

Solution of Low Frequency Open-Boundary Problems by means of FEM-QSDBCI Method

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This paper describes a variant of the FEM-DBCI (Finite Element Method – Dirichlet Boundary Condition Iteration) method for the solution of low-frequency electromagnetic problem in open boundaries. The unbounded free space around the system is truncated by means of a fictitious boundary and the Finite Element Method (FEM) is applied in the bounded domain. An unknown Dirichlet condition is assumed to hold on this boundary. This condition is expressed by means of an integral equation which involves the free space Green's function. The integration surface is selected in the middle of the most external layer of finite elements, very near to the truncation boundary, so that the integral equation becomes quasi singular (Quasi Singular DBCI, QSDBCI). The hybrid global algebraic system is solved by means of an iterative algorithm, by which a first guess for the unknown Dirichlet condition is iteratively improved. The method is suitable for the solution of electrostatic, magnetostatic, skin effect and eddy current problems. Comparisons are made with the FEM-BEM, concluding that FEM-QSDBCI is very competitive in terms of accuracy and computing time.

Index Terms—Finite element method, integral equations, open-boundary problems, electrostatics, skin effect, GMRES.

I. INTRODUCTION

IN computational electromagnetics various techniques exist to compute electromagnetic fields in unbounded domains by means of the FEM [1], such as the Coordinate Transformations (CT) [2,3] and the hybrid FEM-BEM (Boundary Element Method) [4-6].

In addition to these, the authors have devised the hybrid FEM-DBCI (Dirichlet Boundary Condition Iteration) method [7,8], in which a fictitious truncation boundary, say Γ_T , enclosing all the conductors, non-homogeneous objects and distributed sources, is introduced in order to obtain a bounded domain. By assuming an initial guess for the Dirichlet boundary condition of the unknown scalar or vector field on Γ_T , a standard FEM solution is easily obtained. Starting from this solution, a new estimate of the field on Γ_T is obtained by means of integrals over another closed surface, say Γ_M , strictly enclosed by Γ_T , the integration involving the free-space Green's function. This procedure is iterated until convergence takes place. This method has been successfully applied to the solution of various static and quasistatic electromagnetic problems.

This paper presents a modified version of the FEM-DBCI which allow us to put the integration surface Γ_M very near to the truncation boundary Γ_T in order minimize the FEM mesh of free space. In this way the integral equations are not singular (as in the FEM-BEM), but quasi singular. From this the name FEM-QSDBCI, Quasi Singular DBCI.

The paper is organized as follows: in Sect. 2 the FEM-QSDBCI formulation is described by considering a simple electrostatic problem and the solution of the resulting global system is outlined; in Sect. 3 numerical examples are given. The authors' conclusions follow in Sect. 4.

II. FEM-QSDBCI FOR ELECTROSTATIC PROBLEMS

Consider an electrostatic system constituted by volume charge distributions, voltage conductors and localized

dielectric objects, embedded in free space. In order to solve numerically this unbounded electrostatic problem, a truncation boundary Γ_T is introduced, enclosing all the conductors and non-homogeneities, but possibly leaving out some (lumped or distributed) charges. In the domain the Poisson equation holds:

$$-\epsilon_0 \nabla \cdot (\epsilon_r \nabla v) = \rho \quad (1)$$

where ϵ_0 is the free space electric permittivity, ϵ_r is the relative permittivity and ρ is the volume charge density. An unknown Dirichlet condition is assumed on Γ_T . Discretizing the domain by means of (high order) nodal finite elements, the linear algebraic system is derived:

$$\mathbf{A} \mathbf{v} = \mathbf{b}_0 - \mathbf{A}_T \mathbf{v}_T \quad (2)$$

where: $\mathbf{v}$ and $\mathbf{v}_T$ are the vectors of the nodal values of the potential v inside the domain and on Γ_T , respectively, $\mathbf{A}$ and $\mathbf{A}_T$ are sparse matrices depending on geometry and materials, $\mathbf{b}_0$ is the known term vector due to the conductor voltages and internal charges.

The unknown Dirichlet condition on the truncation boundary can be expressed through the integral equation:

$$v(\mathbf{r}) = v_{\text{ext}}(\mathbf{r}) + \int_{\Gamma_M} \left(v(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} - G(\mathbf{r}, \mathbf{r}') \frac{\partial v(\mathbf{r}')}{\partial n'} \right) ds' \quad (3)$$

where Γ_M is a closed surface enclosing the system but strictly enclosed by Γ_T , $\mathbf{r}$ and $\mathbf{r}'$ are the coordinate vectors of points on Γ_T and Γ_M , respectively, v_{ext} is the potential due to (lumped or distributed) external sources, if any, $\mathbf{n}'$ is the outward normal vector to Γ_M and $G(\mathbf{r}, \mathbf{r}')$ is the free-space Green's function:

$$G(\mathbf{r}, \mathbf{r}') = 1 / (4\pi |\mathbf{r} - \mathbf{r}'|) \quad (4)$$

The surface Γ_M is constituted by patches connecting the middle points of the edges leaving from Γ_T , as shown in Fig.1 in a mesh of second order triangles. In 2-D the patches are segments both for triangular and quadrangular finite elements, whereas in 3-D they are triangles and parallelograms for tetrahedra and quadrangles for bricks. In this way the free space in between Γ_T and Γ_M reduces to half a layer of elements (see Fig. 1).

