Virtual Element Method Applied to a 2D Transient Eddy Current Problem

A. Branchu-Harel and N. Galopin

Univ. Grenoble Alpes, CNRS, Grenoble INP, G2Elab, F-38000 Grenoble, France
nicolas.galopin@univ-grenoble-alpes.fr

The virtual element method have the ability in handling meshes with polygonal elements and thus offers greater flexibility than the finite element method for meshing complex geometries. We propose here to apply the method to a transient induced current problem and apply it to the case of an axisymmetric magnetic core.

***Index Terms*—Virtual element method, transient eddy current**

I. Introduction

THE VIRTUAL element method (VEM) is a recent numerical method which can be view as a generalisation of the classical finite element method [1]. VEM allows very general domain discretizations with the flexibility to handle meshes with almost arbitrary polygonal (2D) or polyhedral (3D) shaped elements, which can even be non-convex and with curved edges or faces [2]. The use of polygonal/polyhedral meshes provides several benefits, including a substantial simplification of domain partitioning, even for highly complex geometries, and a decrease in the complexity of mesh adaptation algorithms, as no special manipulation of hanging nodes is necessary to ensure mesh conformity. The use of arbitrary polygonal/polyhedral elements, and the requirement for compatible interconnection of elements along their sides, implies functional spaces that extend beyond just polynomial functions. However, unlike the finite element method (FEM), in VEM the space of local shape functions in each element is defined implicitly. Therefore, to define an approximation of the variational formulation, a set of projection operators, computed from the degrees of freedom (DoFs) of the field, is used to projects the shape functions of each element onto a space of polynomials up to a given order k. To overcome the rank-deficiency due to the projection of primary variables onto polynomial spaces, a stabilization term is added that relies on additional functions, generally non-polynomial, that are solutions to a boundary value problem.

The method has been applied in a wide variety of problems, principally in computational mechanics [3]-[4], but few in computational electromagnetics [5]. Here we propose an application of the method to the problem of transient induced currents in a two-dimensional framework.

II. The Continuous Problem

We are interested in solving the following Maxwell's equations on a domain Ω in the so-called magnetoquasistatic regime [6].

$$\nabla \times \boldsymbol{h} = \boldsymbol{j},\ \nabla \cdot \boldsymbol{b} = 0,\ \nabla \times \boldsymbol{e} = -\partial_t \boldsymbol{b},\ \boldsymbol{h} = \boldsymbol{\nu} \boldsymbol{b},\ \boldsymbol{j} = \boldsymbol{\sigma} \boldsymbol{e}. \quad (1)$$

In (1), $\boldsymbol{b}$, $\boldsymbol{h}$, $\boldsymbol{e}$ and $\boldsymbol{j}$ are the magnetic flux density, magnetic field, electric field and electric current density, respectively, and $\boldsymbol{\nu}$ and $\boldsymbol{\sigma}$ are the magnetic reluctivity and electrical conductivity tensors. It is assumed that only temporal variations in $\boldsymbol{b}$ are the source of eddy currents. In the case of an axisymmetric problem and considering a magnetic vector potential formulation, the variational formulation associated with (1) in an open domain $\Omega \subset \mathbb{R}^2$ is written:

Find $\tilde{a} \in V$ such that:

$$\int_\Omega \nabla \widetilde{\delta a} \cdot \boldsymbol{\nu} \cdot \nabla \tilde{a} \frac{1}{r} d\Omega - \int_\Omega \frac{\sigma}{r} \widetilde{\delta a} \frac{\partial \tilde{a}}{\partial t} d\Omega - \int_\Omega \widetilde{\delta a} j_{s_\theta} d\Omega = 0 \quad (2)$$

for all test function $\widetilde{\delta a}$ in an appropriate functional space V, where time interval $I = [t_0, t_{end})$, with $t \in I$. $\tilde{a} = r a_\theta$ with a_θ the angular component of the magnetic vector potential and j_{s_θ} the angular component of the source current density.

III. VEM Discretization

Let a mesh Ω_h be a decomposition of the domain Ω into finite number of non-overlapping polygon elements $K \in \mathbb{R}^2$. We denote by $E \in \partial K$ the generic edge and $|K|$ the area of polygon K.

