

Nonlinearity-Degree-Reduction for Efficient Solution of Highly Nonlinear Poisson's Equations by The Cell Method

Gaojia Zhu^{1,2}, *Member, IEEE*, Longnv Li², *Member, IEEE*, Dingguo Cai¹, Jianhua Huang¹, Yunhui Mei², *Senior Member, IEEE*, and Jianguo Zhu³, *Senior Member, IEEE*

¹PEARL Electric Ltd., Guangzhou, 511400 China, zhugaojia@tiangong.edu.cn

²School of Electrical Engineering, Tiangong University, Tianjin, 300387 China, lilongnv@tiangong.edu.cn

³School of Electrical and Computer Engineering, The University of Sydney, NSW 2006, Australia, jianguo.zhu@sydney.edu.au

Numerical investigations on Poisson's equations with nonlinear parameters play a critical role in modeling various physical problems, such as electric fields in nonlinear dielectrics and magnetic fields in ferromagnetic domains. This study proposes a nonlinearity-degree-reduction method for solving highly nonlinear Poisson's equations by the cell method, aiming to reduce the computational complexity. The proposed approach characterizes a dependent nonlinear physical parameter at the midpoint of a line in a cell complex using weighted combinations of the independent variable values at the line's endpoints. By carefully selecting these combination conditions, the nonlinearity in the derived discrete equations can be propagated from the entire solution domain to the external boundaries. This redistribution reduces the overall computational burden by lowering the degrees of nonlinearity. To validate the method, a nonlinear electrostatic problem is solved using the proposed method and a conventional fixed-point code. Simulation results and iteration steps are compared to demonstrate the accuracy and efficiency of the proposed approach.

Index Terms—Nonlinear problems, Poisson's equation, nonlinearity-degree-reduction, numerical technique.

I. INTRODUCTION

POISSON’S equation is one of the most widely used partial differential equations (PDEs) in modeling modern physical fields [1]. In many applications, such as magnetic fields in ferromagnetic domains, physical properties are highly sensitive to independent variables. It is crucial to solve these equations in their nonlinear forms accurately and efficiently [2].

For solving nonlinear problems, the commonly used iterative methods, such as the fixed-point and Newton-Raphson methods, are based on the approach to linearize the nonlinear parameters and are thus computationally heavy [3-5]. To deal with this problem, a nonlinearity marginalization method was developed to speed up the analysis of thermal fields with temperature-dependent thermal conduction parameters [6]. This approach is, however, limited to thermal fields with quadratic-interpolated parameters. Efficient algorithms for solving highly nonlinear Poisson's equations are yet to be developed.

This paper proposes a nonlinearity-degree-reduction method to solve highly nonlinear Poisson's equations. A polynomial function is used to interpolate the physical parameters at the midpoint of a certain line in a cell complex, and each term of the interpolation function is characterized by different weighted combinations of the independent variables at the two endpoints. By reasonably selecting the combination structures, the derived discrete equations can completely separate the physical quantities at the two endpoints. By introducing a new independent variable, the nonlinear characteristics can be recursively propagated from the entire solution domain to the outer boundary locations, thereby reducing the nonlinear degree. As an example, the method was utilized in solving the electrostatic field in a nonlinear dielectric, and compared with a conventional fixed-point code to verify the effectiveness and efficiency.

II. THE NONLINEARITY-DEGREE-REDUCTION METHOD

A. Discrete Form of Poisson's Equations

For numerical investigation of static fields expressed by Poisson's equations, such as electric, magnetic, thermal, and gravitational fields, the discretization of the field quantities in tessellated grids to find the algebraic governing equations is the key process. By the cell method, the discrete conservation relation in point P of the cell complex shown in Fig. 1 can be expressed as:

$$\sum C_{P-M} \tau_P - \sum C_{P-M} \tau_M = S_P \quad (M = W, E, S, \text{ and } N) \quad (1)$$

where C_{P-M} is the general coefficient relating the line connecting point P to its nearby point M , τ the general field quantity, and S the general source.

