

Three-dimensional Coil Shape Optimization Using Stream Function Method with ACA-Based Acceleration

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The stream function method is recognized as an effective technique for designing coil shapes capable of generating highly accurate magnetic fields, particularly for complex geometries. However, this method is subject to limitations due to its reliance on computationally expensive matrix operations, such as Truncated Singular Value Decomposition, and its restriction to current density design on two-dimensional curved surfaces. This study addresses the computational cost challenge by introducing the Adaptive Cross Approximation algorithm to accelerate stream function computation. Our research objective is to enable the application of this accelerated method within global optimization frameworks, thereby facilitating the efficient and practical design of manufacturable three-dimensional coil geometries.

Index Terms—ACA, CMA-ES, coil design optimization, stream function

I. INTRODUCTION

In various applications that require precise control of magnetic field distributions such as MRI, nuclear fusion systems, particle accelerators, and induction heating systems, it is essential to highly optimize the coil geometry. By optimizing the coil shape, it becomes possible to generate arbitrary magnetic fields and efficiently shape the magnetic field distribution. Design methods for coil geometry include topology optimization using the Finite Element Method (FEM) [1], the target field method [2], and the stream function method. Among these, the stream function method is known as one of the effective techniques for designing coils with complex geometries, capable of generating highly accurate magnetic fields [3].

In the stream function method, the desired magnetic field distribution $\mathbf{B}$ and the evaluation surface on which this field is assessed are first defined. Then, a curved surface is designated for coil placement, this is referred to as the coil design surface. Both the evaluation surface and the coil design surface are meshed for discretization. Letting $\mathbf{J}$ be the current density vector on the coil design surface and $\mathbf{n}$ the surface normal vector, the stream function φ is defined such that $\mathbf{J} = \nabla\phi \times \mathbf{n}$. The relationship between φ and the resulting magnetic field $\mathbf{B}$ is governed by the Biot-Savart law. Discretization of this law yields a matrix A , which serves as the systems kernel. The stream function φ is then obtained by applying the pseudoinverse of A to the desired field $\mathbf{B}$. Once φ is computed, contour lines are interpolated from it, and by appropriately connecting these contours, the coil pattern is generated [3]. This approach is known as the stream function method. Although effective, this method traditionally constrains the coil design surface to a two-dimensional manifold embedded in three-dimensional space. As a result, the resulting coil geometry may not be optimal for achieving the desired magnetic field in arbitrary 3D regions. If the design requirements are not fulfilled, manual redefinition of the coil design surface followed by re-optimization is required. Consequently, the resulting coil geometry may not be optimal in arbitrary three-dimensional space. If the design requirements are not satisfied, it becomes necessary to manually redefine the coil design surface and repeat the design process. To address this limitation, we are conducting research aimed at integrating global optimization

techniques with the stream function method in order to enable automatic three-dimensional optimization of coil geometries, including the optimization of the coil design surface itself [4], [5]. However, the use of a pseudoinverse matrix in the stream function method for coil shape optimization is computationally expensive. As a result, performing global optimization iteratively leads to significantly prolonged computation times. Efficient strategies for reducing the computational burden are therefore essential for the practical application of such optimization frameworks. However, by applying the Adaptive Cross Approximation (ACA) method [4], the computational cost can be significantly reduced, enabling its application to global optimization.

Fig. 1 shows an examples of a coil designed using the stream function method to generate a gradient magnetic field. Fig. 1(a) illustrates the case where the coil design surface is a flat surface, while Fig. 1(b) shows the case where the coil design surface is a sphere. In the planar case of Fig. 1(a), the contour lines are densely clustered near the center. If such a pattern is converted into an actual coil, it becomes difficult to ensure sufficient spacing between adjacent wires. On the other hand, in the spherical design of Fig. 1(b), there are no regions where the contour lines are densely packed throughout the entire surface. As a result, the coil pattern is relatively easier to fabricate [6].

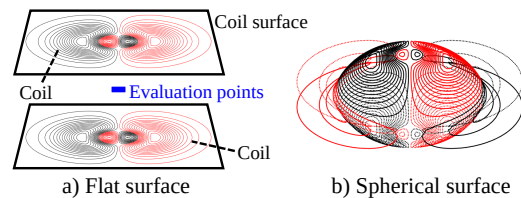


Fig. 1. Coil Design Surface and Contour Lines.

II. CONVENTIONAL STREAM FUNCTION METHOD

Fig. 2 shows the computational flow of Truncated Singular Value Decomposition (TSVD). In the stream function method, the coil design surface and the magnetic field evaluation surface are defined by the user. Let M represent the total number of magnetic field components to be evaluated, which is given by the number of discretized points on the evaluation

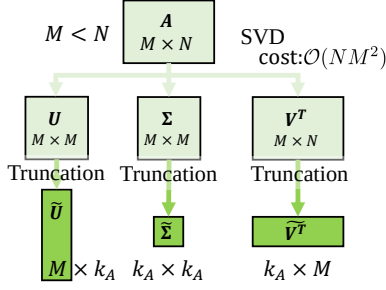


Fig. 2. Computation Process Using TSVD

surface multiplied by the number of field components considered at each point. Let N denote the number of discretization points defined on the coil design surface. The stream function, as introduced in the preceding section, can be formulated accordingly.

$$\mathbf{A}_{(M \times N)} \varphi_{(N \times 1)} = \mathbf{B}_{(M \times 1)} \quad (1)$$

Here, $\mathbf{A}$ is a matrix with the Biot-Savart law as its kernel, $\mathbf{B}$ represents the magnetic field distribution generated by the designed coil, and φ is the discretized stream function. The numbers in parentheses indicate the number of rows and columns of the matrices (or vectors), respectively. In general, problems handled using the stream function method satisfy the condition $M < N$. The parameter k_{mode} not only eliminates the ill-posedness of the matrix but also plays a role in limiting the magnitude of φ (i.e., the magnitude of the current density). Considering the trade-off between the accuracy of the generated magnetic field and the magnitude of the current, it is necessary to determine an engineering-optimal value of k_{mode} . Since TSVD uses SVD, the computational complexity is $\mathcal{O}(NM^2)$.