A. Virtual element spaces

In the VEM, the discrete space V_h is always distributed over the mesh, but its restriction on an element is no longer a polynomial space, it is defined as follows [1]:

$$V_h := \left\{ v_h \in H_0^1(\Omega) : v_{h|K} \in V_k(K)\, \forall K \in \Omega_h \right\} \quad (3)$$

with

$$V_k(K) := \{ v \in H^1(K) \cap C^0(\partial K) : \Delta v_{|K} \in \mathbb{P}_{k-2}(K), \\ v_{|E} \in \mathbb{P}_k(E)\, \forall E \subset \partial K \} \quad (4)$$

where $\mathbb{P}_k(E)$ is the space of polynomials of degree up to k on E and Δ denotes the Laplace operator. The space V_h always contains polynomials of degree k, but it is enhanced by other unknown functions that satisfy two requirements: their Laplacian is a polynomial of degree $k-2$ and they are polynomials of degree k on ∂K. For $k = 1$, the space $V_k(K)$ is a space of harmonic functions which are piecewise linear and continuous on ∂K.

It is possible to show that a set of degrees of freedom (DoFs) for space $V_k(K)$ are: the value of v at the vertices of K; the value of v at $k-1$ points inside each edge $E \in \partial K$; the internal scaled moments of v inside the polygon K: $\frac{1}{|K|}\int_K v\, m_\alpha\, dK$ with $\{m_\alpha\}$ the base of the scaled monomials [7].

B. Construction of the bilinear form

As the functions of V_h are not explicitly known within the element domain K, the local discrete bilinear form $a_K(\bullet,\bullet)$, related to (2), cannot be computed using standard numerical integration, as is usually done in FEM. To do this, VEM relies on the split of the magnetic vector potential $\tilde{a}_h \in V_h$ as:

$$\tilde{a}_h = \Pi_k^\nabla \tilde{a}_h + \left(\tilde{a}_h - \Pi_k^\nabla \tilde{a}_h\right) \tag{5}$$

where Π_k^∇ is a projection operator on a polynomial space $\Pi_k^\nabla : V_k(K) \to \mathbb{P}_k(K)$ defined as follows:

$$\begin{cases} \displaystyle\int_K \Pi_k^\nabla \tilde{a}_h \cdot \nabla p_k dK = \int_K \nabla \tilde{a}_h \cdot \nabla p_k dK \;\; \forall p_k \in \mathbb{P}_k(K) \\ \displaystyle\int_{\partial K} \Pi_k^\nabla \tilde{a}_h dE = \int_{\partial K} \tilde{a}_h dE \end{cases} \tag{6}$$

The projection $\Pi_k^\nabla \tilde{a}_h$ is a polynomial of degree k and can be computed using the DoFs of $\tilde{a}_h$, meaning that we can calculate $\Pi_k^\nabla \tilde{a}_h$ without explicitly knowing the function $\tilde{a}_h$ itself. From the bi-linearity of $a_K(\bullet,\bullet)$ and the orthogonality property, consistency and stability properties [1] lead to the local discrete bilinear form of (2), for all $K \in \Omega_h$ and for all $\tilde{a}_h, \widetilde{\delta a}_h \in V_k(K)$, defined as:

$$\begin{aligned} a_K\left(\tilde{a}_h, \widetilde{\delta a}_h\right) = {} & a_K\left(\Pi_k^\nabla \tilde{a}_h, \Pi_k^\nabla \widetilde{\delta a}_h\right) \\ & + s_K\left(\tilde{a}_h - \Pi_k^\nabla \tilde{a}_h, \widetilde{\delta a}_h - \Pi_k^\nabla \widetilde{\delta a}_h\right) \end{aligned} \tag{7}$$

In equation (7), the first term is the consistency term, as it maintains the method's accuracy. The second term is a stability term, which ensures the appropriate convergence rate of the solution. This stability term can be any positive definite symmetric bilinear form satisfying [1]:

