

An Alternating Explicit-Implicit Time Domain Difference Method Based on the $\mathbf{J} - \phi$ Volume Integral Formulation for Electromagnetic Simulations of High-Temperature Superconductors

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When calculating the induced currents in second-generation high-temperature superconductors (HTS), the strong nonlinear resistivity of HTS poses a critical challenge. This issue is typically addressed using the Newton-Raphson method, which requires extensive computation of Jacobian matrices. However, the introduction of the $\mathbf{J} - \phi$ volume integral formulation enables explicit time domain difference method, thereby avoiding the handling of nonlinear resistivity. Nevertheless, this method is conditionally stable and requires precise control of the time step to prevent divergence of results. In this paper, we propose an alternating explicit-implicit time domain difference method based on the $\mathbf{J} - \phi$ volume integral formulation. By inserting implicit calculations into the original explicit process, this method prevents the rapid accumulation of perturbation errors during the time-stepping process. It overcomes the limitations of explicit time domain difference method, offering unconditional stability. Compared to the implicit time domain difference method, it significantly reduces the number of Jacobian matrix computations. Additionally, we present an adaptive second-order explicit-implicit time domain difference method, which determines whether to insert implicit calculations by estimating errors and their growth rates. This approach not only improves computational accuracy but also demonstrates high stability.

Index Terms—High-temperature superconductors, integral equations, nonlinear equations, numerical analysis.

I. INTRODUCTION

IN recent years, the 2D integral formulation proposed by E. H. Brandt in 1996 to calculate the induced current in HTS subjected to time-varying external magnetic fields has received much attention [1]. The method not only has the advantage of integral equations, but also effectively avoids solving for the nonlinear resistivity of superconductors. In the field of computational electromagnetics, a numerical technique similar to this one is the volume integral formulation with current density $\mathbf{J}$ and potential ϕ as state variables [2]. When nonlinear resistivity is involved, the Newton-Raphson method is required to solve the system of nonlinear equations. The computational cost of the Jacobian matrix is very large, and when the number of unknowns is very high, even though some variants of Newton's method can simplify this calculation. Therefore, integral methods that do not require the computation of Jacobian matrix are very attractive.

In this paper, we present a Jacobian-free version of the $\mathbf{J} - \phi$ volume integral formulation. It employs an explicit time domain difference method, which has the drawback of achieving convergence only with small time steps. To address this issue, we propose an alternating explicit-implicit time domain difference method. The core idea is to use the implicit difference method to suppress the amplification of perturbation errors during the propagation process caused by the explicit difference method. This significantly reduces the traditional volume integral method's reliance on the Jacobian matrix. The proposed method is validated through the calculation of induced currents in 3D HTS, specifically YBCO tape, under time-varying external magnetic fields, demonstrating stable performance even with large time steps. Moreover, we further

propose an adaptive second-order explicit-implicit time domain difference method, which offers higher computational accuracy.

II. NUMERICAL METHODOLOGY

Under the assumption of magneto quasi-static approximation, the $\mathbf{J} - \phi$ volume integral formulation can be expressed as,

$$\rho(\mathbf{j})\mathbf{j} = -\frac{\mu_0}{4\pi} \frac{\partial}{\partial t} \int \frac{\mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3r' - \frac{\partial \mathbf{a}_a}{\partial t} - \nabla\phi, \quad (1)$$

$$\nabla \cdot \mathbf{j} = 0, \quad (2)$$

where μ_0 is the magnetic permeability in vacuum, $\mathbf{j}$ is the current density, ϕ is the electrical potential, and $\mathbf{r}$, $\mathbf{r}'$ are the field and source points, respectively. $\mathbf{a}_a$ is the vector potential characterizing the applied magnetic field $\mathbf{B}_a$, and under the assumption of a uniform field, automatically satisfies $\nabla^2 \mathbf{a}_a = 0$. ρ is the resistivity of the YBCO layer, which is expressed in terms of $\mathbf{E} - \mathbf{J}$ power law as,

$$\rho(\mathbf{j}) = \frac{E_c}{J_c} \left(\frac{|\mathbf{j}|}{J_c} \right)^{n-1}, \quad (3)$$

where E_c is the electric field criterion for determining the critical current, J_c is the critical current density, and n is power law exponent.

The current density $\mathbf{J}$ and potential ϕ are expanded using Whitney facet elements and piecewise constant basis functions, respectively. The matrix system generated by the $\mathbf{J} - \phi$ volume integral formulation is obtained through the Galerkin's method. The explicit difference method, specifically the θ -method and

$\theta = 0$, is used to avoid the computation of the Jacobian matrix.

$$\mathbb{R}(\mathbf{J}^t)\mathbf{J}^t = -\mathbb{M}\frac{\mathbf{J}^{t+1} - \mathbf{J}^t}{\Delta t} - \frac{\mathbf{A}_a^{t+1} - \mathbf{A}_a^t}{\Delta t} - \mathbb{G}\Phi^t, \quad (4)$$

$$\mathbb{D}\mathbf{J}^t = 0. \quad (5)$$

The current density $\mathbf{J}$ needs to be given an initial value condition. However, equation (5) is not appropriate because it does not allow the current density at the moment $t + 1$ to satisfy the continuity equation (2). An effective solution is to differentiate both sides of equation (2) with respect to time t , that is $\nabla \cdot (\partial \mathbf{j} / \partial t) = 0$. In this case, equation (5) transforms into that $\mathbb{D}(\mathbf{J}^{t+1} - \mathbf{J}^t) / \Delta t = 0$. If the initial value of the current density satisfies $\mathbb{D}\mathbf{J}^t = 0$, then $\mathbb{D}\mathbf{J}^{t+1} = 0$.

