again be established. Further note that (9) allows to express $\delta \mathbf{J}^n$ locally as a function of $\text{curl} \delta \mathbf{A}^n$ and $\mathbf{g}^n$. This allows to eliminate $\delta \mathbf{J}^n$ from the system which results in a single linear equation

$$\text{curl}(\tilde{\nu}^n \delta \mathbf{A}^n) = \tilde{\mathbf{f}}^n \quad (10)$$

with appropriate incremental reluctivity tensor $\tilde{\nu}^n$ and right hand side $\tilde{\mathbf{f}}^n$. The main computational effort in every step thus consists in solving a standard linear magnetostatic problem.

IV. SETUP OF THE TEST PROBLEM

For illustrating the feasibility and performance of the proposed method, we now present some preliminary numerical results for a typical test problem. We consider a ferromagnetic ring which is wrapped by a winding. The two-dimensional cross-section which is used for the computations is depicted

in the left plot of Figure 1. As ferromagnetic material law, we use the energy-based hysteresis model with 5 pinning forces taken from [3]. For the remaining parts of the domain we consider a linear relation $\mathbf{H} = \nu_0 \mathbf{B}$. The simulation is started from a completely demagnetized state ($\mathbf{J}_{k,p} = 0$) and the excitation current is set to $\mathbf{j}_s(t) = \pm \mathbf{j}_0 \sin(t)$ with different current amplitudes $\mathbf{j}_0$. The total current in the coil with area C is given by $I = C \mathbf{j}_0$. In every step of the load cycle, the minimization problem (3) is solved by the Newton method described above. In Figure 1, we sketch the geometry of the test problem and a hysteresis loop obtained in our simulations.

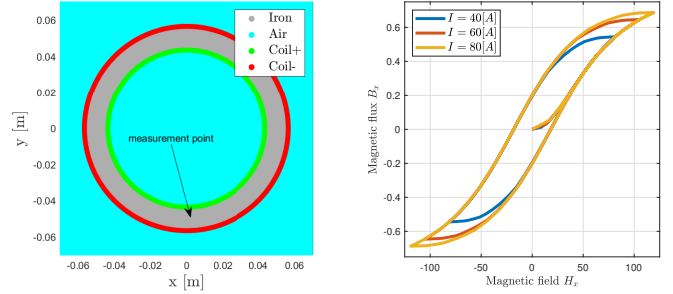


Fig. 1. Geometry (left) and Hysteresis loop (right) and the measurement point for a sinusoidal current excitation with varying total current I .

V. NUMERICAL RESULTS

The computed magnetic fields clearly show the expected hysteresis curves. Since the physical states change incrementally during the load cycle, convergence of the Newton solver for (3) was typically obtained after only 3-5 iterations. Convergence was also observed for the block coordinate descent method, but the number of iterations was very high, so the method seems not competitive with the Newton solver. Further details on the numerical solution, results for a more comprehensive test problem, and a comparison with the strategy of [6] will be presented in the full version of the manuscript.

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