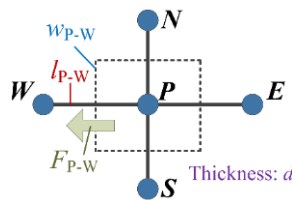


Fig. 1. A typical cell complex.

In the discrete algebraic equation, for example, the items related to points P and W mean the general flux (F_{P-W}) passing through point P to point W , as follows:

$$F_{p-w} = C_{p-w}(\tau_p - \tau_w) = \xi_{p-w} D_{p-w}(\tau_p - \tau_w) \quad (2)$$

where ξ_{P-W} is the general physical parameter relevant to the general independent variable value τ , and D_{P-W} is the general size parameter. For instance, in a nonlinear electrostatic field, τ is the electric potential φ , and ξ_{P-W} is the φ -related dielectric constant ε_{P-W} . The general flux F_{P-W} is the electric flux Ψ_{P-W} expressed as follows:

$$\Psi_{P-W} = \frac{\varepsilon_{P-W} w_{P-W} d}{l_{P-W}} (\phi_P - \phi_W) \quad (3)$$

As ε_{P-M} is related to φ nonlinearly, iterative methods are commonly used to solve the problem.

B. Reduction of Nonlinearity Degree

To reduce the computational complexity, the most effective way is to reduce the nonlinearities in the discrete general flux equation (2) and the overall algebraic governing equations. When ignoring the effect of hysteresis, the nonlinear parameter ζ_{P-W} can be described by an n^{th} -order interpolation function as:

$$\zeta_{P-W} = k_0 + k_1 \tau_{P-W} + k_2 \tau_{P-W}^2 + \dots + k_n \tau_{P-W}^n \quad (4)$$

where k_n is the n^{th} -order interpolation coefficient. In (4), the basis polynomial is co-related to the independent variable value τ_{P-W} at the midpoint of line l_{P-W} and can be functioned by the values of τ_P and τ_W at the two endpoints of the line as follows:

$$\begin{aligned} \zeta_{P-W} = & k_0 + k_1 \frac{\tau_P + \tau_W}{2} + k_2 \frac{\tau_P^2 + \tau_P \tau_W + \tau_W^2}{3} + \dots \\ & + k_n \frac{\tau_P^n + \tau_P^{n-1} \tau_W + \tau_P^{n-2} \tau_W^2 + \dots + \tau_W^n}{n+1} \end{aligned} \quad (5)$$

The general flux function can then be expressed as:

$$\begin{aligned} F_{P-W} = & (k_0 + k_1 \frac{\tau_P + \tau_W}{2} + k_2 \frac{\tau_P^2 + \tau_P \tau_W + \tau_W^2}{3} + \dots \\ & + k_n \frac{\tau_P^n + \tau_P^{n-1} \tau_W + \tau_P^{n-2} \tau_W^2 + \dots + \tau_W^n}{n+1}) D_{P-W} (\tau_P - \tau_W) \\ = & D_{P-W} [(k_0 \tau_P + \frac{k_1 \tau_P^2}{2} + \frac{k_2 \tau_P^3}{3} + \dots + \frac{k_{n-1} \tau_P^n}{n} + \frac{k_n \tau_P^{n+1}}{n+1}) \\ & - (k_0 \tau_W + \frac{k_1 \tau_W^2}{2} + \frac{k_2 \tau_W^3}{3} + \dots + \frac{k_{n-1} \tau_W^n}{n} + \frac{k_n \tau_W^{n+1}}{n+1})] \end{aligned} \quad (6)$$

