

Treatment of curved faces with the generalized dual complex

Silvano Pitassi¹, Francesco Trevisan², and Ruben Specogna²

¹IMAG, CNRS, University of Montpellier, Montpellier, France

²DPIA, EMCLab, University of Udine, Udine, Italy

We present a new geometrical numerical method that is able to deal with grids with curved (i.e., non-planar) faces. The fundamental problem with this kind of grids is that up-to-now no low-order convergent numerical method for diffusion-like problems is able to use only one discrete flux unknown for every curved face. Our new numerical schemes achieves exactly this result. The basic idea behind our method is to devise a new dual grid, which, contrary to standard approaches, is not constructed according to barycentric subdivision but instead is determined from the very geometry of curved faces. Once such a “generalized dual grid” is available, the methodology of our numerical method is exactly the same of the standard discrete geometric approaches, in the sense that the new dual grid is used in the scheme just like the standard barycentric one. Hence, all the solid foundations of geometric methods are immediately available also to our new numerical method.

Index Terms—barycentric dual grid, curved faces, discrete geometric approaches, mimetic finite difference methods

I. INTRODUCTION

TETRAHEDRAL and structured hexahedral grids have been the major type of grid used in engineering simulations since they are the simplest to generate and manage. Nonetheless, one their main limitation relies in the limited approximation of complex simulations domains like the ones having sharp and irregular surfaces. In order to deal with such complex simulation domains, nowadays polyhedral grids have been used in a growing number of numerical methods. As a matter of fact, polyhedral elements can be joined, split, or modified by introducing additional points, edges and faces to model complex geometric features. Moreover, we point out that polyhedral grids offer many additional advantages over tetrahedral and structured grids besides the above mentioned applications, like for instance in adaptive mesh refinement strategies [1].

Despite of many advantages of polyhedral grids, many applications require also grids with *curved* (i.e. non-planar) faces. In this kind of applications, grid elements are not actually polyhedrons since their boundary does not decompose as a disjoint union of polyhedra. Instead, they can be formally defined as the image of a polyhedron under a continuous map subject to suitable regularity assumptions and we refer to them as *generalized polyhedral elements*; see Fig. 1.

Unfortunately, most of the existing low-order numerical methods for polyhedral grids cannot be directly applied to generalized polyhedral grids. In fact, as demonstrated in [4], [5] and as we will also numerically prove in Section III, the standard *Mimetic Finite Difference (MFD)* method [1] and *Discrete Geometric Approaches (DGA)* [2] do not converge on such grids. Instead, to deal with generalized polyhedral grids the only low-order mimetic method that we are aware is [4], [5]. However, it uses three flux unknowns for every curved face, hence three times more the number of flux unknowns used by the standard MFD and DGA methods for the discretization of a diffusion problem.

In this contribution, we present a new numerical method on generalized polyhedral grids which uses only one discrete

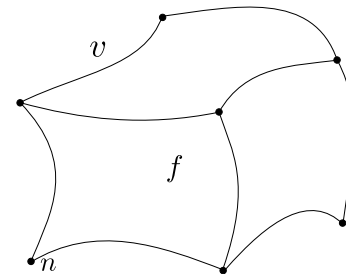


Fig. 1: A generalized polyhedron v obtained as a continuous deformation of a cube. The face f is in general non-planar since its four vertices do not lie all in a common plane.

flux unknown for every curved face and we present it in Section II. The fundamental idea at the core our approach is the construction of a new dual grid structure which generalizes the standard barycentric one. Indeed, for a polyhedral grid, the new dual grid reduces to the standard barycentric one. Instead, for a generalized polyhedral grid, it allows the new numerical method to perform much better than the method in [4], [5] given the reduced number of discrete flux unknowns, while also achieving the same expected first order of convergence. The methodology of our numerical method then is exactly the same of the standard discrete geometric approaches, in the sense that the new dual grid is used in the scheme just like the standard barycentric one. Hence, all the solid foundations of geometric methods are immediately available also to our new numerical method. The present contribution represents the beginning of our analysis of this new promising discretization methodology, as shown by preliminary numerical examples in Section III.

II. GENERALIZED DUAL GRID

To introduce the new dual grid structure let us first recall the definition and fundamental properties of the barycentric dual grid.

Let K be a polyhedral grid. The 1-skeleton of the dual grid $\hat{K}$ of K is constructed by duality process as follows. To each element v corresponds the dual node $\tilde{n}_v \in \mathbb{R}^3$. To each face

f in the boundary of v , corresponds the segment $(\tilde{\mathbf{n}}_v, \mathbf{b}_f)$ of $\mathbb{R}^3$ joining $\tilde{\mathbf{n}}_v$ with the barycenter $\mathbf{b}_f$ of f . We denote by $\tilde{\mathbf{e}}_{f|v}$ the edge vector associated with segment $(\tilde{\mathbf{n}}_v, \mathbf{b}_f)$ oriented with respect to the internal orientation of face f according to the right-hand rule. The vectors $\tilde{\mathbf{e}}_{f|v}$ constructed from the barycentric dual grid [3], [6] satisfy the following fundamental property.

Theorem 1. *The following tensor identity holds*

$$\sum_{f \in F(v)} \tilde{\mathbf{e}}_{f|v} \otimes \mathbf{f} = |v| \mathbb{I}_3, \quad (1)$$

where $\mathbf{f}$ is the face vector of face f , $|v|$ is the volume of element v , and $\mathbb{I}_3$ is the identity matrix of order 3.

