

Quasi-Stationary Solutions to the Nonlinear Schrödinger Equation

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In these notes, we examine a family of dissipative Schrödinger equations with cubic nonlinearity. Equations of this type are used to describe the propagation of waves in weakly nonlinear, dispersive and dissipative media. The study focuses on determining exact solutions to equations of the considered class. The general solution to this problem is given through Weierstrass elliptic functions when the parameter that characterizes the dissipation is real. Another class of exact solutions is specified within the framework of the so-called small-phase approximation.

Index Terms—Dispersive and dissipative media, electromagnetic wave propagation, quasi-stationary states, nonlinear Schrödinger equation

I. INTRODUCTION

THE WORK is devoted to the study of the three-parameter family of cubic nonlinear Schrödinger equations

$$i \frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2} + (\gamma + i\beta) \varphi + \eta |\varphi|^2 \varphi = 0, \quad \beta, \gamma, \eta \in \mathbf{R}, \quad (1)$$

where $\varphi(x, t)$ is assumed to be a complex-valued function of the real variables x and t . These equations describe optical phenomena resulting from the interaction of light and matter. They adequately capture the basic features of electromagnetic wave propagation in weakly nonlinear, dispersive, and dissipative media, reflecting nonlinear effects. Noteworthy, the regarded equations are the subject of study in many branches of physics: in the electromagnetic field theory, nonlinear optics, physics of electrical transmission lines, laser and plasma physics, the theory of phase transitions, etc. (see, e.g., [1]–[10] and the references therein).

II. QUASI-STATIONARY SOLUTIONS

In the current notes, we are interested in the so-called quasi-stationary solutions of (1), which are assumed to have the form

$$\varphi(x, t) = \psi(x) \exp[-i\lambda t], \quad (2)$$

where $\psi(x)$ is a complex-valued function of the real variable x only, and λ is a real constant.

The quasi-stationarity assumption (2) when inserted into the time-dependent Schrödinger equation (1) leads to the following complex eigenvalue problem

$$\frac{d^2 \psi}{dx^2} + \eta |\psi|^2 \psi + (\alpha + i\beta) \psi = 0, \quad (3)$$

where $\alpha = \gamma + \lambda$.

Using the representation

$$\psi(x) = \rho(x) \exp[i\theta(x)], \quad (4)$$

where $\rho(x)$ and $\theta(x)$ are real-valued functions, (3) reduces to the following system of two ordinary differential equations

$$\rho'' - \rho \theta'^2 + \alpha \rho + \eta \rho^3 = 0, \quad \theta'' + \frac{2\rho'}{\rho} \theta' + \beta = 0, \quad (5)$$

Here and below, the primes denote differentiation with respect to the variable x .

A. Exact Solutions in the Case $\beta = 0$

If $\beta = 0$, then the system (5) has two first integrals:

$$\mathcal{I}_1 := 2\rho'^2 + 2\rho^2 (\alpha + \theta'^2) + \eta \rho^4, \quad \mathcal{I}_2 := \rho^2 \theta' \quad (6)$$

that remain constant along the non-constant solutions of the regarded system. These first integrals imply

$$\theta = -c_2 \int \rho^{-2} dx + c_3, \quad (7)$$

$$4\rho^2 \rho'^2 + 4c_2^2 + 2c_1 \rho^2 + 4\alpha \rho^4 + 2\eta \rho^6 = 0. \quad (8)$$

Here, c_1 , c_2 and c_3 are constants of integration. Thus, the non-constant solutions to the eigenvalue problem (3) are determined by the solution of the single first-order equation (8).

In terms of the new dependent variable

$$r = -\frac{1}{6} (2\alpha + 3\eta \rho^2), \quad (9)$$

equation (8) takes the form

$$r'^2 = 4r^3 - rg_2 - g_3, \quad (10)$$

where

$$g_2 = \frac{4\alpha^2}{3} - \eta c_1, \quad g_3 = \frac{8\alpha^3}{27} - \frac{1}{3} \alpha \eta c_1 + \eta^2 c_2^2. \quad (11)$$

Then, following [11, §20.6, §20.22 and §21.73], one can easily prove that each real-valued solution of the equation (10) may be cast in the form

$$r(x) = r_0 + \frac{1}{4} \frac{g_2 - 12r_0^2}{r_0 - \wp(x + x_0; g_2, g_3)}, \quad (12)$$

where $\wp$ is the Weierstrass elliptic function with elliptic invariants g_2 and g_3 , x_0 is an arbitrary real constant, and r_0 is a real root of the polynomial

$$P[r] = 4r^3 - rg_2 - g_3 \quad (13)$$

appearing in the right-hand side of the equation (10). Let us remark that depending on the values of parameters α , η , c_1 and c_2 determining the elliptic invariants g_2 and g_3 via formulas (11), the cubic polynomial $P[r]$ may have one or three real roots but, in any case, should have at least one such root.