Therefore, by defining a new independent variable H_M as the following:

$$H_M = k_0 \tau_M + \frac{k_1 \tau_M^2}{2} + \frac{k_2 \tau_M^3}{3} + \dots + \frac{k_{n-1} \tau_M^n}{n} + \frac{k_n \tau_M^{n+1}}{n+1} \quad (7)$$

the overall discrete governing equation can be expressed as:

$$\begin{aligned} S_P = & \sum C_{P-M} \tau_P - \sum C_{P-M} \tau_M \\ = & (D_{P-W} + D_{P-E} + D_{P-N} + D_{P-S}) H_P - D_{P-W} H_W \\ & - D_{P-E} H_E - D_{P-N} H_N - D_{P-S} H_S \end{aligned} \quad (8)$$

As illustrated in (8), the discrete governing equations are linearized for interior points, and the original variable τ_P can be computed by (7) after the new variable H_P is obtained. It is worth noting that, for the Dirichlet boundary, $\tau_{\Gamma 1} = C_1$, applied in the original problem, it is still in the Dirichlet form in the new model, $H_{\Gamma 1} = k_0 C_1 + (k_1 C_1^2)/2 + \dots + (k_n C_1^{n+1})/(n+1)$, and the field equation is totally linearized. When the Neumann boundary, $d\tau_{\Gamma 2}/dn = C_2$, or the Robin boundary, $d\tau_{\Gamma 3}/dn + C_{31} \tau_{\Gamma 3} = C_{32}$, where $\mathbf{n}$ is the normal direction, is applied, these boundary equations become nonlinear iterative ones as follows:

$$\frac{dH_{\Gamma 2}}{dn} = k_0 C_2 + k_1 C_2 \tau_{\Gamma 2} + k_2 C_2 \tau_{\Gamma 2}^2 + \dots + k_n C_2 \tau_{\Gamma 2}^n \quad (9)$$

$$\begin{aligned} \frac{dH_{\Gamma 3}}{dn} = & k_0 C_{32} + (k_1 C_{32} - k_0 C_{31}) \tau_{\Gamma 3} + (k_2 C_{32} - k_1 C_{31}) \tau_{\Gamma 3}^2 \\ & + \dots + (k_n C_{32} - k_{n-1} C_{31}) \tau_{\Gamma 3}^n - k_n C_{31} \tau_{\Gamma 3}^{n+1} \end{aligned} \quad (10)$$

As nonlinearity propagates from the entire domain to the external boundaries, the overall nonlinear degree is reduced.

III. NUMERICAL INVESTIGATIONS AND VERIFICATION

A simple electrostatic problem in a Kerr nonlinear dielectric, as illustrated in Fig. 2, is analyzed using the proposed approach and the fixed-point method. Table I compares the numerical results with the analytical solution. The fixed-point method iterates three steps to get a converged solution, while the proposed method has no iteration.

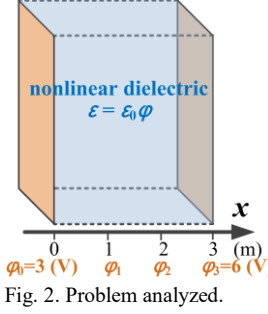


Fig. 2. Problem analyzed.

TABLE I
RESULT COMPARISONS

Location	$x_1=1$ m	$x_2=2$ m
Analytical model	$\varphi = 3\sqrt{x+1}$	
	$\varphi_1=4.2426$	$\varphi_2=5.1962$
Fixed-point method	$\varphi_{11}=4.0000$	$\varphi_{21}=5.0000$
	$\varphi_{12}=4.2427$	$\varphi_{22}=5.2092$
	$\varphi_{13}=4.2436$	$\varphi_{23}=5.1965$
Proposed method	$H_1=9.00\varepsilon_0$	$H_2=13.50\varepsilon_0$
	$\varphi_1=4.2426$	$\varphi_2=5.1962$
	No iteration	

IV. CONCLUSION

This paper proposes a novel nonlinearity-degree-reduction method that propagates the nonlinearity from the entire solution domain to its external boundary to reduce the computational complexity of solving highly nonlinear Poisson's equations by the cell method. The accuracy and efficiency of the proposed method are validated by comparing the numerical solutions of an electrostatic problem in a nonlinear dielectric by the proposed method and the conventional fixed-point method with the analytical solution.

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