Now suppose that K is a generalized polyhedral grid. To define the corresponding 1-skeleton of the generalized dual grid the idea is to mimic the properties of the barycentric dual grid. To start with, to each face f in the boundary of a generalized polyhedron, corresponds the segment $(\tilde{\mathbf{n}}_v, \mathbf{b}'_f)$ of $\mathbb{R}^3$, where now point $\mathbf{b}'_f$ is an unknown to be determined later. Next, we denote by $\tilde{\mathbf{e}}'_{f|v}$ the edge vector associated with segment $(\tilde{\mathbf{n}}_v, \mathbf{b}'_f)$ and oriented with respect to f as described before. Finally, we require that vectors $\tilde{\mathbf{e}}'_{f|v}$ satisfy (1) for all generalized polyhedrons v of K . It can be proven, as we will thoroughly discuss in final version of this contribution, that the set of all these tensor identities can be encoded into a linear system of equations having as unknowns the set of points $\{\mathbf{b}'_f\}_{f \in F}$. In fact, note that each vector $\tilde{\mathbf{e}}'_{f|v}$ depends linearly on unknown point $\mathbf{b}'_f$. Hence, the construction of the generalized dual grid hinges on the solution of a special linear system depending on the geometry of the generalized polyhedral grid.

III. PRELIMINARY NUMERICAL RESULTS

We focus on a stationary current conduction problem in a cubic domain $\Omega = [0, 1]^3$ which reads

$$\begin{aligned} \mathbf{J} &= \sigma \mathbf{E}, \\ \nabla \cdot \mathbf{J} &= 0, \\ \nabla \times \mathbf{E} &= \mathbf{0}, \end{aligned}$$

with $\mathbf{J}$ the current density, $\mathbf{E}$ the electric field and σ the electric conductivity. We construct a generalized polyhedral grid on Ω as follows. Starting from an initial polyhedral grid on Ω made of hexahedral elements, we randomly perturb internal nodes position. In this way, each element has curved faces since nodes of at least one face of every element do not all lie on a plane in $\mathbb{R}^3$. The boundary conditions have been set to generate a uniform current density of amplitude 1 A and pointing down the vertical axis.

In Fig. 2 we show the resulting reconstructed vector field, where we see that the expected uniform current density is reconstructed exactly in each generalized polyhedron. In Fig. 3 we show the associated generalized dual grid structure. We note that computed points $\{\mathbf{b}'_f\}_{f \in F}$ do not coincide with true face barycenters as in the standard barycentric dual grid. This is the crucial aspect of the new dual grid construction. As a matter of fact, if a standard barycentric dual grid is used like

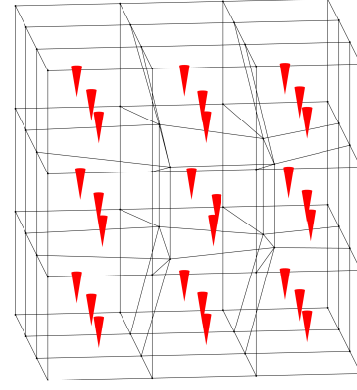


Fig. 2: Generalized polyhedral grid partitioning the cubic resistor where all internal faces are curved. Uniform current density reconstructed inside each generalized polyhedral element: it coincides with the uniform exact field.

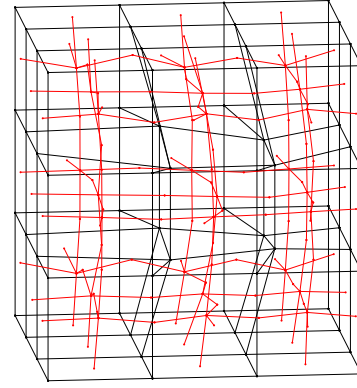


Fig. 3: Generalized polyhedral grid partitioning the cubic resistor where all internal faces are curved. In red, generalized dual grid structure associated with the computed points $\{\mathbf{b}'_f\}_{f \in F}$.

in the DGA method, then we do not achieve consistency in that the reconstructed solution field is not uniform. In order to quantitatively measure such a consistency error, we have considered the relative error on the numerical approximation of the dissipated power of the resistor with respect to its analytical value. For the considered curved grid shown in Fig. 3, we get a 3% relative error.

REFERENCES

- [1] K. Lipnikov, G. Manzini, and M. Shashkov, "Mimetic finite difference method," *J. Comput. Phys.*, vol. 257, pp.1163-1227, 2014.
- [2] L. Codecasa, R. Specogna, and F. Trevisan, "Base functions and discrete constitutive relations for staggered polyhedral grids," *Comp. Meth. Appl. Mech. Eng.*, vol. 198, pp.1117-1123, 2009.
- [3] S. Pitassi, R. Ghiloni, and F. Trevisan, and R. Specogna, "The role of the dual grid in low-order compatible numerical schemes on general meshes," *J. Comput. Phys.*, vol. 436, art. no. 110285, 2021.
- [4] F. Brezzi, K. Lipnikov, and M. Shashkov, and V. Simoncini, "A new discretization methodology for diffusion problems on generalized polyhedral meshes," *Comp. Meth. Appl. Mech. Eng.*, vol. 196, pp. 3682-3692, 2007.
- [5] F. Brezzi, K. Lipnikov, and M. Shashkov, "Convergence of mimetic finite difference method for diffusion problems on polyhedral meshes with curved faces," *Math. Mod. Meth. Appl. Scien.*, vol. 16, pp.275-297, 2006.
- [6] S. Pitassi, R. Ghiloni, and R. Specogna, "New Magic Formula Demonstration Shows Unexpected Features of Geometrically Defined Matrices for Polyhedral Grids," *IEEE Trans. Magn.*, vol. 57, pp. 1-4, 2021