III. METHOD COMBINING TSVD AND ACA+

Fig. 3 shows the computational flow when ACA+ is implemented in addition to the TSVD calculation. As in the previous section, we deal with the problem formulated by (1). First, the matrix $\mathbf{A}$ is decomposed into two matrices, $\mathbf{C}$ and $\mathbf{D}$, using ACA+. Compared to decomposition by SVD, the matrix decomposition using ACA+ offers significantly lower computational cost.

$$\mathbf{A}_{(M \times N)} = \mathbf{C}_{(M \times k_{ACA})} \mathbf{D}_{(k_{ACA} \times N)} \quad (2)$$

Here, k_{ACA} is the rank of the low-rank approximation obtained by ACA+, and its value depends on the approximation accuracy specified by the user. In general, $k_{ACA} \ll N, M$. First, the matrix $\mathbf{C}$ is decomposed using SVD. The subscript C indicates that the matrix is generated from matrix C via SVD. We then define a new matrix $\mathbf{E}$ as follows.

$$\mathbf{E}_{(k_{ACA} \times M)} \equiv \Sigma_{\mathbf{C}}^T \mathbf{V}_{\mathbf{C}}^T \mathbf{D}_{(k_{ACA} \times M)} \quad (3)$$

The matrix $\mathbf{E}$ is decomposed using SVD. The subscript E indicates that the matrix is generated from matrix E via SVD. By introducing a truncation parameter k_{mode} (where $k_{\text{mode}} < M$) and truncating the singular value decomposition accordingly, the stream function φ can be approximated as follows:

$$\varphi_{(M \times 1)} \approx \tilde{\mathbf{V}}_{\mathbf{E}}^T \Sigma_{\mathbf{E}}^{-1} \tilde{\mathbf{U}}_{\mathbf{E}}^T \mathbf{U}_{\mathbf{C}}^T \mathbf{B}_{(N \times 1)} \quad (4)$$

The computational time is primarily determined by the TSVD operations on $\mathbf{C}$ and $\mathbf{E}$, with a total complexity of $\mathcal{O}(Mk_{\text{mode}}^2)$. By incorporating ACA+, the computation can be accelerated by approximately a factor of $(M/k_{\text{mode}})^2$.

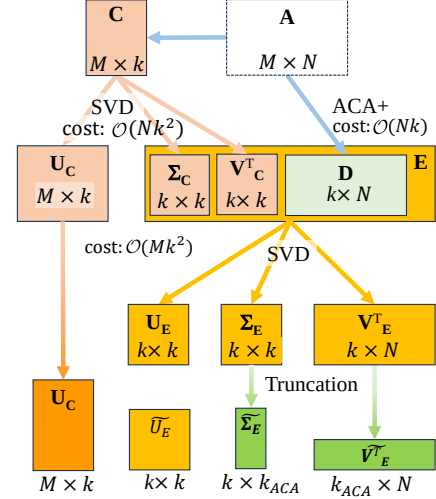


Fig. 3. Using ACA+, the matrices $\mathbf{C}$ and $\mathbf{D}$ are generated. SVD is applied to $\mathbf{C}$ to obtain $\Sigma_{\mathbf{C}}$ and $\mathbf{V}_{\mathbf{C}}$. Using these along with $\mathbf{D}$, the matrix $\mathbf{E}$ is defined, and TSVD is then performed on $\mathbf{E}$.

TABLE I
DIFFERENCE IN STREAM FUNCTIONS OBTAINED USING
TSVD AND THE PROPOSED METHOD

	TSVD	TSVD + ACA+	k_{ACA}	Calculation error (%)
Matrix (s)	5.60	0.04	15	0.08
SVD (s)	5.31	0.61	20	0.02

Comparison of calculation speed

Calculation error with TSVD

IV. NUMERICAL COMPUTATION PROBLEM

The benchmark problem used for evaluating computational time is illustrated in Fig. 1. The magnetic field evaluation region is a square area of $1.5 \text{ mm} \times 1.5 \text{ mm}$, discretized into a 51×51 regular grid. In accordance with standard NMR coil design practices, only the B_z component of the magnetic field is considered. Therefore, the total number of evaluation points is $M = 2601$. In this study, we consider the design problem of generating a linear gradient magnetic field such that $B_z \propto x$ at the specified evaluation points. The coil design surface is defined as a square region of $150 \text{ mm} \times 150 \text{ mm}$, discretized into a 71×71 grid, resulting in $N = 5401$ design elements. As shown in table I, the computation time and computational error required for matrix generation and decomposition are compared between the conventional TSVD method and the proposed ACA acceleration method.

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