

Multilevel Monte Carlo and Precise Integration Time Domain Methods for Electromagnetic Problems Involving Random Parameters

Xiaojie Zhu¹, Xikui Ma², Luca Di Rienzo¹, Lorenzo Codecasa¹, Liang Ma², Mingjun Chi², and Jiawei Wang²

¹Dipartimento di Elettronica, Informazione e Bioingegneria, Politecnico di Milano, Milan, Italy, xiaojie.zhu@polimi.it

²School of Electrical Engineering, Xi'an Jiaotong University, Xi'an, China, jwwang@xjtu.edu.cn

In this work the precise integration time domain (PITD) method is coupled with Monte Carlo (MC) and multilevel Monte Carlo (MLMC) methods to release the constraint of the Courant–Friedrichs–Lewy (CFL) condition in the finite difference time domain (FDTD) method for electromagnetic problems with random materials. Compared to the MC-FDTD method, the MC-PITD method can provide accurate estimations of mean and variance of electromagnetic field quantities using a large time step size. In contrast to the MLMC-FDTD method, where the allowable time step size decreases with increasing levels (i.e., finer grids) due to the stricter CFL condition, the MLMC-PITD method enables all model levels to use the maximum time step size determined by the coarsest grid, allowing the MLMC method to fully leverage its advantages.

Index Terms— Courant–Friedrichs–Lewy (CFL) condition, finite difference time domain, multilevel Monte Carlo, precise integration.

I. INTRODUCTION

DUE to its advantages of efficiency, robustness, and simplicity, the Monte Carlo finite difference time domain (MC-FDTD) method has been widely applied in computational electromagnetics [1]. Recently, to inherit the advantages of MC-FDTD and improve its convergence rate, the multilevel Monte Carlo finite difference time domain (MLMC-FDTD) method has been proposed [2].

However, for a long-time integration problem, an inherent defect of FDTD is that the time step size is limited by the space step size through the Courant–Friedrichs–Lewy (CFL) condition [3]. If the single FDTD simulation is not fast enough, the total computational time of MC-FDTD becomes prohibitive. More importantly, in MLMC-FDTD the time step size is typically dictated by the smallest space grid on each level to comply with the CFL condition, so that it becomes increasingly inefficient on finer space grids, limiting the advantages of the MLMC method [4], [5].

It is worth noting that the precise integration time domain (PITD) method has been proposed to break through the restriction of the CFL condition limit of FDTD [6], [7]. In this paper, PITD is coupled with the MC and MLMC methods. MC-PITD is proposed to accelerate the single deterministic simulation compared to MC-FDTD due to the capability of allowing for a large time step size. In addition, MLMC-PITD is proposed to mitigate the bottleneck of MLMC-FDTD in which the time step size is limited by the space step size on each level according to the CFL condition. PITD allows for the adoption of the largest time step size at all levels, further enhancing the advantage of MLMC. Numerical results show that MLMC-PITD exhibits even higher efficiency compared to MC-PITD and MLMC-FDTD.

II. FORMULATIONS

In PITD, Maxwell's equations for random materials are written in a matrix form with random parameters depending on θ , i.e.,

$$\frac{d\mathbf{X}(\theta)}{dt} = \mathbf{M}(\theta)\mathbf{X}(\theta) + \mathbf{f}(t) \quad (1)$$

where $\mathbf{X}(\theta)$ is a vector consisting of all the electromagnetic field components dependent on θ ; $\mathbf{M}(\theta)$ is the coefficient matrix determined by the random parameters, i.e., permittivity $\varepsilon(\theta)$, permeability $\mu(\theta)$, and conductivity $\sigma(\theta)$; $\mathbf{f}(t)$ is a column vector introduced by the external source.

The recursive solution of (1) can be expressed as

$$\mathbf{X}_{n+1}(\theta) = \mathbf{T}(\theta)\mathbf{X}_n(\theta) + \int_{t_n}^{t_{n+1}} e^{(\mathbf{M}(\theta)(t-s))} \mathbf{f}(s)ds \quad (2)$$

where $t_n = n\Delta t$ ($n = 0, 1, 2, \dots$) in which Δt is the time step size, and the exponential matrix $\mathbf{T}(\theta) = \exp(\mathbf{M}(\theta)\Delta t)$ can be computed with the help of the precise integration technique [6].

