

A Low-Frequency-Stable Two-Step Formulation of Maxwell's Equations Using Generalized Tree-Cotree Gauges

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In this work, we discuss a novel tree-cotree-based regularization technique which corresponds to a generalized Coloumb gauge considering the material distribution. The method is explored in the context of a low-frequency stabilized full Maxwell formulation.

Index Terms—Computational Electromagnetics, Electromagnetic induction, Frequency, Maxwell equations

I. INTRODUCTION

MANY applications require accurate simulation results of electromagnetic devices over a wide frequency range, e.g., to access electromagnetic compatibility. This commonly includes static, low frequency and high frequency regimes. In such cases, static or quasi-static approximations are insufficient to capture the full functionality of the device, see e.g. [1], [2], [3], [4]. In this work, we propose a new variant of the two-step full Maxwell formulation [5] which exploits a tailor-made tree-cotree gauge.

II. THE TWO-STEP APPROACH

We start from the two-step procedure introduced in [5]. It decouples capacitive and inductive effects by first solving the electroquasistatic problem for an electric scalar potential φ represented by the degrees of freedom (DOFs) $\mathbf{u}$. Inductive effects are included in a second step by computing a vector potential $\mathbf{A}$ through its DOFs $\mathbf{a}$. This leads to the two systems

$$\mathbf{K}_\kappa \mathbf{u} = i\omega \mathbf{q}_s \quad (1)$$

$$\mathbf{W} \mathbf{a} = \mathbf{j}(\mathbf{u}) \quad (2)$$

with $\mathbf{W} = \mathbf{K}_\nu + i\omega \mathbf{M}_\kappa$ and $\mathbf{j}(\mathbf{u}) = \mathbf{j}_s - \mathbf{G}_\kappa \mathbf{u}$ when an appropriate Ritz-Galerkin discretization is employed. We write $\kappa = \sigma + i\omega\epsilon$ for the complex conductivity with conductivity σ , permittivity ϵ and angular frequency ω . The reluctivity is denoted as ν . Then, $\mathbf{K}_\star$ are the stiffness matrices arising from the Laplace and the curl-curl operators. With $\mathbf{M}_\star$ we refer to mass matrices and $\mathbf{G}_\star$ originate from testing the gradient of φ , weighted by material $\star$, with lowest-order edge-element basis functions. Additionally, we define a discrete, material-weighted divergence $\mathbf{S}_\star$ for which $\mathbf{S}_\star = -\mathbf{G}_\star^\top$ holds when appropriate boundary conditions are chosen.

A. Low-Frequency Breakdown

For small ω , the strong formulation introduced in [5] and its discrete counterpart (1)-(2) suffer from a low-frequency breakdown. One can show that the matrices $\mathbf{K}_\kappa$ and $\mathbf{W}$ are singular for $\omega = 0$. This impacts the conditioning already when $\omega \rightarrow 0$ and thus leads to a low-frequency breakdown.

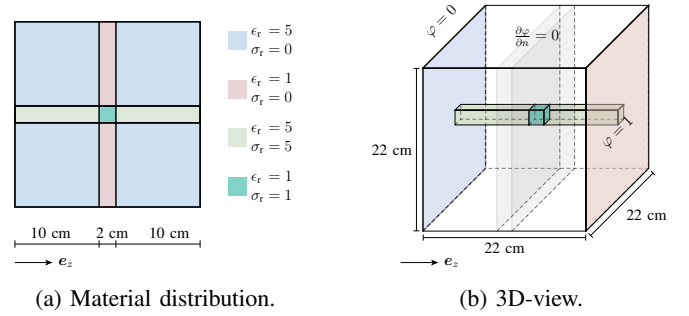


Fig. 1: Academic test example problem setup.

Modifications to stabilize (1) are already proposed in [6], which we employ in the first simulation step. The focus of this contribution is the stabilization of the second step (2) and correspondingly $\mathbf{W}$, which can be written as

$$\mathbf{W} = \mathbf{K}_\nu - \begin{bmatrix} \omega^2 \mathbf{M}_\epsilon^{(CC)} - i\omega \mathbf{M}_\sigma^{(CC)} & \omega^2 \mathbf{M}_\epsilon^{(CA)} \\ \omega^2 \mathbf{M}_\epsilon^{(AC)} & \omega^2 \mathbf{M}_\epsilon^{(AA)} \end{bmatrix}. \quad (3)$$

The superscripts denote a splitting of the DOFs belonging to conducting regions (C) with $\sigma > 0$ and non-conducting, e.g. air, regions (A) with $\sigma = 0$. For $\omega = 0$, the contributions from $i\omega \mathbf{M}_\kappa$ vanish and only the singular $\mathbf{K}_\nu$ remains.

III. LOW-FREQUENCY STABILIZATION TECHNIQUE

In order to overcome the low-frequency breakdown, we explicitly include the generalized Coulomb gauge $\text{div}(\kappa \mathbf{A}) = 0$ which is only implicitly fulfilled by the original approach [5]. This gauge apparently vanishes inside non-conducting regions for $\omega = 0$. This motivates the inclusion of a frequency-scaled discrete gauge condition

$$\mathbf{S} \mathbf{a} = \begin{bmatrix} (\mathbf{S}_\sigma^{(CC)} + i\omega \mathbf{S}_\epsilon^{(CC)}) & \mathbf{S}_\epsilon^{(CA)} \\ \mathbf{S}_\epsilon^{(AC)} & \mathbf{S}_\epsilon^{(AA)} \end{bmatrix} \begin{bmatrix} \mathbf{a}^{(C)} \\ \mathbf{a}^{(A)} \end{bmatrix} = \mathbf{0} \quad (4)$$

as an explicit constraint. In the following, we investigate how to incorporate (4) using tree-cotree gauging instead of Lagrange multipliers.