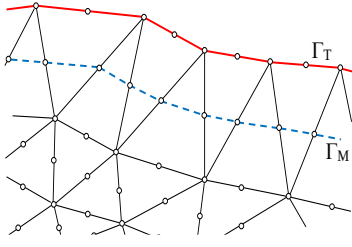


Fig. 1 - Truncation boundary Γ_T and integration curve Γ_M (dashed line) in a mesh of triangular finite elements of the second order.

In numerical form, equation (3) reads

$$v_i = v_{\text{ext}}(\mathbf{r}_i) + \sum_k \sum_m v_m \left(h_{im}^{(k)} - g_{im}^{(k)} \right) \quad (5)$$

with:

$$h_{im}^{(k)} = \int_{S_k} \alpha_m(\mathbf{r}') \frac{\partial G(\mathbf{r}_i, \mathbf{r}')}{\partial n'} ds', \quad g_{im}^{(k)} = \int_{S_k} \frac{\partial \alpha_m(\mathbf{r}')}{\partial n'} G(\mathbf{r}_i, \mathbf{r}') ds' \quad (6)$$

where S_k is the k -th patch of Γ_M , α_m is the shape function of the m -th node of the finite element in which the patch lies, v_i is the potential value at the node $\mathbf{r}_i$ on Γ_T . The integrals (6) are regular and can be computed by means of the Gauss numerical quadrature with varying accuracy. However, if node $\mathbf{r}_i$ is very near to the patch S_k , analytical formulas have been derived in 2D and 3D. In matrix form, equations (5) read:

$$\mathbf{v}_T = \mathbf{v}_{\text{ext}} + \mathbf{H} \mathbf{v}_T + \mathbf{G} \mathbf{v} \quad (7)$$

where $\mathbf{H}$ and $\mathbf{G}$ are dense matrices; matrix $\mathbf{H}$ is square by construction. Note that, unlike FEM-DBCI where equation (7) is explicit with respect to $\mathbf{v}_T$, in the method described here it is not, due to dense matrix $\mathbf{H}$, which collects all the h_{im} and g_{im} coefficients relative to the potentials v_m of nodes on Γ_T .

In order to solve the global system (2)-(7) a Conjugate Gradient (CG)-type solver for non-symmetric linear systems could be used. However this approach is very costly, due to the presence of dense parts in the global matrix. Instead consider the reduced system:

$$\mathbf{M} \mathbf{v}_T = \mathbf{k} \quad (8)$$

where: $\mathbf{M} = \mathbf{I} - \mathbf{H} + \mathbf{G} \mathbf{A}^{-1} \mathbf{A}_T$ $\mathbf{k} = \mathbf{v}_{\text{ext}} + \mathbf{G} \mathbf{A}^{-1} \mathbf{b}_0$ (9)

Vector $\mathbf{k}$ is simply built: assume $\mathbf{v}_T = \mathbf{0}$, solve (2) for $\mathbf{v}$ by the CG, compute $\mathbf{k} = \mathbf{v}_{\text{ext}} + \mathbf{G} \mathbf{v}$. Similarly, matrix $\mathbf{M}$ can be virtually used to perform matrix-vector multiplications $\mathbf{M} \mathbf{v}_T$: solve (2) for $\mathbf{v}$ with $\mathbf{b}_0 = \mathbf{0}$, compute $\mathbf{M} \mathbf{v}_T = (\mathbf{I} - \mathbf{H}) \mathbf{v}_T - \mathbf{G} \mathbf{v}$.

Thus, several non-stationary CG-type solvers for non-symmetric matrices can be used, but GMRES [9] should be preferred because it minimizes the number of matrix-by-vector multiplications, which are quite expensive in this context.

The FEM-QSDBCI method can be also applied to the analysis of other kinds of electromagnetic problems, as for example 2D time-harmonic skin effect ones.

III. NUMERICAL EXAMPLE.

Consider a transmission line made up of two copper conductors of circular cross-section of radius $R=5$ mm and centers distanced of $D=20$ mm, carrying opposite unitary sinusoidal currents at a frequency of $f=1$ kHz. This example is presented in order to compare FEM-QSDBCI and FEM-BEM.

Since this problem does not exhibit an analytical solution, we assume as reference solution the FEM-CT one in which the radius of the internal circular domain (centered at the origin) is set at a sufficiently high value $R_{CT} = 6 R = 30$ mm. The global algebraic system is solved by means of the CG solver with a computing time of $T_S=0.26$ s on a small PC.

FEM-BEM is applied by selecting the air-copper interface as the truncation boundary. In the wire the mesh is formed with 128 second-order triangles, exactly as in the previous analysis. The problem is solved by means of the GMRES within 4 iterations with an end-iteration tolerance 0.1%. The computing time rises to $1.52 T_S$ whereas the accuracy decreases to 0.08%.

FEM-QSDBCI is applied by selecting the truncation boundary Γ_T as the half circumference concentric with the wire, having a radius $R_T = 6.0$ mm. The air gap between the wire and Γ_T is discretized by means of one layer of finite elements (for a total of 32 elements), whereas the wire is discretized as before. Again GMRES reaches convergence in 4 iterations: the computing time is $1.14 T_S$ and the accuracy decreases to 0.11%.

IV. CONCLUSIONS

The above results show that FEM-QSDBCI is competitive with FEM-BEM in terms of accuracy and computing time.

FEM-QSDBCI can be seen as a kind of FEM-BEM, where the normal derivatives are not explicitly considered, being expressed via FEM shape functions. This reduces the number of unknowns and the computing time, but slightly decreases the accuracy, as previously observed for FEM-DBCI [10]. Moreover, singularities are avoided in the integral equation.

The FEM-QSDBCI method has been implemented in ELFIN [11], an FEM research code developed by the authors.

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