

Hybrid Transmission Conditions for the Schwarz Domain Decomposition Method for RF Accelerating Cavities

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We introduce a new transmission condition for the domain decomposition method with application in particle accelerator problems. The new transmission condition combines a modal decomposition approach for guided-wave propagation with the commonly used technique based on the free-space radiation condition. This significantly improves the convergence properties of the domain decomposition method in guided wave structures, while avoiding the need to repeatedly solve costly auxiliary problems. As an important application, we consider the computation of electromagnetic fields in a large accelerator cavity. This example illustrates the accuracy of the method and demonstrates its numerical efficiency in the simulation of electromagnetically large structures.

Index Terms—domain decomposition, Maxwell, particle accelerator, transmission condition.

I. Introduction

The computation of beam coupling impedances is a crucial step in the design and optimization of particle accelerators. Traditionally, this task has been accomplished by employing so-called wakefield solvers, which solve Maxwell's equations in the time domain for ultra-relativistic beams. However, this approach is not well suited for highly resonant structures [1], [2].

A promising alternative is the finite element method in the frequency domain. The main challenge here lies in solving the large system of equations. This can be addressed using the Schwarz domain decomposition method (DDM). The convergence rate of this technique depends strongly on the transmission conditions (TC) applied between the domains. Typically, these TCs are based on either polynomial [3]–[6] or rational [4] approximations of the symbol of the exact absorbing boundary operator. Since these TCs physically model radiation into free space, we refer to them as open TCs (OTCs).

For guided wave structures, however, the OTCs do not achieve optimal convergence. Therefore, in a previous work, we introduced the modal TC (MTC) [7]. The limitation of this method is the large number of modes (one per degree of freedom on the interface) that must be considered. This limitation can be addressed by truncating the number of modes. However, this introduces some error. Therefore, we propose in this contribution a hybridization of the MTC with the OTCs. The MTC is truncated to couple only a few modes, such as the propagating modes, while the OTC describes the remaining tangential electric field at the interface. Thus, we refer to this new TC as hybrid TC (HTC).

We demonstrate the effectiveness of this approach by computing the electromagnetic fields of a TESLA cavity, driven by the current of an ultra-relativistic particle beam passing through the DDM interfaces.

II. The Electromagnetic Problem

The problem to be solved is the electromagnetic wave equation in the frequency domain

$$\begin{aligned} \nabla \times \nabla \times \mathbf{E} - k^2 \mathbf{E} &= -j\omega\mu\mathbf{J} && \text{in } \Omega, \\ \mathbf{n} \times \mathbf{E} &= 0 && \text{on } \Gamma_{\text{PEC}}, \\ \mathbf{E} - \sum_{m=1}^M \alpha_m^{\text{TX}} \mathbf{e}_m^{\text{TX}} &= \mathbf{E}^{\text{Inc}} && \text{on } \Gamma_{\text{Port}}, \end{aligned} \quad (1)$$

where k is the free-space wave number, ω the angular frequency, μ the permeability, $\mathbf{J}$ the current density of the ultra-relativistic particle beam, and Ω the computational domain. The boundaries are assumed to be either perfectly conducting (PEC) walls or waveguide openings, denoted by Γ_{PEC} and Γ_{Port} , respectively. In (1), TX represents the TE, TM or TEM modes supported by the waveguide ports. Further, $\mathbf{e}_m^{\text{TX}}$ denotes the modal fields at the input and output waveguides, while α_m represents their corresponding amplitudes. The quantity $\mathbf{E}^{\text{Inc}}$ is the incident field that is supported by the particle beam on the waveguide openings.

III. Domain Decomposition

The idea of DDM consists in splitting the computational domain into subdomains Ω_i , where the introduced boundary value problem (1) is solved within each subdomain. This results in problems of substantially smaller size than the original one. Moreover, the subdomain problems can be solved on separate computers, providing an efficient method for parallelizing computations.

The global solution, however, must ensure continuity of the electric fields at domain interfaces. In the non-overlapping Schwarz DDM approach, this is achieved by iteratively solving the subdomain problems with appropriate transmission conditions at the interfaces [4]. In general, the transmission from domain 2 to domain 1 through interface Σ is defined by the condition

$$\begin{aligned} j\mathbf{k}\gamma_1^T(\mathbf{E}_1) + \mathbf{n}_1 \times (\nabla \times \mathbf{E}_1)|_{\partial\Omega_1} \\ = j\mathbf{k}\gamma_2^T(\mathbf{E}_2) + \mathbf{n}_2 \times (\nabla \times \mathbf{E}_2)|_{\partial\Omega_2} := \mathbf{F}_{12} \quad \text{on } \Sigma, \end{aligned} \quad (2)$$

where $\mathbf{E}_i$ represents the electric fields in each subdomain, $\gamma_i^T : v \rightarrow \mathbf{n}_i \times (v|_{\partial\Omega_i} \times \mathbf{n}_i)$ is the tangential trace operator, $\mathbf{k}$ is the wave-number operator, which is defined in the following, and $\mathbf{n}_i$ is the outward-facing normal vector of subdomain i . In the following, we will refer to the quantity $\mathbf{F}_{12}$ as boundary flux. Analogously, the coupling from domain 1 to 2 can be defined.