Finally, on account of the formulas (9) and (12), one can write down the solution of the equation (8) as follows:

$$\rho(x) = \sqrt{-\frac{2\alpha}{3\eta} - \frac{2}{\eta} \left(r_0 + \frac{1}{4} \frac{g_2 - 12r_0^2}{r_0 - \wp(x + x_0; g_2, g_3)} \right)}, \quad (14)$$

where g_2 and g_3 are determined by the relations (11). Thus, in the considered case $\beta = 0$, the solutions of the complex eigenvalue problem (3) are determined by (4), (7) and (14).

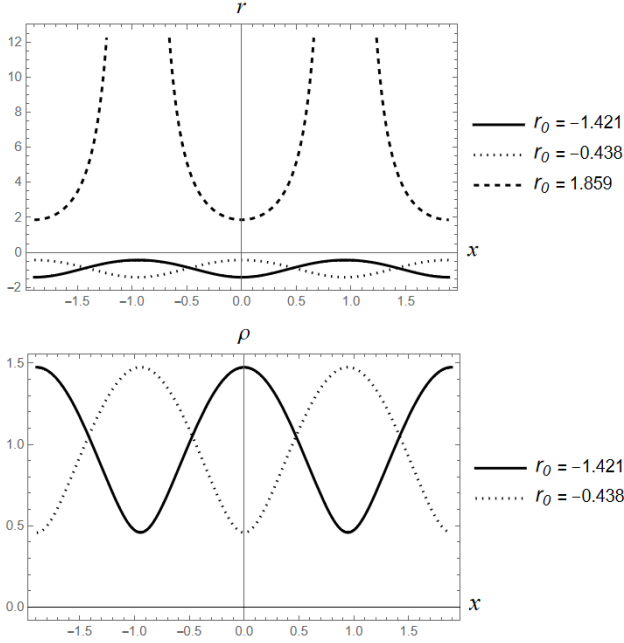


Fig. 1. Graphs of the functions $r(x)$ and $\rho(x)$ given by the formulas (12) and (14), respectively, for $\alpha = \eta = 1$, $c_1 = -10$, $c_2 = 1$, $x_0 = 0$, $r_0 = -1.421, -0.438, 1.859$ in the first case and $r_0 = -1.421, -0.438$ — in the second one.

Graphs of the functions $r(x)$ and $\rho(x)$ given by (12) and (14) for $\alpha = \eta = 1$, $c_1 = -10$, $c_2 = 1$ and $x_0 = 0$, are depicted in Figure 1.

B. Exact Solutions within the Small-Phase Approximation

Now, using the representation

$$\psi(x) = u(x) + iv(x), \quad (15)$$

where $u(x)$ and $v(x)$ are real-valued functions, (3) can be cast in the form of the following system of equations

$$u'' + \alpha u - \beta v + \eta uv^2 + \eta u^3 = 0, \quad (16)$$

$$v'' + \beta u + \alpha v + \eta u^2 v + \eta v^3 = 0, \quad (17)$$

for the functions $u(x)$ and $v(x)$.

Neglecting the terms ηuv^2 and ηv^3 , which might be thought of as a *small-phase approximation*, this system takes the form

$$u'' + \alpha u - \beta v + \eta u^3 = 0, \quad (18)$$

$$v'' + \beta u + \alpha v + \eta u^2 v = 0. \quad (19)$$

Equation (18) implies

$$v = \beta^{-1} (u'' + \alpha u + \eta u^3). \quad (20)$$

Substituting (20) into (19), we arrive at the single forth-order nonlinear ordinary differential equation

$$u'''' + 2(\alpha + 2\eta u^2)u'' + 6\eta u u'^2 + (\alpha^2 + \beta^2)u + 2\alpha\eta u^3 + \eta^2 u^5 = 0. \quad (21)$$

for the function $u(x)$ only.

A direct computation shows that each solution of the first-order equation

$$u'^2 = -\frac{16\alpha^2 + 25\beta^2}{125\eta} - \frac{1}{5}\alpha u^2 - \frac{1}{12}\eta u^4 \quad (22)$$

is a solution of the equation (21) as well. A typical example of such a solution is depicted in Figure 2.

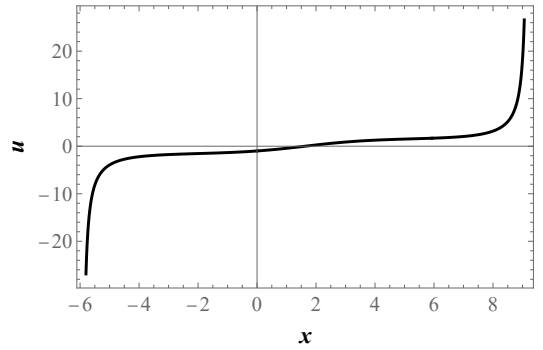


Fig. 2. Graph of the common solution $u(x)$ of the equations (21) and (22) for $\alpha = 2$, $\beta = 0.1$ and $\eta = -1$.

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