

The Kernel Method for the inverse obstacle problem

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In this contribution a new real-time imaging method for the solution of the inverse obstacle problem is proposed. The method has its roots in some peculiar properties of the Neumann-to-Dirichlet operator representing the boundary measurements, recasting the original problem in the form of evaluating eigenpairs of a proper boundary operator.

The method has a very simple and stable numerical implementation and exhibits good performance in the presence of a realistic noise level.

Index Terms—Inverse obstacle problem, Kernel Method, Neumann-to-Dirichlet map, Soft-Field tomography.

I. INTRODUCTION

Soft-field tomography techniques have attracted researchers for decades because of the possibility of obtaining tomographic images with low-cost, portable equipment without the need for ionizing radiation. Despite the interest, very few methods capable of real-time performance have been proposed in the literature so far. The first real-time method to be proposed is the Linear Sampling Method [1], followed by the Factorization Method [2]. The Enclosure Method [3], the MUSIC algorithm [4] and the Monotonicity Principle based method [5] complete the list of available real-time methods.

The focus of this contribution is on the inverse obstacle problem, where the aim is to retrieve the shape, position and dimension of one or more unknown anomalies embedded in a known background material, by probing the system with a low-frequency electromagnetic excitation and making measurements on the boundary. In this context, a new real-time imaging method is proposed. The method is detailed for Electrical Resistance Tomography (ERT), as example of Soft-Field tomography technique, but it is meant for every physical problem modeled by a Laplace-type equation.

In ERT, the electromagnetic excitation is given by a DC electrical current density imposed on the boundary, while the resulting voltage potential on the boundary is measured. This technique finds several applications in different fields ranging from the subsoil inspection [6] to the biomedical area [7], where it is employed for the detection of pulmonary embolism, for example.

The measurement operations in ERT are modeled by the so-called Neumann-to-Dirichlet (NtD) operator, as the operator mapping the applied currents into the applied voltages. This operator plays a key role in developing the imaging method, which has its roots in the peculiar properties of the latter.

In Section II, after a precise statement of the problem, it is shown as the inverse obstacle problem can be led back to an eigenvalue problem involving the NtD operator. Finally in Section III, numerical examples show the effectiveness of the proposed method.

II. THE KERNEL METHOD

Let Ω be the region occupied by the conductive material under tomographic inspection and let $A \subset \Omega$ be the anomalous region (see Figure 1). The related electrical conductivity

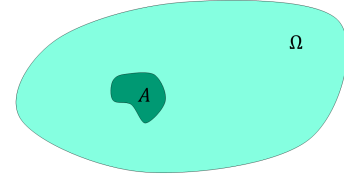


Fig. 1. The inverse obstacle problem: the conductive material occupies the domain Ω , while the unknown anomaly to be reconstructed is denoted by A .

$\sigma_A(x) \in L_+^\infty(\Omega)$ takes the form

$$\sigma_A(x) = \begin{cases} \sigma_{bg} & \text{in } \Omega \setminus A \\ \sigma_a & \text{in } A. \end{cases} \quad (1)$$

The forward problem in ERT, i.e. evaluating the boundary potential when a DC electrical current density g is applied to the boundary $\partial\Omega$, can be given in terms of electrical scalar potential as

$$\begin{cases} \nabla \cdot (\sigma_A(x) \nabla \varphi_A(x)) = 0 & \text{in } \Omega \\ \sigma_A(x) \partial_n \varphi_A(x) = g(x) & \text{on } \partial\Omega, \end{cases} \quad (2)$$

where φ_A is the electrical scalar potential, i.e. the electrical field is $\mathbf{E}_A = -\nabla \varphi_A$, and ∂_n denotes the outer normal derivative defined on the boundary.

In order to state the inverse problem, instead, it is necessary to properly represent the boundary measurements. This can be effectively done by the NtD operator defined as

$$\Lambda_A : g(x) \in L^2(\partial\Omega) \mapsto \varphi_A(x)|_{\partial\Omega} \in L^2(\partial\Omega).$$

In other words, the operator Λ_A maps the applied current on the boundary g into the measured voltages $\varphi_A|_{\partial\Omega}$, when the electrical conductivity is σ_A . Solving the inverse problem means to retrieve A starting from the knowledge of Λ_A .

The proposed method, named Kernel Method, moves from some properties of the NtD operator which reflect some physical properties of the steady conduction. Consider the two configurations depicted in Figure 2, that is, the actual one and the reference configuration without anomalies ($A = \emptyset$) with the related NtD operator Λ_{bg} . It is possible to prove that if there exists a boundary current g such that the measured voltages on the boundary are the same for both configurations, that is $\Lambda_A g = \Lambda_{bg} g$, then the electrical current density flows outside the anomalous region when g is applied to the reference configuration (see Figure 2(b)).