It is easy to do MC-PITD simulations, i.e. choose M random samples $(\varepsilon, \mu, \sigma)$ from the known probability distribution, then repeat PITD simulations M times. Ultimately, the mean and variance estimators of MC-PITD can be expressed as

$$\mathbb{E}^{\text{MC}}[X] = \frac{1}{M} \sum_{m=1}^M X_m \quad (3)$$

$$\mathbb{V}^{\text{MC}}[X] = \frac{1}{M-1} \sum_{m=1}^M (X_m - \mathbb{E}_M^{\text{MC}}[X])^2 \quad (4)$$

where X_m represents the solution X , i.e., any electric or magnetic field component computed by the m th PITD simulation, with $m = 1, \dots, M$.

In MLMC-PITD, Maxwell's equations are discretized by multilevel Yee grids, and the number of samples is related to these grids. Let N_l be the number of space steps, and M_l be the number of samples, respectively, increasing fidelity models, i.e., $N_0 < N_1 < \dots < N_l < \dots < N_L$ and decreasing number of samples, i.e., $M_0 > M_1 > \dots > M_l > \dots > M_L$, where $l = 0, 1, 2, \dots, L$.

Given two positive integers s_N and s_M , we can assume that

$$N_l = s_N^l N_0 \quad (5)$$

$$M_l = s_M^{-l} M_0 \quad (6)$$

The space step sizes, for example, in the x -axis read

$$\Delta x_l = s_N^{-l} \Delta x_0 \quad (7)$$

Since there is no CFL condition limit, the largest time step size Δt_0 of the coarsest grid can be used in all the levels, i.e.,

$$\Delta t_l = \Delta t_0 \quad (8)$$

The MLMC computes most of the samples on the coarsest grid so fewer samples are enabled on the finer grids, improving the convergence rate compared with the MC. The estimator of

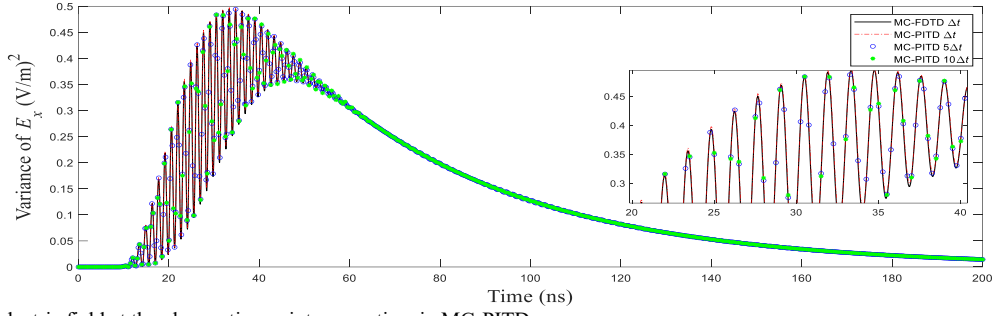


Fig. 1 Variance of the electric field at the observation point versus time in MC-PITD.

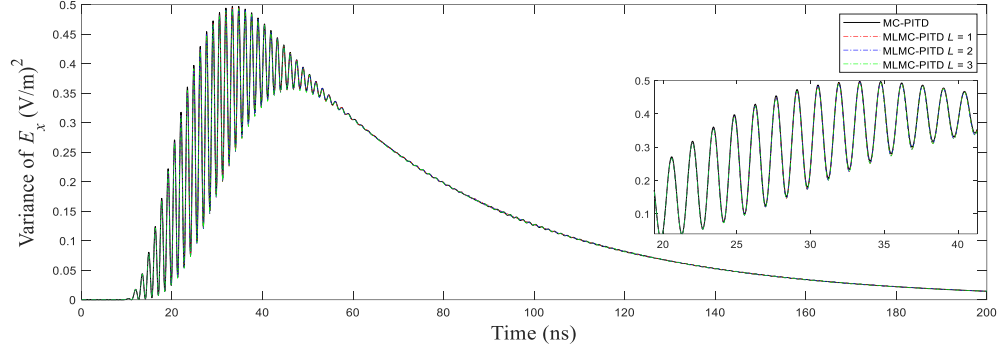


Fig. 2 Variance of the electric field at the observation point versus time in MLMC-PITD.

MLMC-PITD is equal to the estimator on the coarsest grid plus a sum of corrections, i.e., the differences of the solutions on consecutive grids. Ultimately, the mean estimator of the MLMC-PITD can be expressed as

$$\mathbb{E}^{\text{ML}}[X_{N_L}] = \mathbb{E}_{M_0}^{\text{MC}}[X_{N_0}] + \sum_{l=1}^L \mathbb{E}_{M_l}^{\text{MC}}[X_{N_l} - X_{N_{l-1}}] \quad (9)$$

Variance estimator of the MLMC-PITD can be expressed as

$$\mathbb{V}^{\text{ML}}[X_{N_L}] = \mathbb{V}_{M_0}^{\text{MC}}[X_{N_0}] + \sum_{l=1}^L (\mathbb{V}_{M_l}^{\text{MC}}[X_{N_l}] - \mathbb{V}_{M_{l-1}}^{\text{MC}}[X_{N_{l-1}}]) \quad (10)$$

where X_{N_l} denotes the approximation of the solution X on the grid with N_l space steps.

