

Discrete Duality Finite Volume scheme to model Superconductor

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In this paper, we present the use of the Finite Volume Method to compute the nonlinear behavior of superconducting material. In this approach, the *curl-curl* operator is discretized using the diamond scheme and we solve the H-formulation of the electromagnetic problem in superconducting material. An example of a 2D cable with six superconducting filaments is proposed to show the efficiency of this approach.

Index Terms—Finite volume methods, High-temperature superconductors, Modeling, Nonlinear equations

I. INTRODUCTION

WITH the recent development of High-Temperature Superconductors (HTS), projects based on their properties are under study in strategic domains. Many of these are related to energy production technology such as fusion power, HTS generator/rotor [1]. HTS cables are also a key component of smart grids which should be more resilient and power areas more efficiently [2], [3], [4]. Those projects need efficient and high-performance modeling tools to design superconducting systems. Most of them, open source [5] or commercial software (COMSOL, Ansys), use Finite Element Method (FEM) to perform their computations. Due to large complex models and the nonlinear behavior of superconducting materials, computations can be very costly and time consuming. This motivates us to develop a parallel Finite Volume Methods solver.

Finite Volume (FV) methods are extensively used in Computational Fluids mechanics to solve linear or nonlinear differential problems. They are based on numerical fluxes that describe exchange between mesh cells. They are well suited for parallel computations and massively parallel solvers have been developed for fluids dynamics problems [6], [7].

In this paper, we propose a Finite Volume method to compute electromagnetic fields in superconducting materials. Our work is based on the diamond scheme discretization technique [8], [12]. We also present an adaptation of the classical diamond scheme that better handles discontinuities [9], [10]. The diamond scheme used in fluid dynamics requires approximation of the *gradient* operator. In our case, with PDEs related to Maxwell's equations, an approximation of the *curl* operator is proposed. The nonlinearity of the superconductor power law is linearized following the same principle as [11] and a Newton-Raphson algorithm is implemented.

II. PROBLEM STUDIED

In this paper we consider the *H*-formulation (1) of the electromagnetic problem of superconducting materials obtained with Maxwell's equations and the constitutive power law (2) that describes the behavior of superconducting materials:

$$\mu_0 \frac{\partial H}{\partial t} + \nabla \times (\rho(J) \nabla \times H) = 0 \quad (1)$$

$$E(J) = \rho(J)J \quad \text{with} \quad \rho(J) = \frac{E_c}{J_c} \left\| \frac{J}{J_c} \right\|^n \quad (2)$$

where H is the magnetic field, $J = \nabla \times H$ is the current density, E_c , J_c , n are the superconductor parameters that we consider constant in this paper. For most HTS material, the power law exponent $n \in [5, 40]$.

The nonlinearity of the problem is dealt with the following approximation of the power law relationship in the superconducting domain:

$$E(J) = E(\tilde{J}) + A_H(J - \tilde{J})$$

that can be rewritten as:

$$\rho(J) \nabla \times H = \rho(\tilde{J}) \nabla \times \tilde{H} + A_H \nabla \times (H - \tilde{H})$$

with A_H the Jacobian matrix $\frac{\partial E}{\partial J}$, $\tilde{J}$ and $\tilde{H}$ the iterative variables of the Newton-Raphson algorithm.

III. DISCRETIZATION

We denote by Ω our domain and its mesh $\mathcal{T}_h = \bigcup K$, where K are the mesh cells. The flux expressions at the edges of K are obtained from Green's theorem:

$$\int_K \nabla \times (\rho \nabla \times H) = \int_{\partial K} \nu \times (\rho \nabla \times H) \quad (3)$$

where, ∂K are the edges of K and ν the outward normal vector. To approximate $\rho \nabla \times H$ on edges of K , we use a Discrete Duality Finite Volume (DDFV) scheme.

A. Diamond scheme

We present the diamond scheme in 2D. From any inner edge σ , we note S^+ and S^- the nodes defining its bounds and τ^+ , τ^- the barycenter of the cells on each side of σ . Those four points allow us to define the diamond $\mathcal{D}_\sigma$ with the polygon $S^+ \tau^+ S^- \tau^-$ represented by Fig. 1. We note $|\mathcal{D}_\sigma|$ its measure. Two normals are also introduced, ν_S for the segment $\tau^+ \tau^-$ and ν_σ for the edge σ . Their signs are defined by the inequality $\det(\tau^+ - \tau^-, S^+ - S^-) > 0$.