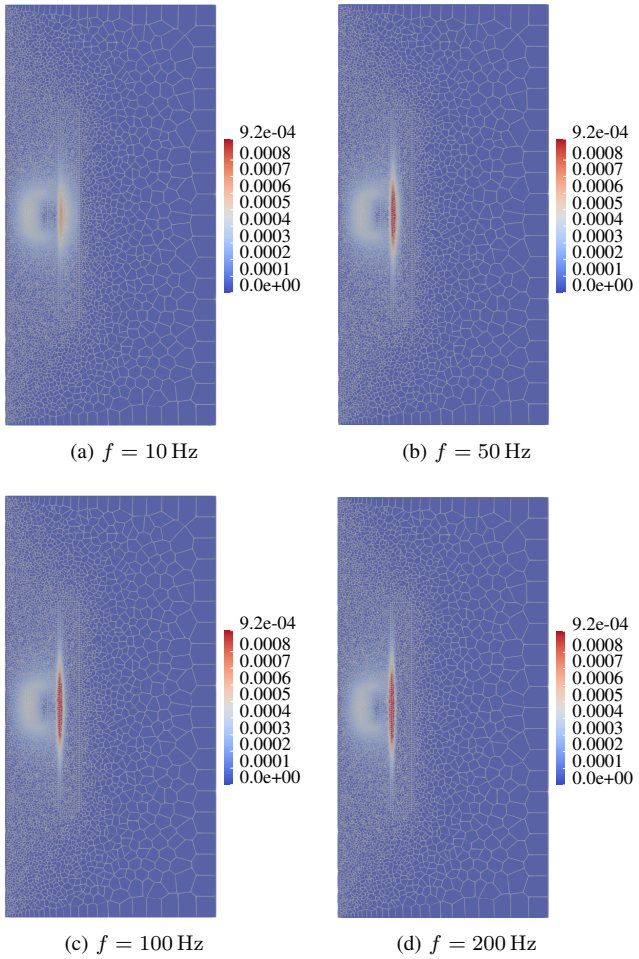
$$c_0\, a_K(v,v) \le s_K(v,v) \le c_1\, a_K(v,v) \quad \forall v \in V_k(K) \tag{8}$$

with $\Pi_k^\nabla v = 0$, c_0 and c_1 two positive constants.

IV. Results

As a test case, we consider a metallic conductor tube whose central axis is oriented along the axis $\boldsymbol{u}_z$, with a height of $10\,\mathrm{cm}$ and a thickness of $1\,\mathrm{cm}$. Inside the cylinder, there is a cylindrical inductor with a height of $2\,\mathrm{cm}$ and a thickness of $1\,\mathrm{cm}$. The tube is made of a linear and isotropic material, with a relative magnetic permeability of $\mu_r = 1000$ and an electrical conductivity of $\sigma = 1 \times 10^{-6}\,\mathrm{S\,m^{-1}}$. A sinusoidal current density is applied to the inductor at frequencies of $10\,\mathrm{Hz}$ and $100\,\mathrm{Hz}$. For temporal resolution, an implicit Euler scheme is used. The time step is set according to the frequency as: $\Delta t = \frac{1}{10f}$.

The evolution of magnetic flux density distribution in the tube for different frequencies f and at $t = \frac{T}{4}$ is shown in Figure 1. The effect of the presence of induced currents, as frequency increases, on the magnetic flux density distribution at the periphery of the tube's inner surface is clearly visible.



(a) $f = 10\,\mathrm{Hz}$ (b) $f = 50\,\mathrm{Hz}$

(c) $f = 100\,\mathrm{Hz}$ (d) $f = 200\,\mathrm{Hz}$

Fig. 1: Magnetic flux density distribution $\boldsymbol{B}$ distribution for different frequencies f at time $t = \frac{T}{4}$ ($T = \frac{1}{f}$).

In the full version of the paper we will further develop the construction of the discrete weak form and apply it to the modelling of an inductor.

References

[1] L. Beirão Da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L. D. Marini and A. Russo, "Basic principles of virtual element methods," *Math. Models Methods Appl. Sci.*, vol. 23(1):199–214, 2013.

[2] L. Beirão Da Veiga, A. Russo and G. Vacca, "The virtual element method with curved edges," *ESAIM: Math. Model. Numer. Anal.*, vol. 53(2):375–404, 2019.

[3] A. L. Gain, C. Talischi, G. H. Paulino," On the virtual element method for three-dimensional linear elasticity problems on arbitrary polyhedral meshes," *Comput. Methods Appl. Mech. Engrg.*, vol. 282:132–160, 2014.

[4] B. Hudobivnik, F. Aldakheel and P. Wriggers, "A low order 3D virtual element formulation for finite elasto–plastic deformations," *Comput. Mech.*, vol. 63:253–269, 2019.

[5] F. Dassi, P. Di Barba and A. Russo, "Virtual element method and permanent magnet simulations: potential and mixed formulations," *IET Sci. Meas. Technol.*, vol. 14:1098-1104, 2020.

[6] G. Meunier, *The finite element method for electromagnetic modelling,* 1st ed., John Wiley & Sons, Ltd, 2008.

[7] L. Beirão Da Veiga,, F. Brezzi, L. Marini and A. Russo. "The hitchhiker's guide to the virtual element method," *Math. Models Methods Appl. Sci.*, vol. 24(08):1541–1573, 2014.