III. RESULTS AND CONCLUSION

A 1-D dielectric cavity of length equal to 60 mm characterized by random relative permittivity and conductivity assuming normal distributions, i.e., $\epsilon_r \sim \mathcal{N}(50, 2^2)$ and $\sigma \sim \mathcal{N}(0.01 \text{ S/m}, (0.001 \text{ S/m})^2)$ is considered. An electric line source with a Gaussian pulse is imposed as the source. Perfect electric conductor (PEC) condition is applied to both sides. The source and observation points are 15 mm and 30 mm away from the left PEC, respectively. The time step size of MC-FDTD is chosen as $\Delta t = 6.25$ ps while the maximum value is 8.14 ps considering CFL condition. Reference results are obtained with MC-FDTD with a fine grid ($N = 160$) and a high number of samples ($M = 2 \times 10^4$). MC-PITD is implemented with the same grid and different time step sizes. MLMC-PITD is implemented with $L = 1, 2, 3$, $s_N = 2$, $s_M = 10$, and with the largest time step size Δt_0 of the coarsest grid in all the levels. Fig. 1 shows that the variances of MC-FDTD and MC-PITD agree well with each other. Fig. 2 shows that the variance of MLMC-PITD agrees with that of MC-PITD. The relative errors are computed in the infinite norm. TABLE I and II show the high degree to which MLMC-PITD allows to overcome the limitations of MC-FDTD by exploiting the advantages of both MLMC and PITD.

TABLE I
COMPUTATIONAL INFORMATION ON FDTD METHODS

Method	Relative Error on Mean (%)	Relative Error on Variance (%)	Execution Time (s)
MC-FDTD (Δt)	Ref.	Ref.	7188
MLMC-FDTD ($L=1$)	0.494	0.969	4598
MLMC-FDTD ($L=2$)	0.869	1.506	2350
MLMC-FDTD ($L=3$)	0.927	1.620	1180

TABLE II
COMPUTATIONAL INFORMATION ON PITD METHODS

Method	Relative Error on Mean (%)	Relative Error on Variance (%)	Execution Time (s)
MC-PITD (Δt)	Ref.	Ref.	16158
MC-PITD ($5\Delta t$)	0.292	0.756	5090
MC-PITD ($10\Delta t$)	0.802	1.143	3932
MLMC-PITD ($L=1$)	0.976	1.756	3510
MLMC-PITD ($L=2$)	0.984	1.761	678
MLMC-PITD ($L=3$)	1.027	1.809	162

REFERENCES

- [1] T. T. Zygididis, "A short review of FDTD-based methods for uncertainty quantification in computational electromagnetics," *Math. Probl. Eng.*, vol. 2017, Art. no. 9247978.
- [2] X. Zhu, L. Di Rienzo, X. Ma, and L. Codecasa, "Multilevel Monte Carlo FDTD method for uncertainty quantification," *IEEE Antennas Wireless Propag. Lett.*, vol. 21, no. 10, pp: 2030-2034, Oct. 2022.
- [3] F. L. Teixeira, C. Sarris, Y. Zhang, et al. "Finite-difference time-domain method," *Nat Rev Methods Primers* vol. 3, no. 75, pp: 1-19, Oct. 2023.
- [4] M. B. Giles, "Multilevel Monte Carlo methods," *Acta Numer.*, vol. 24, pp. 259–328, May 2015.
- [5] M. J. Grote, S. Michel, and F. Nobile, "Uncertainty quantification by multilevel Monte Carlo and local time-stepping for wave propagation," *SIAM/ASA J. Uncertainty*, vol. 10, no. 4, pp: 1601-1628, Dec. 2022.
- [6] X. Ma, X. Zhao, and Y. Zhao, "A 3-D precise integration time domain method without the restraints of the Courant-Friedrich-Levy stability condition for the numerical solution of Maxwell's equations," *IEEE Trans. Microw. Theory Techn.*, vol. 54, no. 7, pp. 3026–3037, Jul. 2006.
- [7] L. Jiang, Z. Chen, and J. Mao, "On the numerical stability of the precise integration time-domain (PITD) method," *IEEE Microw. Wireless Compon. Lett.*, vol. 17, no. 7, pp. 471–473, Jul. 2007.