

Quantized Tensor Train for Topology Optimization of Integral Equation Based Models

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Topology Optimization (TO) presents a significant computational challenge due to its reliance on the design domain mesh, where finer discretizations allow for more detailed configurations but exponentially increase computational demands. This paper explores the application of Quantized Tensor Train (QTT) decompositions to mitigate these challenges in large-scale TO problems for electromagnetics. By efficiently approximating and compactly storing the dense matrices generated from the discretization of Integral Equation (IE) operators, QTT drastically reduces the computational burden. This innovative approach enables the practical use of high-resolution grids, opening new avenues for creating highly detailed and computationally efficient electromagnetic designs.

Index Terms—Integral equations, quantized tensor train (QTT), topology optimization, EMI.

I. INTRODUCTION

TOPOLOGY optimization (TO) has recently emerged as a powerful tool for generating innovative designs of electromagnetic components, driven by the advanced manufacturing capabilities introduced by additive manufacturing (AM) technologies [1], [2].

TO belongs to the category of large-scale optimization problems, as it involves a significant number of design variables. This complexity arises from TO's direct operation on the mesh of the design domain, where increasing the number of elements enables highly detailed configurations. For structured grids, tensor decomposition techniques, such as the Quantized Tensor Train (QTT), offer a promising approach to significantly reduce the computational burden associated with storing the dense matrices generated during the discretization of Integral Equation (IE) operators in electromagnetics.

In this paper, the effectiveness of QTT for the TO of a magnetic shield is evaluated.

II. ELECTROMAGNETIC MODEL

In this section, the EM model used to evaluate the shielding effectiveness (SE) of a magnetic screen is described. The SE is defined as the ratio of the transmitted $\mathbf{H}_t$ and incident $\mathbf{H}_i$ magnetic field amplitudes:

$$SE_{dB} = 20 \log_{10} \left(\frac{H_t}{H_i} \right), \quad (1)$$

By discretizing the Magnetic Field Integral Equation (MFIE) in the magnetostatic limit, and introducing *star basis functions* [3] to discretize the magnetization (i.e., the problem unknown), the following system of equations is obtained:

$$\mathbf{M}(\mu_r)\mathbf{u} = (\mathbf{R}(\mu_r) + \mathbf{A})\mathbf{u} = \mathbf{b}, \quad (2)$$

where $\mathbf{u}$ is the array storing the Degrees of Freedom related to the magnetization in the magnetic shield, $\mathbf{A}$ is the dense matrix that discretizes the Green's function operator, $\mathbf{b}$ is the

rhs term related to the problem excitation, and $\mathbf{R}$ is the sparse matrix given by

$$R_{i,j} = \int \mathbf{w}_i^s(\mathbf{r}) \cdot \mathbf{w}_j^s(\mathbf{r}) \frac{1}{\mu_0(\mu_r(\mathbf{r}) - 1)} d\Omega. \quad (3)$$

With this discretization, the number of unknowns coincides with the number of mesh elements, thus simplifying the application of the QTT described in the next section. Moreover, deflation [4] is applied to (2) to solve its ill-conditioning.

Finally, by removing the logarithm dependence from (1) and exploiting that $\mathbf{H}_t = \mathbf{H}_i - \mathbf{H}_{scat}$, where $\mathbf{H}_{scat}$ is the field generated by the shield, the objective function can be written as a linear operator $\mathbf{C}^T$ acting on $\mathbf{u}$, i.e., $f_{obj} = \mathbf{C}^T \mathbf{u}$.

III. TOPOLOGY OPTIMIZATION

The TO deals with the automatic distribution of a material within a design domain by acting on the design variables x_i associated with the mesh elements. Particularly, for continuous design variables i.e., $x_i \in [0, 1]$, the element is void if “void” if $x_i = 0$, “filled” when $x_i = 1$ (i.e., $x_i = 1 \rightarrow \mu_r = 1000$, $x_i = 0 \rightarrow \mu_r = 1 + \epsilon$). Introducing a discretization of N elements for the domain, the binary TO problem is written as:

$$\begin{aligned} \max \quad & f_{obj}(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in [0, 1]^N \\ & g(\mathbf{x}) = \sum_{i=1}^N x_i vol_i - \alpha V_\Omega \leq 0, \end{aligned} \quad (4)$$

where the constraint $g(\mathbf{x})$ is added and α is a user-defined value to impose the final volume fraction.

A. Updating Design Variables

This paper updates the design variables via the generalized optimality criteria method (GOCM) [5]. The approach is based on the evaluation of the following scale factor for each design variable at k -th iteration:

$$D_i = - \frac{\frac{\partial f_{obj}(\mathbf{x})}{\partial x_i^k}}{\lambda \frac{\partial V(\mathbf{x})}{\partial x_i^k}}, \quad (5)$$

from which the value of x_i^{k+1} is computed as

$$x_i^{k+1} = x_i^k D_i^{1/(\beta^k q_0)}, \quad (6)$$

where β and q_0 are user-defined parameters [5].

