

# Physically Interpretable RC Cauer Ladder Network Representation for the Heat Transfer Analysis

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This paper proposes a derivation of a physically interpretable RC Cauer Ladder Network (CLN) for heat transfer analysis. Originally, CLN method was formulated for eddy current problems, represented as a voltage-driven RL-CLN. Although a generalized CLN formulation for parabolic partial differential equations was later established, it was also based on RL-CLN and lacked a clear physical interpretation. This paper revisits the CLN formulation for thermal circuits, deriving a current-driven RC-CLN representation that offers a more explicit physical meaning than a simple transfer function based CLN. Additionally, we demonstrate that an analytical expression for the RC-CLN can be derived for simple two-dimensional thermal problems, and verify its accuracy by reproducing results obtained through FEM analysis. By utilizing this thermal RC-CLN, we present its effectiveness in achieving efficient computation through reduced calculation time, particularly in cases with complex pulsed input heat waveforms.

**Index Terms**— Cauer ladder network, Induction heating, model order reduction, parabolic PDE, transient analysis.

## I. INTRODUCTION

Coupled eddy current and thermal analysis are essential for the optimal design of various electrical apparatus, with applications ranging from induction heating to magnetic levitations. Accurately predicting the temperature distribution, especially under dynamic electromagnetic fields, is crucial for ensuring performance, reliability, and safety. For instance, in induction heating, understanding the interplay between induced eddy currents and the resulting temperature distribution is vital for optimizing heating efficiency and preventing overheating.

Thermal circuit analysis, with its lower computational cost, remains a popular alternative to FEM for thermal design. However, this method requires expert knowledge and provides only average temperatures. To leverage the advantages of both methods, approaches combining FEM and thermal circuit analysis have been proposed [2]. Among various MOR techniques, the CLN method, proposed in 2017, shows promise for efficient coupled simulations. Initially targeting eddy current fields using a voltage-driven RL-CLN [4], as shown in Fig. 1. Extending the CLN method to thermal analysis offers a significant advantage: both electromagnetic and thermal domains can be seamlessly connected through their respective circuit representations. This facilitates a more integrated and efficient approach to coupled electro-thermal simulations, potentially reducing computational cost and complexity. In 2020, a generalized formulation called CVN, applicable to parabolic Partial Differential Equations, was proposed [5,6], with reported applications to thermal problems [7]. However, in thermal analysis, these circuits, based on voltage-driven RL-CLN, lacked clear physical interpretations.

This paper revisits the CLN formulation specifically for thermal circuits. Instead of simply replicating the transfer function with CLN, we derive a current-driven RC-CLN representation that offers a more explicit physical meaning. While the topology of this circuit differs from conventional thermal circuits, it yields physically meaningful components:

resistances  $R_{th2n}$  [K/W] corresponding to thermal conductivity and capacitances  $C_{2n+1}$  [J/K] representing specific heat capacity. We demonstrate the feasibility of deriving analytical expressions for this thermal RC-CLN in simple two-dimensional problems and validate its accuracy by reproducing results obtained through FEM simulations. Furthermore, we illustrate that this thermal RC-CLN enables efficient computation through reduced calculation time, particularly in scenarios with complex pulsed input heat waveforms.

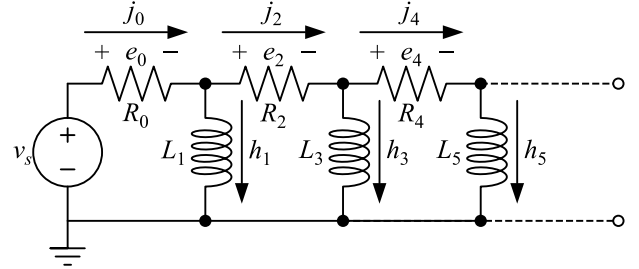


Fig. 1. Voltage source-driven RL-CLN representation for eddy current field analysis.

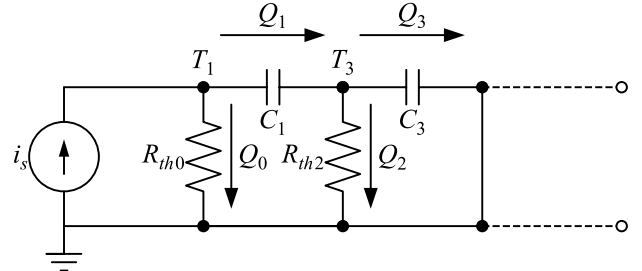


Fig. 2. Current source-driven RC-CLN representation for heat transfer analysis.

## II. FORMULATION

Let  $T$  [K] represent temperature,  $\lambda$  [W/(K·m)] thermal conductivity,  $\rho$  [kg/m<sup>3</sup>] density, and  $c$  [J/(m<sup>3</sup>·K)] specific heat capacity. Considering  $q_s(t)$  [W/m<sup>3</sup>] as a time-varying heat generation density while keeping its spatial distributions, the general linear heat conduction equation is given by:

$$\nabla \cdot (-\lambda \nabla T) + \rho c \frac{\partial T}{\partial t} = q_s(t) \quad (1)$$

By imposing the condition that this equation, when expanded using RC-CLN basis functions, becomes identically equal to the equation derived from the RC-CLN in Fig. 2, and carefully comparing the terms, we derive the pseudo-algorithm presented in Algorithm 1. Due to the limitation of the pages, the detailed derivation will be presented in fullpaper.