A. Transmission Conditions

An exact transmission condition can be defined on a flat interface between subdomains. This condition follows by the choice

$$\mathbf{k}\gamma_i^T(\mathbf{E}) = (k_0^2\mathbf{I} + \Delta_\Sigma)^{-0.5}(k_0^2\mathbf{I} - \mathbf{curl}_\Sigma\mathbf{curl}_\Sigma)\gamma_i^T(\mathbf{E}), \quad (3)$$

where $\mathbf{I}$ denotes the identity operator [4].

The commonly used OTCs are obtained by approximating $\mathbf{k}$. The simplest choice is [4]

$$\mathbf{k}\gamma_i^T(\mathbf{E}) \approx k_0\gamma_i^T(\mathbf{E}). \quad (4)$$

We refer to the resulting TC as OTC of order 0 (OTC0). A more accurate approximation involves the choice

$$\mathbf{k}\gamma_i^T(\mathbf{E}) \approx p^{-1}(k_0^2\mathbf{I} - \mathbf{curl}_\Sigma\mathbf{curl}_\Sigma)\gamma_i^T(\mathbf{E}), \quad (5)$$

where $p \in \mathbb{C}$ is a free parameter [3]. We refer to this case as extended OTC0 (OTC0e).

The modal transmission condition is based on the orthogonal decomposition of the interface field. It is defined by

$$\mathbf{k}\gamma_i^T(\mathbf{E}) = \sum_{m=1}^{\infty} k_m^{\text{TX}} \alpha_m^{\text{TX}} \mathbf{e}_m^{\text{TX}}, \quad (6)$$

where k_m^{TX} are functions that only depend on mode type and frequency [7]. As this TC provides an exact representation of the wave-number operator, it achieves optimal convergence rate. In practice, however, the number of modes has to be truncated to a finite number M , which introduces some error.

In order to ensure convergence to machine precision, the MTC and the OTCs can be combined into a hybrid TC, which is defined by

$$\mathbf{k}\gamma_i^T(\mathbf{E}) = \sum_{m=1}^M (k_m^{\text{TX}}\mathbf{I} - \mathbf{k})\alpha_m^{\text{TX}} \mathbf{e}_m^{\text{TX}} + \mathbf{k}\gamma_i^T(\mathbf{E}). \quad (7)$$

IV. Tesla Cavity

The proposed method is applied to compute the electric field of a TESLA cavity operating at 1.3 GHz. The cavity is decomposed into 11 domains. The resulting CAD model is shown in Fig. 1. The walls are modelled as PEC, while the beam pipes and couplers use waveguide boundary conditions.

In the simulation, the GMRES algorithm is employed as the iteration scheme. The employed TCs are the MTC with all modes, the OTC0 and the OTC0e, and three hybrid TCs. For the OTC0e, $p = k_0(1+j)/\sqrt{2}$ is chosen to ensure convergence of both evanescent and propagating modes. For the HTC0e, $p = k_0$ can also be used, as the propagating

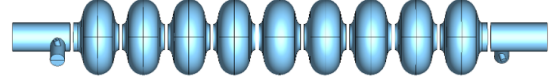


Fig. 1. Decomposition of a TESLA cavity into 11 domains.

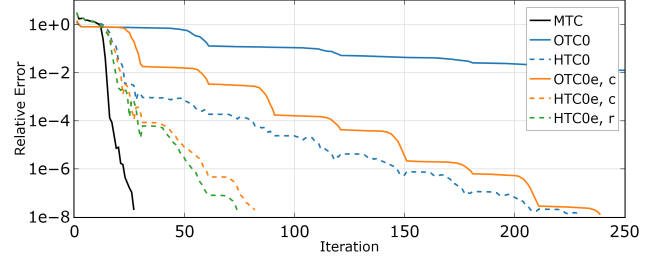


Fig. 2. Convergence histories for different TCs in the Tesla cavity application. In case of the MTC, all modes which are defined by the discretization are used. In all cases of the HTC, 10 TE and 10 TM modes are considered. In case of the OTC0e and the HTC0e, 'c' indicates a complex choice of p , while 'r' indicates a real value.

modes are coupled by the modal component of the TC, leaving only the evanescent modes to be coupled by the free-space component.

The resulting convergence histories are shown in Fig. 2. As shown in the plot, the HTCs with just 10 TE and 10 TM modes significantly outperform their corresponding OTCs. Moreover, selecting p as a real number further enhances the convergence of the HTC0e. In this configuration, nearly the convergence rate of the full MTC including all interface modes is achieved, using a limited number of modes.

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