

An Energy Stable Precise Integration Time Domain Method With Application to Domain Decomposition

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A summation-by-parts precise integration time domain (SBP-PITD) method with provable energy stability is proposed based on modified Yee's cells, where the discrete energy depends entirely on boundary fields. On this foundation, a novel memory-efficient domain decomposition SBP-PITD (DD-SBP-PITD) method is proposed to guarantee the energy stability for coupling different subdomains. Numerical results confirm the effectiveness and advantage of SBP-PITD and DD-SBP-PITD.

Index Terms—Matrix exponential, precise integration time domain (PITD) method, summation by parts (SBP), upwind fluxes.

I. INTRODUCTION

STARTING from spatially semidiscrete Maxwell equations, the precise integration time domain (PITD) method is proposed to break the Courant-Friedrichs-Lewy (CFL) condition. Using the precise integration (PI) technique to evaluate the matrix exponential within machine precision, PITD manifests superior numerical stability and dispersion properties [1], [2]. However, the large and dense matrix exponential leads to prohibitive memory consumption in PITD. Domain decomposition schemes [3] can effectively mitigate this issue by reducing the dimensions of the matrix exponential. However, one main challenge for domain decomposition schemes is that theoretical proofs of the stability for coupling different subdomains are hardly given. Recently, the summation-by-parts (SBP) technique [4], [5] has been proposed to theoretically guarantee the energy stability of semidiscrete systems. SBP operators enable the discrete energy to be fully determined by boundary fields, and the boundary conditions are weakly enforced by simultaneous approximation terms (SATs) [4].

In this digest, an energy stable SBP-PITD method is proposed to enhance the scalability and flexibility of PITD, especially for the application to domain decomposition. In SBP-PITD, we made a few modifications in the node distribution to satisfy the SBP property. Furthermore, we propose a provably stable domain decomposition SBP-PITD (DD-SBP-PITD) method to alleviate the memory requirement of PITD.

II. FORMULATION

A. Energy Stable SBP-PITD Method

For simplicity, considering a linear, homogeneous, and lossless medium, the 1-D Maxwell equations for E_y and H_z are

$$\varepsilon \partial E_y / \partial t = -\partial H_z / \partial x, \quad \mu \partial H_z / \partial t = -\partial E_y / \partial x \quad (1)$$

where ε and μ are the permittivity and permeability, respectively.

As shown in Fig. 1, compared with classical Yee's cells, additional magnetic field nodes are only added on the boundaries. Based on modified Yee's cells, the SBP finite difference operators are used to discretize Maxwell equations spatially as

$$dE_y/dt = -\mathbf{D}_+ H_z, \quad dH_z/dt = -\mathbf{D}_- E_y \quad (2)$$

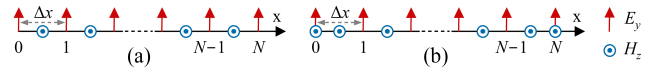


Fig. 1. (a) Classical Yee's cells. (b) Modified Yee's cells. There are N cells of size Δx .

where $\mathbf{E}_y = [E_y|_0, E_y|_1, \dots, E_y|_N]^T$, $\mathbf{H}_z = [H_z|_0, H_z|_{\frac{1}{2}}, \dots, H_z|_{N-\frac{1}{2}}, H_z|_N]^T$, and the SBP operators $\mathbf{D}_+$ and $\mathbf{D}_-$ are

$$\mathbf{D}_+ = \frac{1}{\varepsilon \Delta x} \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & & & \\ -\frac{1}{2} & -\frac{1}{4} & \frac{3}{4} & & & \\ & -1 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & -1 & 1 & \\ & & & & -\frac{3}{4} & \frac{1}{4} & \frac{1}{2} \\ & & & & -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \quad (3a)$$

$$\mathbf{D}_- = \frac{1}{\mu \Delta x} \begin{bmatrix} -1 & 1 & & & & \\ -1 & \frac{1}{5} & & & & \\ -\frac{1}{5} & -\frac{3}{5} & \frac{4}{5} & & & \\ & -1 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & -1 & 1 & \\ & & & & -\frac{4}{5} & \frac{3}{5} & \frac{1}{5} \\ & & & & -1 & 1 & \\ & & & & & -1 & 1 \end{bmatrix} \quad (3b)$$

