

Convergence Analysis of the Parallel Block Jacobi ICCG Solver Through Eigenvalue Distribution

Takeshi Iwashita^{1,3}, *IEEE*, Yasuhito Takahashi², *IEEE*, and Takeshi Fukaya³, *IEEE*

¹Academic Center for Computing and Media Studies, Kyoto University, Kyoto, 606-8501 Japan, iwashita@i.kyoto-u.ac.jp

²Department of Electrical Engineering, Doshisha University, Kyotanabe, 610-0394 Japan, ytakahashi@mail.doshisha.ac.jp

³Information Initiative Center, Hokkaido University, Sapporo, 060-0811 Japan, fukaya@iic.hokudai.ac.jp

In this paper, we investigate the parallel Block Jacobi ICCG (BJ-ICCG) solver. It is well-known that the BJ-ICCG method suffers from the convergence degradation as the number of threads increases. We analyze this phenomenon by computing the eigenvalue distribution of the preconditioned coefficient matrix. Because the calculation of the eigenvalues requires significant computational effort, it has been rarely performed for three-dimensional eddy current analyses with large problem sizes. In our analysis, we observed that many eigenvalues are shifted in an unfavorable direction, resulting in the decline in convergence as the number of threads increases.

Index Terms—Eddy current problem, Eigenvalue distribution, ICCG method, Parallel processing

I. INTRODUCTION

ELECTROMAGNETIC field analysis is an essential tool for designing various electric machines, such as motors and transformers. The finite element method is a standard approach for electromagnetic field analysis, particularly for solving eddy current problems. However, finite element analysis involves solving linear systems, which is a time-consuming process. Although linear iterative solvers are typically employed for this process, the solution time often constitutes the majority of the total computational time. Therefore, a high performance iterative linear solver is always in demand.

In this paper, we focus on the Incomplete Cholesky Conjugate Gradient (ICCG) solver, one of the most commonly used linear solvers for finite element electromagnetic field analyses. When we consider current architectures of CPUs, which are equipped with multiple cores, parallel processing is one of the most critical techniques to accelerate all kinds of computations. However, it is well known that the IC preconditioning cannot be naturally parallelized because it involves forward and backward substitutions (sparse triangular solver). Accordingly, various parallelization methods have been proposed to parallelize IC preconditioning or sparse triangular solvers [1]. Among these methods, block Jacobi IC preconditioning is frequently used in finite element electromagnetic field analyses [2]. This method is also referred to as localized IC or block IC preconditioning in many reports [3]. Although it has the advantage of easy implementation, it suffers from convergence degradation. As the number of threads increases, the effectiveness of the preconditioning gradually declines.

In this paper, we address the convergence deterioration problem of block Jacobi IC preconditioning by analyzing the eigenvalue distribution of the preconditioned coefficient matrix. Although it is well known that the convergence of the CG method is affected by the eigenvalue distribution of the coefficient matrix, such an analysis is rarely conducted for a three dimensional electromagnetic field problem due to the high computational cost involved [4].

II. PROBLEM AND ICCG METHOD

In this paper, we address a linear system of equations:

$$Ax = b, \quad (1)$$

where A is an n -dimensional sparse real symmetric semi-positive definite matrix which is derived from a finite element electromagnetic field analysis. When solving the linear system by using the preconditioned CG method, we solve the preconditioned system instead of (1):

$$P^{-1}AP^{-\top}(P^{\top}x) = P^{-1}b, \quad (2)$$

where $M = PP^{\top}$ is the preconditioner matrix.

When IC, more precisely IC(0), preconditioning is employed, the IC factorization is applied to the coefficient matrix A :

$$A \simeq LL^{\top}, \quad (3)$$

where L is a lower triangular matrix. In the IC(0) factorization, all fill-ins are ignored during the factorization process. Consequently, the nonzero element patterns of the lower triangular part of A and L are identical. In the ICCG method, although P is given by L , the preconditioned linear system (2) is not explicitly constructed. The preconditioning step is performed through the forward and backward substitutions in each iteration as follows:

$$y = L^{-1}r, \quad z = L^{-\top}y, \quad (4)$$

where r is the residual vector. While the CG method can be easily parallelized, the forward and backward substitutions cannot be naturally parallelized.

