

The latest developments in real-time tomography of nonlinear materials

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The electromagnetic tomography of nonlinear materials is a quite new topic in literature. The lack of results can be ascribed to the complexity in dealing with the inverse problem, which is highly nonlinear and ill-posed even in the presence of linear materials. Despite this, nonlinear materials find application in a large variety of different contexts, where tomographic inspection techniques are also needed.

In this contribution, the work of the research group is summarized starting from the mathematical formulation for the inverse problem in the nonlinear scenario ending up to a real-time inversion method based on the Monotonicity Principle.

Index Terms—Inverse obstacle problem, Monotonicity Principle, Nonlinear materials, Soft-Field tomography.

I. INTRODUCTION

Nonlinear materials are nowadays widely spread, for instance superconductors, in applications like energy storage or magnetic levitation; ceramic materials, employed in field-grading applications; semiconductors or even human tissues can exhibit a nonlinear electrical conductivity. In most of these applications, there is a need for low-cost tomographic techniques able to inspect these materials.

However, as quoted in [1] “... the mathematical analysis for inverse problems governed by nonlinear Maxwell’s equations is still in the early stages of development”. Indeed, the inverse problem of retrieving the spatial behavior of some unknown material property, starting from boundary measurements, is severely ill-posed and nonlinear even in the presence of linear materials and the complexity exponentially grows when nonlinear materials are taken into account.

The aim of this contribution is to show the recent progress of the research group on the theme. Specifically a new path to address the inverse problem, based on the discovery of a Monotonicity Principle (MP) [2] in the presence of nonlinear materials, is presented. As application, the inverse obstacle problem for Electrical Resistance Tomography (ERT) is considered, where the aim is to retrieve an anomalous region immersed in an otherwise known material, by probing the system with DC voltages applied to the boundary and making measurements on the some boundary.

The MP is a key property, shared by different physical models, which is at the basis of a class of real-time inversion methods in the linear materials context [3], [4]. These methods show some peculiar features: other than the feasibility for real-time applications, which is a rare feature in this field, they are stable with respect to the measurement noise and the achievable performance in absence of noise are clearly characterized.

The discovery of the MP in the nonlinear scenario allows to set up a generalized treatment for rather arbitrary nonlinearities, in contrast with the current literature which focuses

on monomial type of nonlinear electrical conductivities of the form $\sigma(x, E) = \theta(x)E^{p-2}$ [5], [6] and some of its variants.

In the remaining part of this contribution, the MP for nonlinear materials is briefly summarized (Section II) and the real-time imaging method is presented, together with a characterization of its performance (Section III). Finally, some numerical examples show the effectiveness of the proposed method.

II. THE MONOTONICITY PRINCIPLE

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, the region occupied by the nonlinear conductor, be a Lipschitz domain and let denote with $A \subset \Omega$ the anomalous region, which represents the unknown of the inverse problem. Without loss of generality, the nonlinear electrical conductivity related to the anomalous region $\sigma_a(x, E)$ is considered greater than the one of the surrounding material $\sigma_{bg}(x, E)$, a.e. $x \in \Omega$ and $\forall E > 0$, where E is the amplitude of the electrical field.

A crucial role in dealing with the inverse problem is played by the operators representing the boundary measurements. In [1], it has been recognized the need of introducing a new boundary operator, the *average* DtN operator $\bar{\Lambda}$, to model the measurement operations in this framework. Indeed, the classical DtN operator, mapping the applied DC voltages to the measured DC current density on the boundary, fails to respect a Monotonicity Principle when passing from the linear to nonlinear case. The average DtN, instead, guarantees a MP.

If f is the applied boundary voltage, the average DtN maps f into the mean current density measured on the boundary, where the mean is computed by applying αf and averaging for α ranging from 0 to 1. In this sense, the power product $\langle \bar{\Lambda}(f), f \rangle$ represents a weighted average of the ohmic power absorbed by the body.

Introducing a test anomaly T , as an arbitrary region filled by the same material of the anomaly A , the MP, then, can be

stated in terms of the average DtN operator, i.e. in terms of measured data, as [7]

$$T \subseteq A \implies \langle \bar{\Lambda}_T(f), f \rangle \leq \langle \bar{\Lambda}_A(f), f \rangle \forall f, \quad (1)$$

where $\bar{\Lambda}_A$ and $\bar{\Lambda}_T$ are the average DtNs related to the actual configuration and to the configuration with the test anomaly, respectively. Equation (1) expresses the fact that if the region occupied by the more conductive material is enlarged, the measurements have to grow also, in the sense of the power products.