The energy of the 1-D discrete system (2) is defined as

$$\mathcal{E} = (\mathbf{E}_y^T \mathbf{P}_+ \mathbf{E}_y + \mathbf{H}_z^T \mathbf{P}_- \mathbf{H}_z) / 2 \quad (4)$$

where $\mathbf{P}_+$ and $\mathbf{P}_-$ are the induced-norm matrices

$$\mathbf{P}_+ = \varepsilon \Delta x \cdot \text{diag} \left[\frac{1}{2}, 1, 1, \dots, 1, \frac{1}{2} \right] \quad (5a)$$

$$\mathbf{P}_- = \mu \Delta x \cdot \text{diag} \left[\frac{1}{2}, \frac{1}{4}, \frac{5}{4}, 1, \dots, 1, \frac{5}{4}, \frac{1}{4}, \frac{1}{2} \right]. \quad (5b)$$

Since $\mathbf{D}_+$, $\mathbf{D}_-$, $\mathbf{P}_+$ and $\mathbf{P}_-$ are carefully designed, $\mathbf{Q}_+ = \mathbf{P}_+ \mathbf{D}_+$ and $\mathbf{Q}_- = \mathbf{P}_- \mathbf{D}_-$ satisfy the SBP property as follows:

$$\mathbf{Q}_+ + \mathbf{Q}_-^T = \mathbf{e}_{R+} \mathbf{e}_{R-}^T - \mathbf{e}_{L+} \mathbf{e}_{L-}^T = \begin{bmatrix} -1 & & \\ & & \\ & & 1 \end{bmatrix} \quad (6)$$

where $\mathbf{e}_{L+} = [1, 0, \dots, 0]^T$, $\mathbf{e}_{R+} = [0, 0, \dots, 1]^T$, $\mathbf{e}_{L-} = [1, 0, \dots, 0]^T$, and $\mathbf{e}_{R-} = [0, 0, \dots, 1]^T$. The dimensions of $\mathbf{e}_{L+}$ ($\mathbf{e}_{R+}$) and $\mathbf{e}_{L-}$ ($\mathbf{e}_{R-}$) are $(N+1) \times 1$ and $(N+2) \times 1$, respectively.

Algorithm 1 PI Technique With Dimensional Expanding

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1:  $\tau = \Delta t/2^M$ ,  $M$  is a preselected integer
2: Create  $\mathbf{M}$  and identity matrix  $\mathbf{I}$ ,  $\mathbf{T}_a = \sum_{k=1}^4 (\mathbf{M}\tau)^k/k!$ 
3:  $\mathbf{T}_b = \sum_{k=1}^4 (\mathbf{M}^{k-1}\tau^k)/k!$ ,  $\mathbf{T}_c = \sum_{k=2}^4 (\mathbf{M}^{k-2}\tau^k)/k!$ ,  $a = \tau$ 
4: for iter = 1 :  $M$  do
5:    $\mathbf{T}_c = 2\mathbf{T}_c + a\mathbf{T}_b + \mathbf{T}_a \times \mathbf{T}_c$ ,  $\mathbf{T}_b = 2\mathbf{T}_b + \mathbf{T}_a \times \mathbf{T}_b$ 
6:    $\mathbf{T}_a = 2\mathbf{T}_a + \mathbf{T}_a \times \mathbf{T}_a$ ,  $a = 2a$ 
7: end for
8: return  $\mathbf{T} = \mathbf{I} + \mathbf{T}_a$ ,  $\mathbf{T}_b$ , and  $\mathbf{T}_c$ 

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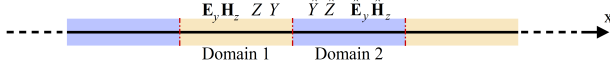


Fig. 2. 1-D domain decomposition. $Z = \sqrt{\mu/\epsilon}$ and $Y = 1/Z$. The quantities without and with $\hat{\cdot}$ are defined in Domain 1 and 2, respectively.