III. BLOCK JACOBI IC PRECONDITIONING AND ITS EFFECTIVENESS

The Block Jacobi IC (BJ-IC) preconditioning is one of the standard parallelization methods for the ICCG method. In this method, both rows and columns of the coefficient matrix are divided based on the number of threads. Then, only the

TABLE I
NUMBER OF ITERATIONS

# threads	1	2	4	8	16	32	64	96
# iterations	51	77	83	91	101	121	142	151

diagonal blocks are used for the IC factorization. The factorized matrix $\hat{\mathbf{A}}$ is defined as follows:

$$\hat{a}_{ij} = \begin{cases} a_{ij} & \text{if } \exists t \in \{1, \dots, n_t\}, s_t \leq i, j \leq e_t \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

where $\hat{a}_{ij}$ and a_{ij} represent the i -th row and j -th column elements of $\hat{\mathbf{A}}$ and $\mathbf{A}$, respectively, and n_t denotes the number of threads. For simplicity, we assume in (5) that n is a multiple of n_t , and s_t and e_t are given by $s_t = n(t-1)/n_t + 1$ and $e_t = nt/n_t$, respectively, for $t = 1, \dots, n_t$. In BJ-IC preconditioning, $\hat{\mathbf{A}}$ is incompletely factorized in parallel, and the substitutions during iterations are independently performed in each thread.

In a finite element electromagnetic field analysis, the shifted IC factorization is typically employed to avoid breakdown during the factorization process. Thus, more precisely, the target matrix for the IC factorization is defined as $\bar{\mathbf{A}} = \hat{\mathbf{A}} + \gamma \text{diag}(\hat{\mathbf{A}})$, where $\text{diag}(\hat{\mathbf{A}})$ is the diagonal part of $\hat{\mathbf{A}}$, and γ is referred to as the acceleration factor in our area. The shifted matrix $\bar{\mathbf{A}}$ is incompletely factorized as

$$\bar{\mathbf{A}} \simeq \bar{\mathbf{L}} \bar{\mathbf{L}}^T, \quad (6)$$

and $\bar{\mathbf{L}}$ is used for $\mathbf{P}$ in (2). We examine the eigenvalue distribution of the preconditioned coefficient matrix $\bar{\mathbf{L}}^{-1} \bar{\mathbf{A}} \bar{\mathbf{L}}^{-T}$.

IV. NUMERICAL RESULTS

We used a linear system derived from a three-dimensional finite element eddy current analysis for the numerical test. Since the A - ϕ formulation was employed without gauge conditions, the resulting coefficient matrix is semi-positive definite. The program of the solver was written in Fortran, and LAPACK routines were used to calculate the eigenvalue distribution of the preconditioned matrix. Figure 1 illustrates the 1/8 model used in the analysis, where an external time-varying magnetic field is applied to a sphere conductor. The linear system was solved using the ICCG method and its parallel version, BJ-ICCG. The convergence criterion was defined as a relative residual norm of less than 10^{-8} . The acceleration factor was set to 0.03.

Table I shows the number of iterations for the BJ-ICCG solver. As the number of threads increases, the number of iterations also rises. This phenomenon, commonly observed in other analyses, is a significant drawback of BJ-IC preconditioning. To investigate this characteristic, we calculated the eigenvalue distribution of the preconditioned coefficient matrix. Figure 2 depicts the eigenvalue distributions of the preconditioned matrix when the number of threads is 1, 16, or 64. In the figure, only positive eigenvalues are shown, though the matrix contains many zero eigenvalues. It can be observed that the condition of the preconditioned matrix deteriorates as the number of threads increases. Specifically, many eigenvalues are shifted

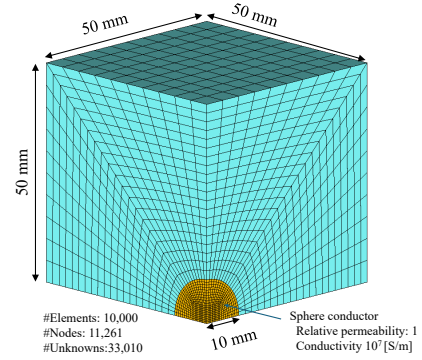


Fig. 1. Model of finite element eddy current analysis

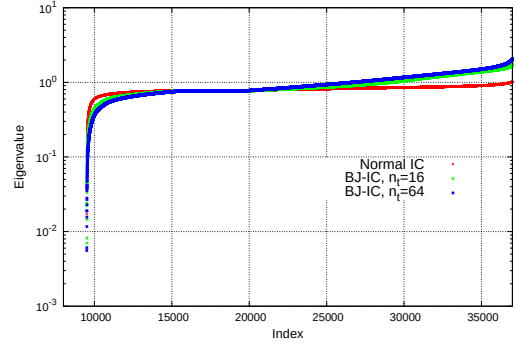


Fig. 2. Eigenvalue distribution

to unfavorable directions: smaller eigenvalues become even smaller, and larger eigenvalues become even larger. As a result, we gained the following insight: it is difficult to mitigate the adverse effect of the BJ-IC preconditioning in a highly parallel environment using convergence acceleration techniques such as the deflation or the subspace correction methods [5].

V. CONCLUSION

We evaluated the eigenvalue distribution of the preconditioned coefficient matrix in a three-dimensional finite element eddy current analysis using a parallel block Jacobi ICCG solver. The numerical results confirm that the matrix becomes increasingly ill-conditioned as the number of threads increases. We will report the results from additional models and provide a more detailed analysis of the eigenvalue distribution.

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