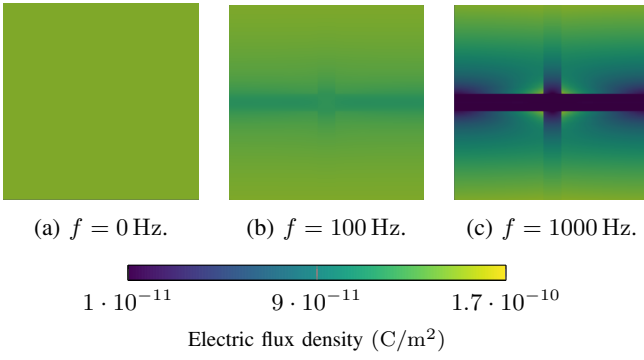


Fig. 2: Electric flux density $\|D\|_2$ for different excitation frequencies.

A. Tree-Cotree Decomposition

Following [7], one identifies the nullspace of K_ν by only using topological mesh information. The DOFs $\mathbf{a}^{(T)}$ spanning the nullspace correspond to edges of a spanning tree in the mesh, while the remaining cotree DOFs $\mathbf{a}^{(R)}$ are determined by the problem. Consequently, one can introduce the splitting

$$K_\nu = \begin{bmatrix} K_\nu^{(RR)} & K_\nu^{(RT)} \\ K_\nu^{(TR)} & K_\nu^{(TT)} \end{bmatrix}, \quad (5)$$

where $K_\nu^{(RR)}$ is invertible. A similar principle [8] can be applied to (4) such that

$$\begin{bmatrix} \mathbf{S}^{(R)} & \mathbf{S}^{(T)} \end{bmatrix} \begin{bmatrix} \mathbf{a}^{(R)} \\ \mathbf{a}^{(T)} \end{bmatrix} = \mathbf{0} \quad (6)$$

with $\mathbf{S}^{(T)}$ having full rank for an adequate tree with appropriate boundary conditions. This enables reformulating (6) into

$$\mathbf{a}^{(T)} = -\left(\mathbf{S}^{(T)}\right)^{-1} \mathbf{S}^{(R)} \mathbf{a}^{(R)} \quad (7)$$

applying the split into tree and remaining cotree DOFs and substituting $\mathbf{a}^{(T)}$ in (2) yields

$$\left(\mathbf{W}^{(RR)} - \mathbf{W}^{(RT)} \left(\mathbf{S}^{(T)}\right)^{-1} \mathbf{S}^{(R)}\right) \mathbf{a}^{(R)} = \mathbf{j}^{(R)}(\mathbf{u}). \quad (8)$$

Consequently, the stabilized method consists of solving (8) before inserting the result in (7) and recovering $\mathbf{a}^{(T)}$.

B. Numerical Test Example

The test problem, from [5], consists of three conducting bars in a dielectric box. The configuration is shown in Fig. 1. It is constructed such that the flux density $\mathbf{D}_e = -\kappa \text{grad } \varphi$ does not depend on ω and is constant in space. Fig. 2 shows simulation results concerning $\|D\|_2 = \|\mathbf{D}_e + \mathbf{D}_m\|_2$ with inductive correction $\mathbf{D}_m = -i\omega\kappa\mathbf{A}$ for different frequencies. We observe the expected constant $\mathbf{D}_e$ for $f = 0$ Hz where no inductive effects play a role. For $f = 100$ Hz and $f = 1000$ Hz, one can see that the flux density is being pulled out of the conductor due to the inductive contribution $\mathbf{D}_m$.

The behavior of the conditioning over f is presented in Fig. 3. It shows the condition number of the original system matrix and the stabilized method. We observe major improvements for low frequencies.

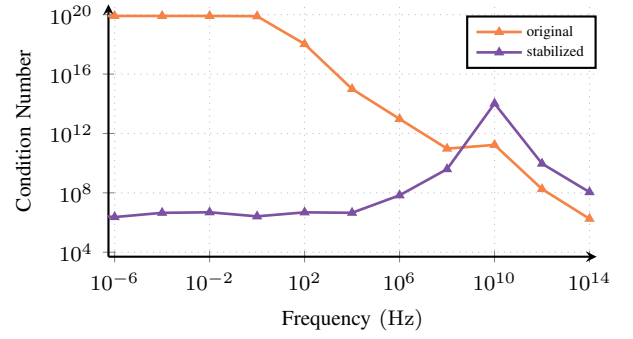


Fig. 3: Condition number of different system matrices of academic test example over frequency.

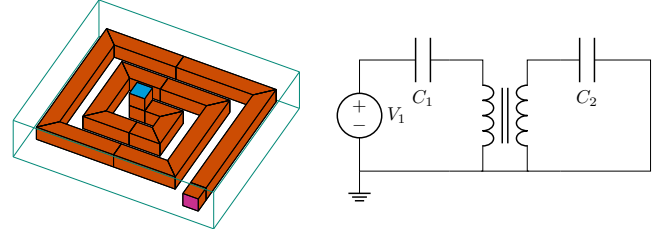


Fig. 4: More industrially relevant simulation examples.

IV. OUTLOOK

The full paper will present numerical experiments with more industrially relevant examples, e.g., configurations with planar coils such as Fig. 4. In particular, configuration where both capacitive and inductive effects are intrinsically coupled, such as sketched in the circuit equivalent, are of interest.

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