

Resolution of Linear Inverse Source Problems using Physics-Informed Neural Networks

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A physics-informed neural network (PINN) is developed to find the bases of the unknown source in the linear inverse sources problems (ISPs). The singular value decomposition (SVD) is applied to the Green’s matrix of the ISP system. The PINN solves a set of bases of the unknown source by using the projection of the measurement data onto the left singular vectors from the SVD as the label data, and the PINN training results show that the right singular vectors from the SVD can serve as a set of bases of the unknown source in the linear ISPs. The conclusions benefit future explorations on the SVD-based solution approaches for linear ISPs.

Index Terms—Magnetostatic Inverse Problems, Physics-Informed Neural Networks, Singular Value Expansion.

I. INTRODUCTION

THE inverse source problems (ISPs) in electromagnetism require to determine the source producing the measured (electromagnetic) fields, under the assumption of complete knowledge of the environment [1]. The linear behavior with respect to the sources of all the components and materials presents in the system can be assumed in many engineering applications, such as in the determination of currents in the power transmission lines, in the quality assessment of quadrupole magnets in accelerators, and in the mapping of magnetization in permanent magnets [2-3]. Developing a stable inverse algorithm is crucial in engineering applications, enabling features including the accelerated computation and the robustness to noisy measurement data.

In this work, the bases of the unknown source are explored to provide a solution approach for linear ISPs. A Green’s matrix which consists of the observations under unit source excitation is constructed first, and a physics-informed neural network (PINN) is developed to identify a set of bases for the unknown source. The identified bases facilitate further solutions approaches and explorations on the ISPs.

II. INVERSE SOURCE PROBLEMS AND SVD

Taking advantage of the linearity and leveraging on the propriety of the mathematical operators linking sources to measurements, the usual approach to the resolution of ISPs is to adopt some finite-dimensional representation of the sources distributions, and, considering the discrete nature of the measurements, to formulate the direct problem as:

$$\mathbf{m}_{m \times 1} = \mathbb{G}_{m \times n} \mathbf{p}_{n \times 1}, \quad (1)$$

where $\mathbf{m}$ is the m -dimensional array of measurements, $\mathbf{p}$ is the n -dimensional array of the degrees of freedom (DOF) describing the source, and the elements of Green’s matrix $\mathbb{G}$ represent the contribution to each measurement by each unit source. Different approaches can lead to different meanings of the DOF. For example, if the unknown sources are represented

by the cables in a bundle, the DOF will correspond to the amplitude of currents in each cable.

As a general rule, one assumes that $m > n$, *i.e.* more measurements than DOF, and $\mathbb{G}$ exhibits the ill-posed nature of the inverse relationship between measurements and DOFs. The singular value decomposition (SVD) of $\mathbb{G}$ yields:

$$\mathbb{G}_{m \times n} = \mathbb{U}_{m \times m} \Sigma_{m \times n} \mathbb{V}_{n \times n}^T, \quad (2)$$

where $\mathbf{u}_i$ in $\mathbb{U}$ and $\mathbf{v}_i$ in $\mathbb{V}$ are the left and right singular vectors, respectively; σ_i in the diagonal matrix Σ is the square root of the singular values. The unknown source of the ISP is obtained by using the pseudo-inverse. However, the noise in the measurement can affect the results significantly due to the ill-posedness of the inverse problem. The truncated SVD method, which exploits the bases of both the measurement data and the unknown source, can be used to enhance the noise rejection capability. Assumed $m > n$ in $\mathbb{G}$, the $\mathbf{u}_i$ can be used as a set of bases for $\mathbf{m}$. One chooses the first K bases, and the unknown source is obtained by

$$\mathbf{p} \approx \sum_{i=1}^K \frac{\langle \mathbf{m}, \mathbf{u}_i \rangle}{\sigma_i} \tilde{\mathbf{v}}_i, \quad (3)$$

where $\tilde{\mathbf{v}}_i$ is the chosen basis to represent the source. By assuming $K < n$, the truncated SVD grants a satisfactory noise rejection capability and allows a reliable ISP resolution. Therefore, determining the bases for the source, *i.e.* $\tilde{\mathbf{v}}_i$ in (3), is the key to the truncated SVD.

III. PINN EXTRACTION OF BASES FOR UNKNOWN SOURCES

A PINN is constructed to find a bases for the source. A system for measuring the magnetization of a permanent magnet is considered as a general study case, as shown in Fig. 1(a). The presented case can be readily extended to other ISPs, provided that relation between the source and the measurement is linear. One considers a cubic magnet with side length of 1 cm, consisting of $2 \times 2 \times 2$ identical magnetically-isolated cubic grains, and each grain is uniform magnetized only in the z direction. Each grain is an independent source; thus the case is

a multi-source scenario. The magnetization of the entire magnetic is denoted as $\mathbf{M}^{\text{true}}$, with element $\mathbf{M}_i^{\text{true}}$ denoting the magnetization of the i^{th} grain. The magnetic field generated by all the grains is measured on 2 square planes above and below the magnet, respectively, as shown in Fig. 2. Both observation planes have the side length of 6 cm. In practical cases, the locations and numbers of observations should be decided based on the geometry of the magnet and the required measurement accuracy; the measurement data can be obtained by Gauss probes. On each plane, the x , y , and z components of the field intensity $\mathbf{H}^{\text{obs}}$ are measured on a uniform 60×60 grid. By applying the Biot-Savart law, the $\mathbf{H}^{\text{obs}}$ at location $\mathbf{r}$ is obtained by calculating the gradient of the magnetic scalar potential φ_m :

$$\varphi_m(\mathbf{r}) = \sum_{i=1}^8 \left[\frac{1}{4\pi} \int_S \frac{\mathbf{M}_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{S}' \right], \quad (4)$$