Substituting (2) into the time derivative of (4), we obtain

$$\begin{aligned} d\mathcal{E}/dt &= -(\mathbf{E}_y^\top \mathbf{P}_+ \mathbf{D}_+ \mathbf{H}_z + \mathbf{H}_z^\top \mathbf{P}_- \mathbf{D}_- \mathbf{E}_y) = -\mathbf{E}_y^\top (\mathbf{Q}_+ + \mathbf{Q}_-) \mathbf{H}_z \\ &= -(E_y|_N H_z|_N - E_y|_0 H_z|_0). \end{aligned} \quad (7)$$

It shows that the variation of discrete energy is controlled by boundary fields, similar to the continuous Poynting theorem.

Considering an external excitation, (2) can be expressed in the compact form

$$\frac{d\mathbf{X}}{dt} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_+ \\ -\mathbf{D}_- & \mathbf{0} \end{bmatrix} \mathbf{X} + \mathbf{f}(t) = \mathbf{M}\mathbf{X} + \mathbf{f}(t) \quad (8)$$

where $\mathbf{X} = [\mathbf{E}_y^\top, \mathbf{H}_z^\top]^\top$, and $\mathbf{f}(t)$ is the excitation source term. The analytical solution of (8) is

$$\mathbf{X}(t) = e^{\mathbf{M}t} \mathbf{X}(0) + \int_0^t e^{\mathbf{M}(t-s)} \mathbf{f}(s) ds. \quad (9)$$

In $[t_n, t_{n+1}]$, $\mathbf{f}(t)$ is approximated by the linear polynomial as $\mathbf{f}(t) = \mathbf{f}_0 + \mathbf{f}_1(t - t_n)$, yielding the time-stepping formula

$$\mathbf{X}^{n+1} = \mathbf{T}\mathbf{X}^n + \mathbf{T}_b \mathbf{f}_0 + \mathbf{T}_c \mathbf{f}_1 \quad (10)$$

where $\mathbf{X}^n = \mathbf{X}(t_n)$, $t_n = n\Delta t$ and Δt is the time-step size. $\mathbf{T} = e^{\mathbf{M}\Delta t}$, $\mathbf{T}_b = \int_0^{\Delta t} e^{\mathbf{M}s} ds$, and $\mathbf{T}_c = \int_0^{\Delta t} (\Delta t - s) e^{\mathbf{M}s} ds$ can be computed by performing Algorithm 1.

B. Application of SBP-PITD to Domain Decomposition

As shown in Fig. 2, the whole domain is decomposed into multiple small subdomains for memory optimization, and the connection between different subdomains is important for stability and accuracy. To suppress spurious modes arising from interfaces, the upwind fluxes with dissipative terms [6], [7] are used as a penalization for SBP-PITD in Domain 1

$$\frac{d\mathbf{E}_y}{dt} = -\mathbf{D}_+ \mathbf{H}_z + \mathbf{P}_+^{-1} \mathbf{e}_{R+} \frac{\hat{Z}(H_z|_N - \hat{H}_z|_0) - (E_y|_N - \hat{E}_y|_0)}{Z + \hat{Z}} \quad (11a)$$

$$\frac{d\mathbf{H}_z}{dt} = -\mathbf{D}_- \mathbf{E}_y + \mathbf{P}_-^{-1} \mathbf{e}_{R-} \frac{\hat{Y}(E_y|_N - \hat{E}_y|_0) - (H_z|_N - \hat{H}_z|_0)}{Y + \hat{Y}} \quad (11b)$$