Although the evaluation of volume constraint derivative is straightforward, the GOCM (5) requires the evaluation of the sensitivity of the objective function i.e., $\partial f_{obj}(\mathbf{x})/\partial x_i$, usually obtained with the Adjoint Variable Method (AVM).

IV. TENSOR TRAIN FORMAT

Tensor-Train (TT) format and Quantized TT (QTT) decomposition are efficient techniques to store multidimensional arrays (tensor) characterized by low-rank properties [6].

A d -dimensional tensor $\mathcal{T}$ is approximated with the TT decomposition if the entries are written as [7]:

$$\mathcal{T}(i_1, i_2, \dots, i_d) \approx \sum_{\alpha_1, \dots, \alpha_{d-1}} G_1(i_1, \alpha_1) G_2(\alpha_1, i_2, \alpha_2) \cdots G_d(\alpha_{d-1}, i_d), \quad (7)$$

where $i_k = 1, \dots, n_k$ with n_k the mode size, G_k are called cores, $\alpha_k = 1, \dots, r_k$, and r_k identify the compression ranks. In QTT format $n_k = 2$ for each mode. Exploiting the decomposition (7), for a tensor $\mathcal{T}$ with $\mathcal{O}(n^d)$ elements, the storage complexity reduces to $\mathcal{O}(dnr_{max}^2)$, where $r_{max} = \max(r_k)$, showing the great advantage of the decomposition if the rank r_{max} is small.

Matrices arising from the discretization of EM integral equations, for example through the Method of Moments (MoM) are characterized by low-rank properties, i.e., can be efficiently represented with a reduced number of entries. The possibility of using the QTT decomposition to such matrices is discussed in [8], [9], where the procedure to embed the matrix arising from the IE discretization, $\mathbf{A} \in \mathbb{R}^{n \times n}$ with $n = 2^d$, into a $2d$ -dimensional tensor $\mathcal{A}$, is described.

It is worth mentioning that in our case, the dimension of matrix $\mathbf{A}$ has a one-to-one correspondence to the number of mesh elements. The possibility of keeping the computational complexity low, while increasing the number of mesh elements is a great advantage to perform the TO on very fine grids.

Following [10], the tensor decomposition (i.e., the cores) is built with the Density Matrix Renormalization Group (DMRG) scheme. It is important to note that the construction of tensorized versions of $\mathbf{M}$ and $\mathbf{b}$ does not require the previous construction of the quantities; however, only a function to build the generic entry (e.g., M_{ij}) is required. Indeed, by using QTT decomposition, only a subset of the entries of $\mathbf{M}$ is generated.

V. NUMERICAL RESULTS

The QTT-based TO problem described previously is applied to a magnetic shield of dimensions $6.4 \times 6.4 \times 0.1 \text{ m}^3$. The objective of the optimization is the maximization of the shielding effectiveness. The numerical algorithm is realized in MATLAB based on the TT-toolbox available at the Github page <https://github.com/oseledets/TT-Toolbox>, for constructing the tensor representation.

A grid with $n = 2^7 \times 2^7 \times 2$ hexahedral elements discretizes the computational domain (i.e., the shield) in x, y, z directions.

TABLE I
PERFORMANCES OF TENSOR DECOMPOSITION

	r_{max}	CR (%)	ϵ (%)
Tensor Decomposition	60	$7 \cdot 10^{-3}$	$6 \cdot 10^{-10}$

The performances of QTT decomposition are summarized in Table I. Particularly, the maximum rank, the compression ratio, and relative error (percentage) are listed, where

$$\epsilon = 100 \times \frac{\|\tilde{\mathcal{U}} - \mathcal{U}\|}{\|\mathcal{U}\|}. \quad (8)$$

$\mathcal{U}$ is the QTT form of the solution vector $\mathbf{u}$. The compression ratio (CR) is evaluated as:

$$\text{CR} = 100 \times \frac{\text{GetSize}(\mathcal{M})}{\text{GetSize}(\mathbf{M})} \quad (9)$$

where $\text{GetSize}(\cdot)$ computes the memory requirements for storing the argument and $\mathcal{M}$ is the QTT representation of matrix $\mathbf{M}$.

VI. CONCLUSIONS

Preliminary results demonstrate that tensor decomposition using the Quantized Tensor Train (QTT) format can effectively reduce the computational cost of Integral Equation (IE)-based approaches for topology optimization (TO) problems. In the full paper, a comprehensive analysis of the results and computational cost associated with the QTT-IE-based TO method will be presented. The QTT approximation significantly alleviates the computational burden caused by the fully populated system matrix inherent in electromagnetic problems, enabling the use of grids with a large number of elements.

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