The alternating explicit-implicit time domain difference method applies the explicit method in the first of every two time steps, while the implicit method (with $\theta = 1$) is used in the second time step.

The second-order explicit time domain difference method employs a prediction-correction system. The prediction system is formulated as follows.

$$\mathbb{R}(\mathbf{J}^t)\mathbf{J}^t = -\mathbb{M}\frac{\bar{\mathbf{J}}^{t+1} - \mathbf{J}^{t-1}}{2\Delta t} - \frac{\mathbf{A}_a^{t+1} - \mathbf{A}_a^{t-1}}{2\Delta t} - \mathbb{G}\Phi^t, \quad (6)$$

$$\mathbb{D}\frac{\bar{\mathbf{J}}^{t+1} - \mathbf{J}^{t-1}}{2\Delta t} = 0. \quad (7)$$

The correction system adopts the $\theta = 0.5$ method, and the predicted values from the prediction system are substituted into the correction system for solving $\mathbf{J}^{t+1}$. The second-order implicit time-domain differencing method is still chosen as the θ -method and $\theta = 0.5$. Since all three are second-order methods, the difference between any two of them can be used to estimate the error.

$$error_{est} = |\bar{\mathbf{J}}^{t+1} - \mathbf{J}^{t+1}| / J_c. \quad (8)$$

The error estimation for the implicit time-domain difference method is based on the norm of the difference between the computed result and $\mathbf{J}^{t+1}$. When the estimated error $error_{est}$ exceeds a given threshold or the error growth rate surpasses a predefined value, the implicit method must be applied for calculation.

III. NUMERICAL RESULTS

The wide plane of the YBCO layer lies in the xoy plane. The perpendicular field is assumed to be $\mathbf{B}_a = B_m \sin(2\pi ft)\mathbf{e}_z$, where $B_m = 0.02$ T and $f = 50$ Hz. The length, width and thickness of the YBCO layer are 10 cm, 4 mm and 1 μm , respectively. The critical current density of the YBCO layer is set to 2.8×10^{10} A/m². The other two parameters of the $\mathbf{E} - \mathbf{J}$ power law are set to $E_c = 1 \times 10^{-4}$ V/m and $n = 38$.

The simulation time is one quarter of a frequency cycle. We adopt the current density value $\mathbf{J}_{ref}$ computed using the $\theta = 0$ method with an extremely small and constant time step $\Delta t = 1 \times 10^{-8}$ s as the reference. The real computational error

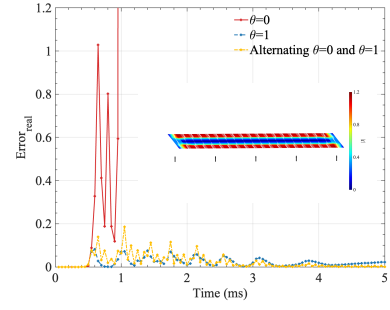


Fig. 1: The actual computational error of the first-order time domain difference method with a constant time step $\Delta t = 0.05$ ms. The induced current density distribution at $t = 0.0025$ s is embedded in the center of the figure.

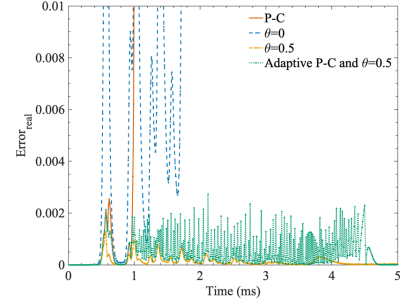


Fig. 2: The actual computational error of the second-order time domain difference method with a constant time step $\Delta t = 0.01$ ms. 'P-C' refers to the prediction-correction method.

is defined as

$$error_{real} = |\mathbf{J}^{t+1} - \mathbf{J}_{ref}^{t+1}| / J_c. \quad (9)$$

As can be seen from figure 1, the explicit difference method ($\theta = 0$) exhibits the largest error, and the results diverge after 1 ms. In contrast, the implicit difference method ($\theta = 1$) shows the smallest computational error, followed by the alternating implicit-explicit difference method. Both of these methods remain stable, indicating that the proposed approach is effective. Furthermore, this method successfully reduces the number of Jacobian matrix computations by half.

As shown in figure 2, the prediction-correction method exhibits significantly smaller calculation errors before divergence compared to the first-order method ($\theta = 0$). However, it diverges earlier than the $\theta = 0$ method, indicating that it is less stable. Therefore, An adaptive second-order explicit-implicit discretization method is proposed to improve the stability of the prediction-correction method. By limiting the estimated error $error_{est}$ and its growth rate, the stability was enhanced. The green curve in the figure 2 illustrates the computational error of this method, which is significantly lower than that of the first-order method ($\theta = 0$) and ultimately achieves stability.

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