$$\mathbf{H}^{\text{obs}}(\mathbf{r}) = -\nabla \varphi_m(\mathbf{r}), \quad (5)$$

where $\mathbf{S}'$ is the outward-directed surface of each grain, $\mathbf{r}'$ is spatial locations inside the magnet. The Biot-Savart law requires no boundary conditions at the outer region. Then, a PINN is constructed to solve the inverse problem, as shown in Fig. 1(b). A fully-connected neural network (NN) is constructed to approximate the source $\mathbf{M}_i^{\text{PINN}}$. The NN inputs are the spatial coordinates of the center of each grain, and the outputs are the $\mathbf{M}_i^{\text{PINN}}$ in the corresponding grain. The φ_m is obtained by using the NN outputs and (4), and the $\mathbf{H}^{\text{PINN}}$ is then obtained by applying the finite difference to (5). The discrepancy between the $\mathbf{H}^{\text{PINN}}$ obtained by PINN and the observation is used to evaluate the training loss and thus guides the NN training.

The goal of the PINN is to identify a set of bases for the source $\mathbf{M}^{\text{true}}$. One first calculates the Green's matrix $\mathbb{G}$ according to the configuration in Fig. 1(a), then performs the SVD to $\mathbb{G}$ and obtains $\mathbb{U}$ and $\mathbb{V}$. One then generates a random $\mathbf{M}^{\text{true}}$ and calculates the resultant magnetic field $\mathbf{H}^{\text{obs}}$ on the measurement planes. As stated in section II, the $\mathbf{u}_i$ can be used as a set of bases for the measurement data $\mathbf{H}^{\text{obs}}$. One projects $\mathbf{H}^{\text{obs}}$ onto each basis:

$$\mathbf{H}_i^{\text{proj}} = \langle \mathbf{H}^{\text{obs}}, \mathbf{u}_i \rangle \mathbf{u}_i, \quad i = 1, \dots, 8. \quad (6)$$

One then builds 8 NNs, each with the label data $\mathbf{H}_i^{\text{proj}}$ and is trained 8000 epochs. After training, the source $\mathbf{M}_i^{\text{PINN}}$ that generates the $\mathbf{H}_i^{\text{proj}}$ is obtained by a forward inference of the NN. According to the linearity, the unknown source is recovered by summing up all $\mathbf{M}_i^{\text{PINN}}$ obtained from the PINN.

Next, one projects $\mathbf{M}^{\text{true}}$ onto $\mathbf{v}_i$ by using:

$$\mathbf{M}_i^{\text{true}} = \langle \mathbf{M}^{\text{true}}, \mathbf{v}_i \rangle \mathbf{v}_i, \quad i = 1, \dots, 8, \quad (7)$$

then compares the 8 trained $\mathbf{M}_i^{\text{PINN}}$ with the $\mathbf{M}_i^{\text{true}}$, the results being summarized in Table I. It is clearly observed that $\mathbf{M}_i^{\text{PINN}}$ agree well with $\mathbf{M}_i^{\text{true}}$. In other words, the $\mathbf{M}_i^{\text{PINN}}$, which is found by the PINN using the projected $\mathbf{H}^{\text{obs}}$ onto bases $\mathbf{u}_i$, is

identical to the $\mathbf{M}_i^{\text{true}}$ which is obtained by projecting $\mathbf{M}^{\text{true}}$ onto $\mathbf{v}_i$. Consequently, the $\mathbf{u}_i$ and $\mathbf{v}_i$ can be used as the bases for the measurement data and the unknown source, respectively.

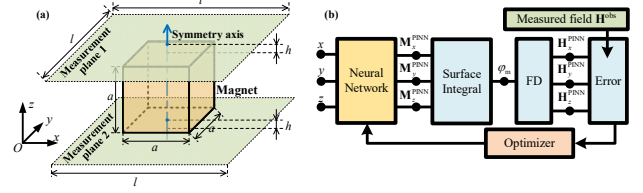


Fig. 1. (a) The system for measuring the magnetization of a permanent magnet, and (b) the architecture of the PINN.

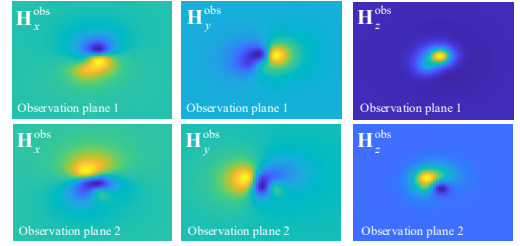


Fig. 2. The magnetic field intensity generated by the random magnetization of the magnet

Mode	Relative error	Mode	Relative error
1	2.69e-3 %	5	2.17e-3 %
2	6.03e-4 %	6	1.44e-3 %
3	7.01e-3 %	7	7.63e-4 %
4	9.99e-4 %	8	5.17e-4 %

IV. DISCUSSION

The approach presents a number of advantages, which will be discussed in detail in the full paper:

- 1) The impact of noise can be easily reduced by using a reduced number of “modal networks”, which amounts to truncating the expansion (2);
- 2) After the training, each modal network will have built an internal representation of the modal term in the source;
- 3) Not all the networks are requested to have the same level of complexity: modal NN associated with lower order modes can be simpler than those ones associated with higher modes;
- 4) Once the modal NN ensemble is trained, to solve the ISPs for a different sets of measurements $\mathbf{m}'$, it will be sufficient to “rescale” the output corresponding to $\mathbf{m}$ according to the factor $\langle \mathbf{m}', \mathbf{v}_i \rangle / \langle \mathbf{m}, \mathbf{v}_i \rangle$.

The latter point is a direct consequence of the linearity, and is equivalent to the re-normalization of modal coefficients in (2).

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