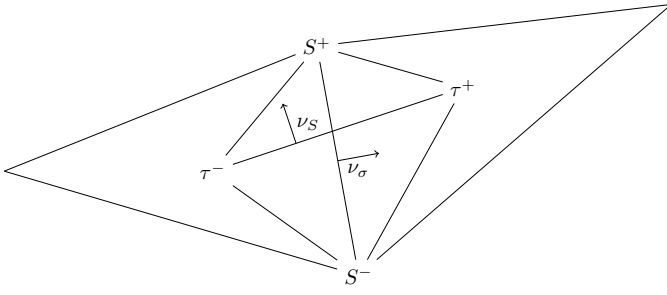


Fig. 1. Diamond scheme between cells τ^+ and τ^- .

The approximation of $\nabla \times H$ on σ is done using the value of H on the diamond:

$$|\mathcal{D}_\sigma| (\nabla \times H)|_\sigma \approx \int_{\mathcal{D}_\sigma} (\nabla \times H) dV$$

Applying the Green-Gauss theorem and some factorisation, we have the following formula:

$$(\nabla \times H)|_\sigma = \frac{1}{|\mathcal{D}_\sigma|} \left(\nu_\sigma \times \frac{H_{\tau^+} - H_{\tau^-}}{2} + \nu_S \times \frac{H_{S^+} - H_{S^-}}{2} \right)$$

To have the discretization on the edges, required in (3), we apply the following rules

- When ρ is a piecewise constant in cells τ^- and τ^+ , we choose the average value $\rho_\sigma = \frac{\rho_{\tau^-} + \rho_{\tau^+}}{2}$
- In the superconductor, we use the average current density at the edge $J_\sigma = \frac{J_{\tau^-} + J_{\tau^+}}{2}$ to compute $\rho(J_\sigma)$ and the Jacobian matrix A_H

For faces where ρ value on each side are too different or are computed in a different manner, an adaptation of the diamond scheme yields better results [9].

IV. APPLICATION

In this example, we consider a HTS cable constituted by six superconducting filaments embedded in a resistive matrix. The matrix diameter is $d_M = 0.1 \text{ mm}$ and its resistivity $\rho_M = 10^{-6} \Omega m$. The superconducting filament diameter is $d_S = 10 \text{ } \mu m$, and the superconducting material parameters are $E_c = 10^{-4} \text{ V.m}^{-1}$, $J_c = 10^7 \text{ A.m}^{-2}$ and $n = 10$.

The HTS cable is subjected to a constant external transverse magnetic field with an amplitude 1 mT . The current density distribution obtained after 80 iterations with a time step $\delta t = 10^{-5} \text{ s}$ is plotted in Fig. 2. Our result is in conformity to that obtained from FEM computation with Open Source GetDP [5].

V. CONCLUSION

In this paper we have introduced a DDFV scheme to model superconducting materials by solving H-formulation. An example on a 2D superconducting cable is proposed as an illustration. More details will be presented in the extended version of this paper, especially with regard to 3D developments.

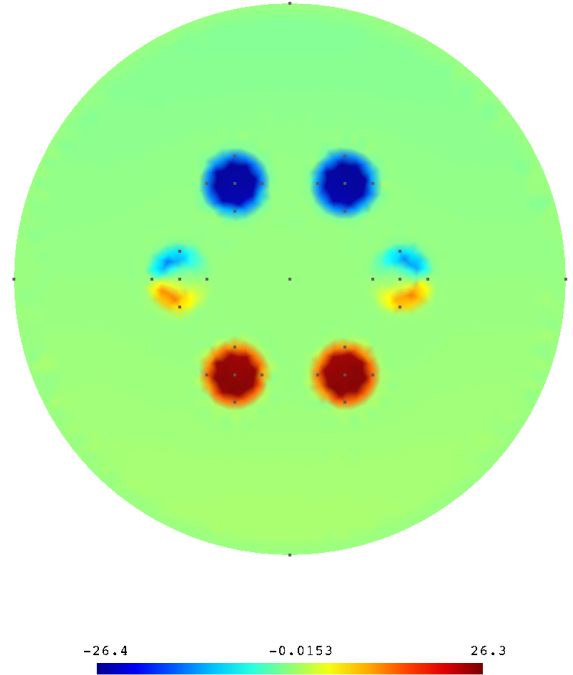


Fig. 2. View of the current density J (Amm^{-2}) of 6 HTS filaments.

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