III. THE INVERSION METHOD

The real-time inversion method is meant for nonlinear anomalies embedded in a linear background, i.e. $\sigma_{bg}(x, E) = \sigma_{bg}(x)$ in this section. It is derived from the negation of Equation (1) [3]

$$\exists f \mid \langle \bar{\Lambda}_T(f), f \rangle > \langle \bar{\Lambda}_A(f), f \rangle \implies T \not\subseteq A. \quad (2)$$

The above equation gives a very simple tool, a monotonicity test, to exclude from the reconstruction the test anomalies which are not included in the actual anomaly, starting only from boundary measurements. The idea is, then, to construct a covering $\{T_k\}_k$ of Ω made by test anomalies and repeating for each test anomaly the monotonicity test in (2). The reconstructed region is given by the union of all test anomalies that have not been discarded, i.e.

$$\tilde{A} = \bigcup_k \{T_k \mid \langle \bar{\Lambda}_T(f), f \rangle \leq \langle \bar{\Lambda}_A(f), f \rangle \forall f\}. \quad (3)$$

In absence of noise, the above reconstruction rule gives a set which always is between the actual anomaly A and its outer support, i.e. A plus some of its holes which are not electrically connected to the boundary [8]. In other words, anomalies without holes can be exactly reconstructed.

However, real data are always corrupted by noise. In this case, a form of regularization of the (3) is needed. In [8], after giving a proper regularized reconstruction rule, it is proved that the latter is intrinsically stable with respect to noise, that is (i) the reconstructed region $\tilde{A}$ became smaller when the noise level is decreased and (ii) for the noise level tending to zero the reconstructed set tends to a deterministic set which is the one given by (3) in absence of noise.

Finally, in [9], a *practical* implementation of the imaging method is developed. Equation (3) requires to verify the monotonicity condition for every possible excitation f , being $\bar{\Lambda}$ a nonlinear operator, which is not feasible in practice. By physical reasoning it is possible to prove that an optimal set of boundary potential can be pre-computed before any measurement and used to perform the monotonicity test. This set is formed by a finite and reasonable small number of elements, that are the boundary voltages needed to recognize if a test anomaly have to be excluded from the reconstruction.

The method as described in [9] has been tested in a numerical environment to retrieve nonlinear anomalies with the electrical conductivity in Figure 1, embedded in common steel. The results are reported in Figure 2. The overall quality is good and they have been obtained in just 30 ms on a common laptop.

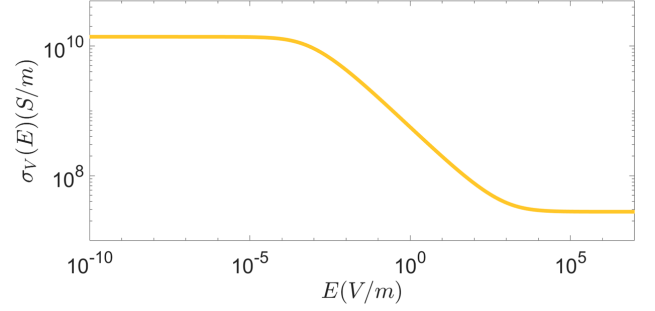


Fig. 1. Nonlinear electrical conductivity $\sigma_a(E)$ for the numerical examples.

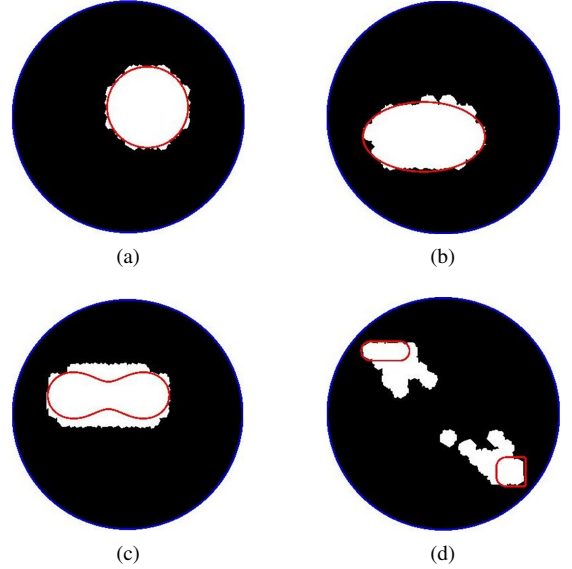


Fig. 2. Reconstructions obtained by the proposed method. The numerical data are corrupted by the noise model, and related noise level, provided by 2002 8½-Digit High Performance Multimeter by Keithley

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