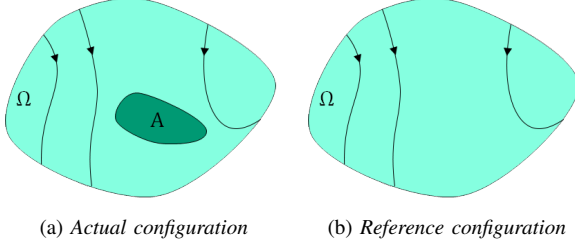


Fig. 2. Left: actual configuration. Right: reference configurations. Both configurations are driven by the same boundary current g , such that $\Lambda_A g = \Lambda_{bg} g$

Hence, the imaging method consists in (i) finding a boundary current g such that $\Lambda_A g = \Lambda_{bg} g$, (ii) solving problem (2) for an electrical conductivity equal to σ_{bg} and g as boundary data and (iii) retrieving the anomaly as the region in which the current density is vanishing.

Unfortunately, due to the Unique Continuation Principle [8] such boundary data does not exist. However, it is possible to approximate the sought boundary data with an eigenfunction of $\Lambda_A - \Lambda_{bg}$ related to a sufficiently small eigenvalue. Hence, the imaging method is modified as follows

- Compute experimentally or numerically Λ_{bg} ;
- measure Λ_A ;
- compute the eigenfunctions and the eigenvalues of $\Lambda_A - \Lambda_{bg}$;
- select an eigenfunction $\bar{g}_n$ related to a sufficiently small eigenvalue $\bar{\lambda}_n$ of $\Lambda_A - \Lambda_{bg}$;
- compute the solution of problem (2) when the electrical conductivity is σ_{bg} and the applied boundary data is $\bar{g}_n$;
- retrieve the unknown anomaly A as the region in which the electrical current density is vanishing.

III. NUMERICAL EXAMPLES

The effectiveness of the proposed method is shown by some examples developed in a numerical environment. Specifically, the conductive domain Ω is a circle with radius 10 cm, while different geometries for the unknown inclusion A are tested. The background conductivity is equal to $\sigma_{bg} = 200 \text{ S/m}$, while the anomalous region is characterized by an electrical conductivity equal to $\sigma_a = 1 \text{ S/m}$. The case of anomalies less conductive than the background is particularly significant in applications, where fatigue defects in metals shown a conductivity lower than the health material.

The simulations are carried out by an in-house FEM code. The domain Ω is discretized by means of triangular elements

and first-order shape functions have been employed. The voltage on the boundary is represented by a piece-wise linear function, while the electrical current density by a piece-wise constant function. The discrete counterpart of the operator Λ is, then, a linear operator $\mathbf{R} : \mathbb{R}^{N_b-1} \rightarrow \mathbb{R}^{N_b-1}$, where N_b are the number of elements on the boundary. The simulated data are, then, corrupted by a synthetic noise on the third significant digit of the simulated noise-free data.

In Figure 3 the obtained reconstructions are reported. As it can be seen, the overall quality is good, showing the capability of distinguishing different components of a non connected anomaly.

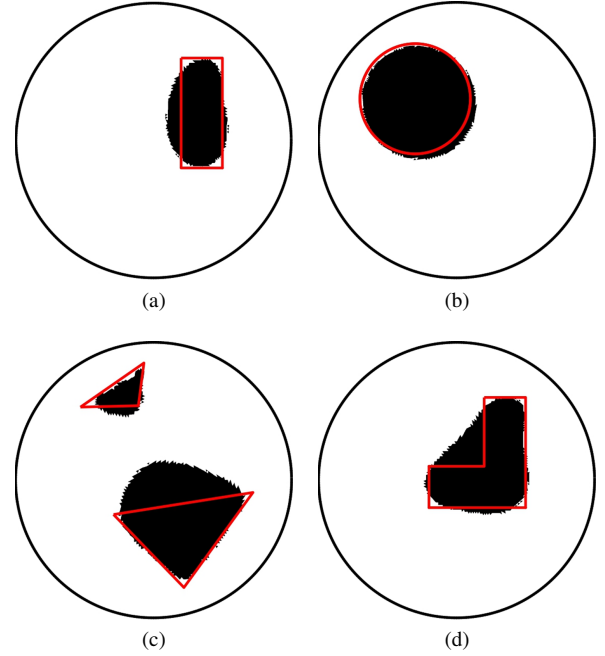


Fig. 3. Reconstructions obtained by the Kernel Method in the presence of synthetic noise on the simulated data. The noise affects the third significant digit of the noise-free data

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