and in Domain 2, we have

$$\frac{d\hat{\mathbf{E}}_y}{dt} = -\hat{\mathbf{D}}_+ \hat{\mathbf{H}}_z - \hat{\mathbf{P}}_+^{-1} \hat{\mathbf{e}}_{L+} \frac{Z(\hat{H}_z|_0 - H_z|_N) + (\hat{E}_y|_0 - E_y|_N)}{Z + \hat{Z}} \quad (12a)$$

$$\frac{d\hat{\mathbf{H}}_z}{dt} = -\hat{\mathbf{D}}_- \hat{\mathbf{E}}_y - \hat{\mathbf{P}}_-^{-1} \hat{\mathbf{e}}_{L-} \frac{Y(\hat{E}_y|_0 - E_y|_N) + (\hat{H}_z|_0 - H_z|_N)}{Y + \hat{Y}}. \quad (12b)$$

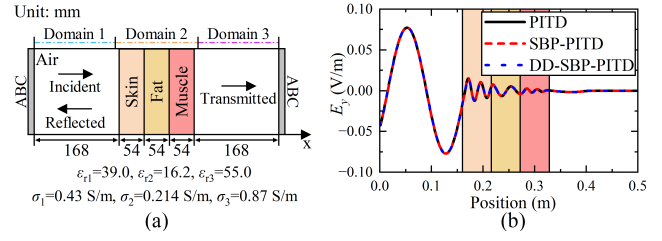


Fig. 3. (a) 1-D tissue model. (b) Distribution of E_y along x -axis at $t = 4$ ns.

TABLE I
COMPUTATIONAL INFORMATION OF DIFFERENT METHODS.

Method	PITD	SBP-PITD	DD-SBP-PITD
CPU Time (s)	33.22	33.42	16.27
Memory (MB)	252.43	252.74	84.45

Hence, the energy change rate can be expressed as

$$\begin{aligned} d\mathcal{E}/dt &= -E_y|_N H_z|_N + \hat{E}_y|_0 \hat{H}_z|_0 \\ &\quad + E_y|_N \cdot [\hat{Z}(H_z|_N - \hat{H}_z|_0) - (E_y|_N - \hat{E}_y|_0)] / (Z + \hat{Z}) \\ &\quad + H_z|_N \cdot [\hat{Y}(E_y|_N - \hat{E}_y|_0) - (H_z|_N - \hat{H}_z|_0)] / (Y + \hat{Y}) \\ &\quad - \hat{E}_y|_0 \cdot [Z(\hat{H}_z|_0 - H_z|_N) + (\hat{E}_y|_0 - E_y|_N)] / (Z + \hat{Z}) \\ &\quad - \hat{H}_z|_0 \cdot [Y(\hat{E}_y|_0 - E_y|_N) + (\hat{H}_z|_0 - H_z|_N)] / (Y + \hat{Y}) \\ &= -(E_y|_N - \hat{E}_y|_0)^2 / (Z + \hat{Z}) - (H_z|_N - \hat{H}_z|_0)^2 / (Y + \hat{Y}) \leq 0. \end{aligned} \quad (13)$$

Thus, the energy stability of DD-SBP-PITD can be guaranteed.

III. NUMERICAL RESULTS

A 1-D three-layered biological model is considered in Fig. 3(a). A 2 GHz sinusoidal plane wave is introduced from the left air region. Set $\Delta x = 0.3$ mm and $\Delta t = 5$ ps. Fig. 3(b) shows that the results of PITD, SBP-PITD, and DD-SBP-PITD are in good agreement. Their computational information is listed in Table I. Compared to the conventional PITD, SBP-PITD has negligible extra overheads in memory and runtime since the additional boundary nodes are far less than the total spatial components. More importantly, memory-efficient DD-SBP-PITD requires 1/2 of CPU time and 1/3 of memory of the conventional PITD. Therefore, SBP-PITD provides more possibilities and flexibility for theoretically stable domain decomposition schemes.

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