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**Algorithm 1** CLN method for heat transfer analysis

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$$T_{-1} = 0$$

$$q_0 = q_s(\mathbf{r})$$

**for**  $n = 0$  to  $N$  **do**

$$\text{Solve: } \nabla \cdot (-\lambda \nabla \tilde{T}) = q_{2n}$$

$$R_{th2n} = \int \frac{(-\lambda \nabla \tilde{T})^2}{\lambda} d\Omega$$

$$T_{2n} = \frac{\tilde{T}}{R_{th2n}}$$

$$T_{2n+1} = T_{2n-1} + T_{2n}$$

$$C_{2n+1} = \int \rho c (T_{2n+1})^2 d\Omega$$

$$q_{2n+2} = q_{2n} - \rho c \frac{T_{2n+1}}{C_{2n+1}}$$

**end for**

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### III. NUMERICAL ANALYSIS

To validate the proposed method, we first present a numerical example of a benchmark problem involving the heating of a two-dimensional circular conductor with radius  $a = 50$  mm, thermal conductivity  $\lambda = 500$  W/(K·m), density  $\rho = 8960$  kg/m<sup>3</sup>, and specific heat capacity  $c = 385$  J/(m<sup>3</sup>·K). The outer boundary is subjected to a Dirichlet boundary condition with a constant temperature  $T = 0$  K. This problem has an analytical RC-CLN solution, and by applying the pseudo-algorithm in Algorithm 1, we obtain:

$$R_{th2n} = \frac{1}{8(n+1)^5 \pi \lambda} [(K \cdot m)/W] \quad (2)$$

$$C_{2n+1} = \frac{(n+1)^2 (n+2)^2 \pi a^2 \rho c}{2n+3} [J/(K \cdot m)] \quad (3)$$

These circuit elements  $R_{th2n}$  and  $C_{2n+1}$  quantities have the same units as the elements of general thermal circuits. This analytical RC-CLN solution includes corresponding basis functions for each circuit element  $R_{th2n}$  and  $C_{2n+1}$ . The temperature distribution can be reconstructed by combining the basic functions with the results of circuit analysis, weighted by their circuit quantities (i.e., heat flux).

Fig. 3 shows the transient waveform of the heat generation density. In applications such as induction heating, the heat generation density can exhibit a pulsed waveform due to controlling the temperature of the work. Analyzing such

situations with FEM requires small time steps, leading to increased computational cost. Fig. 4 compares the temporal variation of the temperature at the center of the conductor obtained using the thermal RC-CLN with varying numbers of stages and FEM. A 4-stage RC-CLN proves to be sufficient for reproducing the FEM results with satisfactory accuracy.

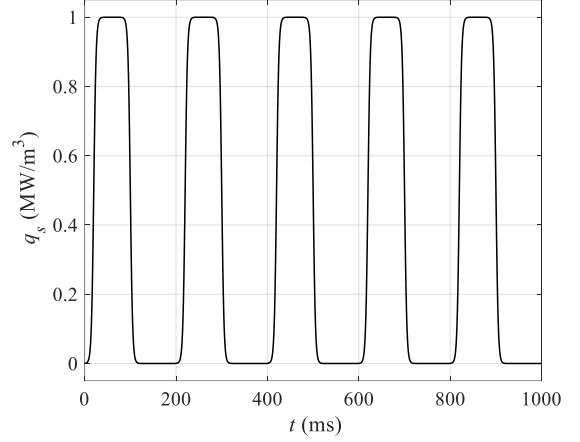


Fig. 3. Temporal variation of heat generation density.

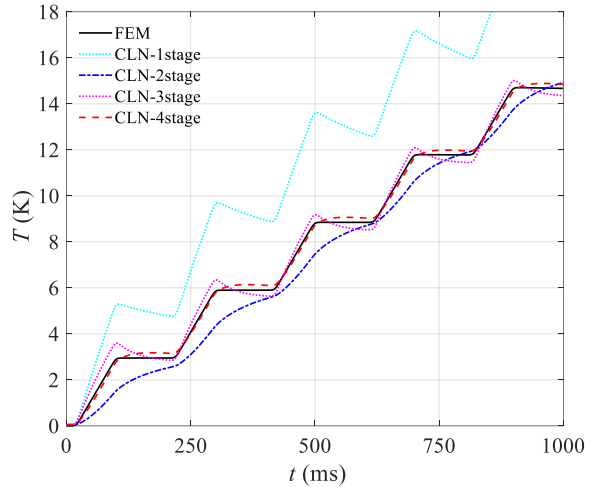


Fig. 4. Transient thermal response at the center of a circular conductor.

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